Estimates for the Fourier transform of the characteristic function of a convex set

INGVAR SVENSSON

University of Lund, Sweden

1. Introduction

Let C be a measurable set in R^{n+1} and set

$$\hat{u}_C(\xi) = \int_C u(x) e^{i<x,\xi>} \, dx \,, \quad \xi \in R^{n+1} \,, \, u \in C_0^\infty(R^{n+1}) \,.$$

The order of magnitude of $\hat{u}_C(\xi)$ when $\xi \to \infty$ is frequently of importance in harmonic analysis, for example in application to analytic number theory. However, even if one assumes that C is the closure of an open set with boundary $\partial C \in C^\infty$ the known results are far from complete. It is known then that

$$\hat{u}_C(\xi) = O(|\xi|^{-(n+2)/2}) \,, \quad \xi \to \infty \,; \quad u \in C_0^\infty \,; \tag{1.1}$$

if and only if the Gaussian curvature of ∂C never vanishes (Herz [1], Hlawka [2], Littman [3]). Randol [4], [5] has also studied the case where C is convex and ∂C is analytic. His result is that the »maximal function»

$$\tilde{u}(\xi) = \sup_{r>0} r^{(n+2)/2} |\hat{u}_C(r\xi)| \,, \quad \xi \in S \tag{1.2}$$

is then in $L^p(S^n)$ for some $p > 2$ if ∂C is analytic. In fact, Randol proved that this is true for precisely those $p > 2$ such that

$$\int_{\partial C} K(x)^{(2-p)/2} \, dS(x) < \infty \tag{1.3}$$

where $K(x)$ is the Gaussian curvature at $x \in \partial C$. The necessity of (1.3) follows easily from the fact that

$$r^{(n+2)/2} |\hat{u}_C(r\xi)| \to c(|u(x_+)| K(x_+)^{-1/2} + |u(x_-)| K(x_-)^{-1/2})$$