Comparison theorems for a generalized modulus of continuity

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1. Introduction

In previous work [18], [19], [20], one of the authors introduced a generalized modulus of continuity of a function $f \in L^p$. Like the usual L^p modulus of continuity it is a function of a positive variable a, but depends also upon a measure σ. By suitable specialization of σ this generalized modulus (written $\omega_{\sigma,p}(f\,;a)$) can serve as a measure either of the smoothness of a function or of the degree to which the function is approximable in L^p norm by its convolution with $(1/a)k(t/a)$, k being a given integrable function. *Comparison theorems* were proved enabling the τ modulus to be estimated in terms of the σ modulus under certain conditions, and this enabled several questions concerning so-called direct and inverse theorems of approximation theory to be studied from a unified viewpoint.

The main reason for writing a new paper on the subject is as follows. In the cited work, only sup norm estimates (i.e. $p = \infty$) were treated in detail, apart from a remark in [20] that identical inequalities were valid when all norms were interpreted in the L^p sense. While this is correct, examination showed that the results so obtained were unsharp for values of p other than 1 and ∞, in many typical cases where one would like to apply the method. Thus, for example, although the theory yielded the sharp Marchaud estimates (see [13], p. 48) for the (sup norm) modulus of continuity in terms of the second order modulus of smoothness, it yielded the identical estimate for all values of p. But it is known from work of A. F. Timan ($p = 2$) and Zygmund (general p) that sharper estimates are valid when $1 < p < \infty$ (more details below in § 6, see also [26], p. 121 and [30]).

The clue to overcoming the difficulty was provided by Zygmund's paper [30]. In this paper (seldom quoted in the literature, although it pioneered a technique which has since found wide application) Zygmund employed a characteristic method based upon the decomposition of the Fourier series into blocks of the type $\sum c_k e^{ikt}$, the summation being from 2^n to $2^{n+1} - 1$, to which he then applied a rather deep inequality due to Littlewood and Paley.