On sums of primes

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1. Introduction

In this paper we prove the following

Theorem. *Every even natural number can be represented as a sum of at most eighteen primes.*

It follows at once that every natural number n with $n>1$ is a sum of at most nineteen primes. The previous best result of this kind is due to Deshouillers [2] who has twentysix in place of nineteen.

Let $N(x)$ denote the number of even numbers n not exceeding x for which n is the sum of at most two primes. Then it suffices to show that

$$N(x) > x/18 \quad (x \geqq 2), \tag{1.1}$$

for then the theorem will follow in the usual manner (for example as in § 6 of [7]).

The proof of (1.1) is divided into three cases according to the size of x. When $\log x \geqq 375$ we use the method described in § 7 of [7], but with an important modification that enables us to dispense altogether with the Brun–Titchmarsh theorem. When $\log x \leqq 27$ the inequality (1.1) is easy to establish. This leaves the intermediate region $27 < \log x < 375$. Here we develop a completely new argument, based partly on sieve estimates and partly on calculation.

2. Some constants

We give here a list of constants that arise in the proof together with estimates for their values. A detailed description of the more difficult calculations is given in § 10.

Let

$$\gamma_k = \lim_{n\to\infty} \left(\sum_{m=1}^{n} m^{-1} (\log m)^k - \frac{(\log n)^{k+1}}{k+1} \right). \tag{2.1}$$