On the regularity of difference schemes
Part II. Regularity estimates for linear and nonlinear problems

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1. Preliminaries

1.1. Discrete regularity estimate

Let L be an elliptic differential operator of second order. Usually, the differentiability of the solution u of

(1.1) $$Lu = f \quad (\Omega), \quad u|_\Gamma = 0,$$

is two orders larger than the order of differentiability of f. This property can be expressed in terms of Sobolev spaces,

(1.2a) $$\|L^{-1}\|_{H^s(\Omega)\to H^{2+s}(\Omega)} \leqq C$$

or in terms of Hölder spaces,

(1.2b) $$\|L^{-1}\|_{C^s(\bar\Omega)\to C^{2+s}(\bar\Omega)} \leqq C \qquad (s > 0, \quad s \neq \text{integer}).$$

For the notation of the various spaces and of the norm, see Section 1.3.

The discretization of the boundary value problem is written as

(1.3) $$L_h u_h = f_h,$$

where h denotes the discretization parameter (usually: grid size). Let $H_h^s(\Omega_h)$ be the discrete analogue of $H^s(\Omega)$ (derivatives replaced by differences). Then we want to prove the counterpart of (1.2a):

(1.4) $$\|L_h^{-1}\|_{H_h^s(\Omega_h)\to H_h^{2+s}(\Omega_h)} \leqq C \quad \text{uniformly in } h.$$