On the spectral synthesis problem for $(n-1)$-dimensional subsets of $\mathbf{R}^n$, $n \geq 2$

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1. Introduction

Let E be a closed subset of $\mathbf{R}^n$ and $K(E)$ the space of all functions in $\mathcal{D}(\mathbf{R}^n)$, vanishing in some neighborhood of E. $\mathcal{F}L^1(\mathbf{R}^n)$ is the Banach space of Fourier transforms of functions in $L^1(\mathbf{R}^n)$. $\mathcal{D}(\mathbf{R}^n) \subset \mathcal{F}L^1(\mathbf{R}^n)$, and we denote by $\overline{K(E)}$ the closure of $K(E)$ in $\mathcal{F}L^1(\mathbf{R}^n)$. The well-known concept of sets of spectral synthesis can be defined as follows: E is a set of spectral synthesis if $\overline{K(E)}$ contains every element in $\mathcal{F}L^1(\mathbf{R}^n)$ that vanishes on E.

C. Herz [3] has proved that $S^1 \subset \mathbf{R}^2$ is a set of spectral synthesis. His proof can unfortunately not be extended to obtain the corresponding result for more general curves. It is however possible to use a different approach to get the desired extension of the result of Herz (cf. [2]). We shall here apply basically the same method to investigate a still more general problem.

As was discovered by L. Schwartz [9], the sphere $S^{n-1} \subset \mathbf{R}^n$ is not of spectral synthesis, if $n \geq 3$. N. Th. Varopoulos [10] has investigated this question in more detail, using methods related to the Herz method for $n = 2$. Let us denote, for any closed set E and any positive integer m, by $J_m(E)$ the space of functions in $\mathcal{D}(\mathbf{R}^n)$, $n \geq 2$, vanishing on E together with all their partial derivatives of order $\leq m - 1$. Taking closures in $\mathcal{F}L^1(\mathbf{R}^n)$, we have then

$$\overline{J_1(S^{n-1})} \supset \overline{J_2(S^{n-1})} \supset \ldots \supset \overline{J_{[(n+1)/2]}(S^{n-1})} = \overline{K(S^{n-1})}\,, \tag{1.1}$$

where all inclusions are strict. It is very easy to understand from this why there is a fundamental difference between the case $n = 2$ and the case $n \geq 3$ in this context.

The cited paper of Varopoulos does however contain a considerably more precise description of the situation than the one given above. Let us by $B_m(S^{n-1})$, $m \geq 1$, denote the linear space spanned by all measures on S^{n-1} with infinitely differentiable