Entire functions of several variables and their asymptotic growth

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§ 1. Introduction

Let $f(z)$ be an entire function of several complex variables of finite order ϱ and normal type (in what follows, we shall always let the variable z represent an n-tuple $(z_1, \ldots, z_n)$ and $\|z\| = (\sum_{j=1}^{n} z_j \bar{z}_j)^{1/2}$ is the Euclidean norm).

Following the classical case of functions of one variable [7], we introduce the functions $h(z) = \overline{\lim_{r\to\infty}} \frac{\ln|f(rz)|}{r^\varrho}$ and $h^*(z) = \overline{\lim_{z'\to z}} h(z')$, and we call $h^*(z)$ the radial indicator (of growth) function [4, 5, 6].

Both $h(z)$ and $h^*(z)$ are positively homogeneous of order ϱ; that is for $t > 0$, $h(tz) = t^\varrho h(z)$ and $h^*(tz) = t^\varrho h^*(z)$. Lelong has further shown that $h(z) = h^*(z)$ except on a set of $\mathbf{R}^{2n}$ Lebesgue measure zero, and since both are positively homogeneous of order ϱ, $h(z) = h^*(z)$ for almost all $z \in S^{2n-1}$, the unit sphere in $\mathbf{C}^n$. The function $h^*(z)$ is plurisubharmonic and is independent of the point in $\mathbf{C}^n$ chosen as origin (thus, if $f(z) \not\equiv 0$, it will always be possible to assume, without loss of generality, that $f(0) \neq 0$).

There are certain properties of the classical indicator function of one variable which have no counterpart for n variables ($n \geqq 2$). The classical indicator function is continuous [7, p. 54], but Lelong [6] has shown that this is no longer necessarily true for $n \geqq 2$. His method was to construct (for all ϱ) a non-continuous plurisubharmonic function complex homogeneous of order ϱ.

For functions of one variable, the growth of the function $f(z)$ is determined by the density and distribution of the zeros [7]. In particular, the regularity of the distribution of the zeros determines the regularity of the function $f(z)$ and the regularity of the indicator function. Our criteria for regularity of growth will be the following: Let E be a measurable set of positive numbers and $E_r = E \cap [0, r)$. If $\lim_{r\to\infty} \frac{\text{meas}(E_r)}{r} = 0$, E is said to be a set of zero relative measure (an E^0-set).