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# Differentiability properties of Orlicz–Sobolev functions

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## 1. Introduction and main results

In this paper we are concerned with the pointwise behaviour of functions in certain classes of weakly differentiable functions.

The ancestor of all modern results dealing with pointwise properties of non-smooth functions is certainly the Lebesgue differentiation theorem, which asserts that if $\Omega$ is an open subset of $\mathbf{R}^n$, $n\geq 1$, and $u$ is a locally integrable function in $\Omega$, then $\lim_{r\to 0^+} \fint_{B_r(x)} u(y)\,dy$ exists and is finite for a.e. $x\in\Omega$, and

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y)-u(x)|\,dy=0 \tag{1.1}$$

for a.e. $x\in\Omega$. Here, $B_r(x)$ is the ball centered at $x$ and having radius $r$, and $\fint_E u(y)\,dy$ stands for $(1/|E|)\int_E u(y)\,dy$, when $E$ is a measurable set with finite Lebesgue measure $|E|$. A point $x$ where (1.1) holds will be called a *Lebesgue point* for $u$, or a point of *approximate continuity* for $u$. (This terminology is borrowed from [AFP], where a comparison with a slightly weaker definition of approximate continuity due to Federer can also be found.) The function defined as the limit of the averages of $u$ at those points where such a limit exists, and 0 elsewhere, is usually referred to as the precise representative of $u$. Henceforth, we shall assume that every locally integrable function is precisely represented.

It has been long known that Sobolev functions fulfill (1.1) in a stronger sense, in that the exceptional set of those points where (1.1) does not hold is considerably smaller. The size of this exceptional set can be properly estimated through the notion of capacity. Indeed, one of the fundamental fine properties of Sobolev functions tells us that any element $u$ from the Sobolev space $W^{k,p}_{\mathrm{loc}}(\Omega)$ of functions endowed with $k$th order weak derivatives in $L^p_{\mathrm{loc}}(\Omega)$ satisfies

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y)-u(x)|^p\,dy=0 \tag{1.2}$$