On the separation properties of the duals of general topological vector spaces

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Consider a non locally convex topological vector space E. Suppose there is a non-zero point x in the subset $E_1 \subset E$ of all points which cannot be separated from the origin by any continuous linear form on E. One might ask whether x can be separated from the origin by a continuous linear form which is defined only on E_1. It will be shown by means of examples that this may be the case. Since E_1 is a linear subspace — namely, the intersection $\bigcap_{f \in E'} f^{-1}(0)$ of all closed hyperplanes through the origin — this fact gives rise to a more general question: Let E_2 be the subspace in E_1 of points which cannot be separated from the origin by any continuous linear form; and then define recursively subspaces $E_3 \supset E_4 \supset \ldots$ In a natural way we thus get a transfinite decreasing »sequence» (indexed by the ordinals) of closed linear subspaces of E. Obviously, this sequence must become stationary at some ordinal $\alpha(E)$. The observation just mentioned shows that this need not happen at once, i.e., we may have $E_1 \neq E_2$, so that $\alpha(E) \geq 2$. Thus we ask: Which ordinal values can be assumed by $\alpha(\cdot)$? Our aim in II below is to answer this question by constructing examples to show that even for locally bounded spaces, $\alpha(\cdot)$ may assume *any* ordinal value.

In I below, we investigate some general properties of the transfinite »sequence» mentioned above.

As a by-product of the constructive methods employed in II, we obtain (Section II.3) a certain isometric imbedding of metric spaces into p-normed spaces, which has universal and functorial properties.

For brevity, we will write tvs for topological vector space(s). Further, E' will denote the (topological) dual of a tvs E, and ω is as usual the least transfinite ordinal. All tvs are supposed to have the same scalar field, which may be the real or the complex number field. A linear subspace of a tvs is always topologized by the subspace topology.