

4 **FRÖBERG'S THEOREM, VERTEX SPLITABILITY AND HIGHER**  
5 **INDEPENDENCE COMPLEXES**  
67  
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910 **ABSTRACT.** A celebrated theorem of Fröberg gives a complete combinatorial classification of  
11 quadratic square-free monomial ideals with a linear resolution. A generalization of this theorem  
12 to higher degree square-free monomial ideals is an active area of research. The existence of  
13 a linear resolution of such ideals often depends on the field over which the polynomial ring  
14 is defined. Hence, it is too much to expect that in the higher degree case a linear resolution  
15 can be identified purely using a combinatorial feature of an associated combinatorial structure.  
16 However, some classes of ideals having linear resolutions have been identified using combinatorial  
17 structures. In the present paper, we use the notion of  $r$ -independence to construct an  $r$ -uniform  
18 hypergraph from the given graph. We then show that when the underlying graph is co-chordal, the  
19 corresponding edge ideal is vertex splittable, a condition stronger than having a linear resolution.  
20 We use this result to explicitly compute graded Betti numbers for various graph classes. Finally,  
21 we give a different proof for the existence of a linear resolution using the topological notion of  
22  $r$ -collapsibility.23 **1. Introduction**24 Let  $G$  be a finite simple graph with  $V(G) = \{x_1, \dots, x_n\}$  as its vertex set and  $E(G)$  be its  
25 edge set. The independence complex of  $G$ , denoted by  $\text{Ind}(G)$ , is the simplicial complex whose  
26 simplices are independent subsets of vertices in  $G$ . The complex  $\text{Ind}(G)$  is an important object in  
27 combinatorics which lies at the crossroads of various fields of mathematics and computer science.  
28 For example, via *edge ideals*, a concept introduced by Villarreal [17], the independence complex  
29 appears in commutative algebra. In particular, let  $R = \mathbb{K}[x_1, \dots, x_n]$  be the polynomial ring in  
30  $n$  variables over a field  $\mathbb{K}$ . Then the edge ideal  $I(G)$  of  $G$  is the quadratic square-free monomial  
31 ideal  $\langle x_i x_j \mid \{x_i, x_j\} \in E(G) \rangle$  of  $R$ . The ideal  $I(G)$  is also the *Stanley-Reisner ideal* of  $\text{Ind}(G)$ .  
32 Determining algebraic and homological properties of the ideal  $I(G)$  in terms of the combinatorial  
33 properties of  $\text{Ind}(G)$  is an active area of research in commutative algebra.34 In 2018 Paolini and Salvetti [16] considered a generalisation of the independence complex in the  
35 context of braid groups, called the  *$r$ -independence complex*  $\text{Ind}_r(G)$  of  $G$ , for any positive integer  $r$ .  
36 A subset  $A \subseteq V(G)$  is called  $r$ -independent if each connected component of the induced subgraph  
37  $G[A]$  has at most  $r$  vertices. The collection of all  $r$ -independent sets forms the simplicial complex  
38  $\text{Ind}_r(G)$ . Note that  $\text{Ind}_1(G)$  is the independence complex of  $G$ .39 The main focus of Paolini and Salvetti [16] was to understand twisted (co)homology groups  
40 of the classical braid groups via relating them to that of  $r$ -independence complexes of certain  
41 graphs. Their results indicated that these complexes are interesting in their own right. Later,

42 Last updated: March 7, 2024.

43 2010 *Mathematics Subject Classification.* 13F55, 05E45.44 *Key words and phrases.* Independence complex, Stanley-Reisner ideal, edge ideal, linear resolution, vertex  
45 splittable, collapsible complex.

1 it was proved in [4] that the  $r$ -independence complexes of cycle graphs and perfect  $m$ -ary trees  
 2 are homotopy equivalent to a wedge of spheres. Extending a result of Meshulam [14], it was also  
 3 proved in [5] that the (homological) connectivity of  $r$ -independence complexes of graphs gives an  
 4 upper bound for the distance  $r$ -domination number of graphs. In the same paper the authors  
 5 proved that the  $r$ -independence complexes of chordal graphs are homotopy equivalent to a wedge  
 6 of spheres for all  $r \geq 1$ . From the perspective of the Cohen-Macaulay property, it was shown in  
 7 [1] that the  $r$ -independence complexes of trees are shellable. Moreover, it was also proved, using  
 8 commutative algebra techniques, that the  $r$ -independence complexes of caterpillar graphs are vertex  
 9 decomposable, a property that implies the Cohen-Macaulay property.

10 In this article we focus on developing the algebraic properties of the  $r$ -independence complexes.  
 11 In particular, we focus on various algebraic and homological invariants of the Stanley-Reisner ideal  
 12  $I_r(G)$  of  $\text{Ind}_r(G)$ . The ideal  $I_r(G)$  can also be viewed as the hyperedge ideal of a hypergraph  
 13 associated to certain subgraphs of  $G$ . Specifically, let  $\text{Con}_r(G)$  denote the hypergraph with the  
 14 vertex set  $V(G)$  and the hyperedges are those  $r+1$ -subsets  $W$  such that the induced subgraph  
 15  $G[W]$  is connected. Then the Stanley-Reisner ideal of  $\text{Ind}_r(G)$  is same as the edge ideal of  $\text{Con}_r(G)$ .  
 16 Note that  $\text{Con}_1(G) = G$ . Thus  $I_1(G)$  is nothing but the usual *edge ideal* of  $G$ . Moreover, if  $G = K_n$ ,  
 17 the complete graph, or if  $G = K_{n_1, \dots, n_t}$ , the complete multipartite graph, then  $\text{Con}_r(G)$  matches  
 18 with the complete hypergraph  $K_n^{r+1}$  and a  $(r+1)$ -complete multipartite hypergraph  $K_{n_1, \dots, n_t}^{r+1}$   
 19 defined by Emtander in [6].

20 In 1988 Fröberg [7] showed that the complement of a graph is chordal if and only if the Stanley-  
 21 Reisner ideal of its independence complex has a linear free resolution. We are interested in  
 22 determining whether or not the Stanley-Reisner ideal of the  $r$ -independence complex of a graph  
 23 has a linear free resolution, given that the complement of the graph is chordal. We show that this  
 24 is indeed true, thus giving a partial generalization of Fröberg's result:

25 **Theorem 1.1.** *If  $G$  is a graph whose complement is chordal, then  $I_r(G)$  has a  $(r+1)$ -linear*  
 26 *resolution.*

27 In fact, we give two different proofs. One proof is by using the notion of vertex splittable ideals  
 28 from commutative algebra (see Corollary 3.14) and another one is by showing that the complex is  
 29  $r$ -collapsible, which is a key concept in topological combinatorics (see Theorem 5.2). Note that the  
 30 converse of Theorem 1.1 does not hold for  $r \geq 2$ ; for more, see Corollary 5.5.

31 One of the most useful ways to study the structure of a module is by analyzing the minimal free  
 32 resolution of the module. Important numerical invariants of the free resolution are its graded Betti  
 33 numbers. Using the fact that vertex splittable ideals admit a recursive formula for the graded Betti  
 34 numbers, we are able to provide explicit formulas for the  $\mathbb{N}$ -graded Betti numbers of  $I_r(G)$  for  
 35 various families of graphs.

36 This paper is organised as follows. In Section 2 we recall the relevant graph theory and Stanley-  
 37 Reisner theory. In Section 3, we use the notion of vertex splittable ideals to give our first proof of  
 38 Theorem 1.1. In Section 4 we deduce results about the graded Betti numbers of  $I_r(G)$  for some  
 39 families of graphs  $G$ . In Section 5 we provide an alternative proof to Theorem 1.1 that uses tools  
 40 from topological combinatorics. Finally, in Section 6 we outline some questions for future research.

41 **Acknowledgements.** The authors would like to thank the referee for a quick and careful reading  
 42 and for some helpful suggestions; especially for suggesting to use the Hilbert series for computations  
 43 of Betti numbers in Section 4 of the paper. Priyavrat Deshpande and Amit Roy are partially  
 44 supported by a grant from the Infosys Foundation. Anurag Singh is partially supported by the  
 45

1 Start-up Research Grant SRG/2022/000314 from SERB, DST, India. Van Tuyl's research is  
 2 partially supported by NSERC Discovery Grant 2019-05412.

## 3 4 2. Stanley-Reisner ideals of higher independence complexes

5 In this section we recall some relevant results from graph theory and Stanley-Reisner theory.  
 6 In particular, we state some basic properties of Stanley-Reisner ideals of higher independence  
 7 complexes. For more on Stanley-Reisner theory and monomial ideals, see [9, 18].  
 8

9 **2.1. Graph terminology.** Throughout this paper,  $G = (V(G), E(G))$  denotes a finite simple  
 10 graph on the vertex set  $V(G) = \{x_1, \dots, x_n\}$  and the set  $E(G)$  of edges which is a collection of  
 11 2-element subsets of  $V(G)$ . If  $x$  is a vertex of  $G$ , then  $|\{y \in V(G) \mid \{x, y\} \in E(G)\}|$  is called the  
 12 degree of  $x$  in  $G$ , and is denoted by  $\deg(x)$ . If  $\deg(x) = 1$ , then  $x$  is called a leaf of  $G$ . For  $x \in V(G)$ ,  
 13  $G \setminus x$  denotes the graph with vertex set  $V(G) \setminus \{x\}$  and edge set  $\{\{u, v\} \in E(G) \mid x \notin \{u, v\}\}$ . The  
 14 complement  $G^c$  of  $G$  is a graph with vertex set  $V(G^c) = V(G)$  and  $E(G^c) = \{\{x, y\} \mid \{x, y\} \notin E(G)\}$ .  
 15 The neighbourhood of  $x$  in  $G$  is defined as  $N_G(x) := \{y \in V(G) \mid \{x, y\} \in E(G)\}$ . The set  $N_G(x) \cup \{x\}$   
 16 is called the closed neighbourhood of  $x$  in  $G$ , and is denoted by  $N_G[x]$ . A connected component of  
 17  $G$  is a (maximal) subgraph of  $G$  such that for every pair of vertices in the subgraph, there is a  
 18 path within the subgraph that connects these vertices. If  $G$  is disconnected and  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$  are  
 19 the connected components of  $G$ , then for each  $i$ ,  $V(\mathcal{C}_i)$  denotes the vertex set of the connected  
 20 component  $\mathcal{C}_i$ .

21 If  $G = (V(G), E(G))$  is a graph and  $A \subseteq V(G)$ , then the induced subgraph of  $G$  on  $A$ , denoted  
 22  $G[A]$ , is the graph with vertex set  $V(G[A]) = A$  and edge set  $E(G[A]) = \{e \in E(G) \mid e \subseteq A\}$ . Let  
 23  $A = \{x_{i_1}, \dots, x_{i_k}\} \subseteq V(G)$  be such that  $G[A] \cong C_k$ , a cycle of length  $k$ . If  $E(G[A]) = \{\{x_{i_j}, x_{i_{j+1}}\} \mid$   
 24  $1 \leq j \leq k-1\} \cup \{x_{i_k}, x_{i_1}\}$ , then we simply write the cycle  $G[A]$  as  $x_{i_1} \cdots x_{i_k}$ .

25 A subset  $W \subseteq V(G)$  is called an  $r$ -independent set of  $G$  if each connected component of  $G[W]$   
 26 has at most  $r$  vertices. Note that a 1-independent set is the usual independent set in a graph  $G$ .  
 27 We call an  $r$ -independent set  $W$  a maximal  $r$ -independent set if  $W$  is maximal with respect to  
 28 inclusion among all  $r$ -independent sets.  
 29

30 **2.2. Simplicial Complexes.** Fix a set of vertices  $V = \{x_1, \dots, x_n\}$ . A simplicial complex  $\Delta$  on  
 31  $V$  is a subset of  $2^V$ , that satisfies the properties that  $\{x_i\} \in \Delta$  for all  $i = 1, \dots, n$  and if  $F \in \Delta$  and  
 32  $G \subseteq F$ , then  $G \in \Delta$ . If  $\Delta = 2^V$ , then  $\Delta$  is called a simplex (or  $(n-1)$ -simplex if we want to highlight  
 33 the number of vertices). An element  $F \in \Delta$  is called a face of the simplicial complex  $\Delta$ . A face in  
 34  $\Delta$  that is maximal with respect to inclusion is called a facet. If  $\{F_1, \dots, F_s\}$  is a complete list of the  
 35 facets of  $\Delta$ , then we sometimes write  $\Delta = \langle F_1, \dots, F_s \rangle$  and say that  $\Delta$  is generated by  $F_1, \dots, F_s$ .  
 36 The dimension of a face  $F$  is  $\dim F = |F| - 1$ , while the dimension of a simplicial complex  $\Delta$  is  
 37  $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$ . A simplicial complex is said to be pure if  $\dim F = \dim \Delta$  for all  
 38 facets of  $\Delta$ . If a simplex has dimension  $d$ , then it is called a  $d$ -simplex. For  $k \leq d$ , the  $k$ -skeleton of  
 39 the  $d$ -simplex  $\Delta$  on the vertex set  $Y = \{x_1, \dots, x_{d+1}\}$  is the collection of all subsets of  $Y$  which  
 40 have cardinality at most  $k+1$ .

41 Given a simplicial complex  $\Delta$  on the vertex set  $V = \{x_1, \dots, x_n\}$  we can associate it with a  
 42 square-free monomial ideal  $I_\Delta$  in the polynomial ring  $R = \mathbb{K}[x_1, \dots, x_n]$  over the field  $\mathbb{K}$  in the  
 43 following way. For  $A \subseteq V$ , we write

$$44 \quad \mathbf{x}_A = \prod_{x_i \in A} x_i,$$

1 to denote the monomial obtain by multiplying together all the variables corresponding to the  
2 vertices in  $A$ . Then the ideal

$$3 \quad I_\Delta = \langle \mathbf{x}_A \mid A \subseteq V \text{ and } A \notin \Delta \rangle$$

4 is called the Stanley-Reisner ideal of  $\Delta$ . The ideal  $I_\Delta$  and the ring  $R/I_\Delta$  (sometimes called  
5 the Stanley-Reisner ring) captures invariants of the simplicial complex. For example, the Krull  
6 dimension of  $R/I_\Delta$  satisfies

$$7 \quad (1) \quad \text{K-dim}(R/I_\Delta) = \dim \Delta + 1$$

8 Expanding this dictionary between the algebraic invariants of  $R/I_\Delta$  and  $\Delta$  when  $\Delta$  is a simplicial  
9 complex constructed from a graph is the focus for the remainder of the paper.

10 **2.3. Stanley-Reisner ideals of higher independence complexes.** We now formally intro-  
11 duce the higher independence complexes of a graph  $G$ , the main object of study in this paper.

12 **Definition 2.1.** Let  $G = (V(G), E(G))$  be a finite simple graph with  $V(G) = \{x_1, \dots, x_n\}$ , and  
13 let  $r$  be any positive integer. Then the  $r$ -independence complex of  $G$ , denoted by  $\text{Ind}_r(G)$ , is the  
14 simplicial complex

$$15 \quad \text{Ind}_r(G) = \{W \subseteq V(G) \mid W \text{ is an } r\text{-independent set}\}$$

16 on the vertex set  $V(G)$ .

17 When  $r = 1$ ,  $\text{Ind}_1(G)$  is the independence complex of  $G$ . If  $r \geq n$ , then  $\text{Ind}_r(G) = \langle \{x_1, x_2, \dots, x_n\} \rangle$   
18 since  $V(G)$  is the  $r$ -independent set. For  $r = n - 1$ , if  $G$  is disconnected then  $\text{Ind}_{n-1}(G)$  is the  
19  $(n - 1)$ -simplex  $\langle \{x_1, x_2, \dots, x_n\} \rangle$  and if  $G$  is connected then  $\text{Ind}_{n-1}(G)$  is the  $(n - 2)$ -skeleton of  
20 the simplex  $\langle \{x_1, x_2, \dots, x_n\} \rangle$ .

21 We now describe the Stanley-Reisner ideal of  $\text{Ind}_r(G)$ .

22 **Theorem 2.2.** Let  $G = (V(G), E(G))$  be a finite simple graph with  $V(G) = \{x_1, \dots, x_n\}$ . Then the  
23 Stanley-Reisner ideal of  $\text{Ind}_r(G)$  in  $R = \mathbb{K}[x_1, \dots, x_n]$  is

$$24 \quad I_{\text{Ind}_r(G)} = \langle \mathbf{x}_A \mid A \subseteq V(G) \text{ with } |A| = r + 1 \text{ and } G[A] \text{ connected} \rangle.$$

25 *Proof.* Let  $J = \langle \mathbf{x}_A \mid A \subseteq V(G) \text{ with } |A| = r + 1 \text{ and } G[A] \text{ connected} \rangle$ . Suppose that  $A \subseteq V$  with  
26  $|A| = r + 1$  and  $G[A]$  connected. Then  $A$  is not an  $r$ -independent set, i.e.,  $A \notin \text{Ind}_r(G)$ , and  
27 consequently,  $\mathbf{x}_A$  is in  $I_{\text{Ind}_r(G)}$ . So  $J \subseteq I_{\text{Ind}_r(G)}$  since all of its generators are in  $I_{\text{Ind}_r(G)}$ .

28 Consider any square-free monomial  $x_{i_1} \cdots x_{i_t} \in I_{\text{Ind}_r(G)}$ . So  $W = \{x_{i_1}, \dots, x_{i_t}\} \subseteq V(G)$  is not in  
29  $\text{Ind}_r(G)$ . This means that  $G[W]$  has a connected component with at least  $r + 1$  vertices. Thus, we  
30 can find a subset  $A \subseteq W$  with  $|A| = r + 1$  such that  $G[A]$  is connected. So  $\mathbf{x}_A \in J$ , and  $\mathbf{x}_A$  divides  
31  $x_{i_1} \cdots x_{i_t}$ . Since both  $I_{\text{Ind}_r(G)}$  and  $J$  are square-free monomial ideals we have  $I_{\text{Ind}_r(G)} \subseteq J$ .  $\square$

32 **Remark 2.3.** The ideal  $I_{\text{Ind}_r(G)}$  is sometimes denoted by  $I_r(G)$ . When  $r = 1$  in the previous  
33 Theorem,  $I_1(G) = I_{\text{Ind}_1(G)} = \langle x_i x_j \mid \{x_i, x_j\} \in E(G) \rangle$  is the well-known *edge ideal*  $I(G)$  of  $G$ . Also  
34 observe that when  $G$  is a connected graph on  $n$  vertices, then  $I_{n-1}(G) = \langle x_1 x_2 \cdots x_n \rangle$ . If  $G$  is  
35 disconnected and  $|V(G)| = n$ , then  $I_r(G) = \langle 0 \rangle$ , for  $r \geq n - 1$ .

36 When  $r = 1$ , i.e., when working with edge ideals, specifying two adjacent vertices is equivalent  
37 to specifying an edge. However, when considering  $r > 1$ , the ideal may not necessarily *specify* the  
38 induced subgraphs on  $r + 1$  vertices; it merely specifies whether or not they are connected. It is  
39 well known that graphs and square-free monomial ideals generated in degree 2 are in one-to-one  
40 correspondence.

1 correspondence. However, for  $r > 1$ , non-isomorphic graphs may have the same  $r$ -independence  
 2 complex (and hence have the same Stanley-Reisner ideals). A trivial example is that any two  
 3 connected graphs on  $n$  vertices will have the same  $(n-1)$ -independence complex. A less trivial  
 4 example is given below.

5 **Example 2.4.** Let  $G$  be the complete graph  $K_5$  on five vertices  $X = \{x_1, \dots, x_5\}$ . Let  $r > 1$   
 6 and  $H$  be the graph obtained from  $K_5$  by removing the edge  $\{x_1, x_2\}$ . Note that the facets of  
 7  $\text{Ind}_r(G)$  are all  $F \subseteq \{x_1, \dots, x_5\}$  such that  $|F| = r$ . Now suppose  $F' \subseteq X$  is a facet of  $\text{Ind}_r(H)$ . By  
 8 definition  $|F'| \geq r$ . If  $\{x_1, x_2\} \not\subseteq F'$ , then for each  $x_i, x_j \in F'$ ,  $\{x_i, x_j\} \in E(H)$ . Hence  $|F'| \leq r$ . If  
 9  $\{x_1, x_2\} \subseteq F'$ , then for any  $i \notin \{1, 2\}$ ,  $H[x_1, x_2, x_i]$  is a connected subgraph of  $H$ . Thus  $|F'| \leq r$ .  
 10 Consequently,  $|F'| = r$ . Hence  $\text{Ind}_r(G) = \text{Ind}_r(H)$ . The graphs  $G$  and  $H$  are non-isomorphic graphs  
 11 since every vertex in  $G$  has degree four whereas the vertex  $x_1$  in  $H$  has degree three.  
 12

13 The previous example is just a special case of a more general phenomena.

14 **Proposition 2.5.** Let  $K_n$  be the complete graph on  $n$  vertices. Suppose  $V(K_n) = \sqcup_{i=1}^k W_i$  with  
 15  $\max_{i=1}^k \{|W_i|\} = s$ . If  $H$  is the graph obtained from  $K_n$  by deleting some edges (maybe all or none)  
 16 from each  $K_n[W_i]$ , then for all  $r \geq s$ ,  $\text{Ind}_r(K_n) = \text{Ind}_r(H)$ .  
 17

18 *Proof.* The statement is true for all  $r \geq n$  by the discussion after Definition 2.1. So, we can assume  
 19  $s \leq r < n$ . It is enough to show that if  $F \subseteq V(K_n)$  is such that  $s \leq |F| = r$ , then  $F$  is a facet of  
 20  $\text{Ind}_r(H)$  and these are all the facets of  $\text{Ind}_r(H)$ . Now suppose  $F$  is a facet of  $\text{Ind}_r(H)$ , then  $|F| \geq r$ .  
 21 If  $W_i \not\subseteq F$  for all  $i$ , then the induced subgraph  $H[F]$  is connected and hence  $|F| \leq r$ . Now suppose  
 22  $W_i \subseteq F$  for some  $i$ . In that case for all  $x \in F$ , with  $x \notin W_i$ ,  $H[W_i \cup \{x\}]$  is a connected subgraph  
 23 of  $H$ . Note that  $|W_i| \leq r$  since  $\max_{i=1}^k \{|W_i|\} = s \leq r$ . Therefore,  $F = W_i$ , if  $|W_i| = r$  and  $H[F]$  is  
 24 connected if  $r > |W_i|$ . Consequently,  $|F| = r$  and hence  $\text{Ind}_r(K_n) = \text{Ind}_r(H)$ .  $\square$   
 25

### 26 3. Fröberg's theorem via vertex splittable ideals

27 In this section, we show that one direction of Fröberg's theorem extends very naturally to the  
 28 Stanley-Reisner ideals of higher independence complexes. On the other hand, the natural converse  
 29 of this theorem does not hold.

30 We first recall the definition of a (linear) resolution. Given any homogeneous ideal  $I \subseteq R =$   
 31  $\mathbb{K}[x_1, \dots, x_n]$ , the *graded minimal free resolution* of  $I$  is the long exact sequence  
 32

$$33 \quad 0 \rightarrow \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p,j}(I)} \rightarrow \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p-1,j}(I)} \rightarrow \dots \rightarrow \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{0,j}(I)} \rightarrow I \rightarrow 0,$$

35 where  $R(-j)$  denotes the polynomial ring with the grading twisted by  $j$  and  $p \leq n$ . The numbers  
 36  $\beta_{i,j}(I)$  are called the  $(i, j)$ -th *graded Betti numbers* of  $I$ . See [18] for more on the graded resolution.

37 **Definition 3.1.** Let  $I \subseteq R$  be a homogeneous ideal, and suppose that all the generators of  $I$  have  
 38 degree  $d$ . Then  $I$  has a *linear resolution* if  $\beta_{i,j}(I) = 0$  for all  $j \neq i + d$ .  
 39

40 **Remark 3.2.** The *regularity* of an ideal  $I$  is defined to be  $\text{reg}(I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}$ . If  $I$  is  
 41 a homogeneous ideal generated in degree  $d$ , then  $I$  has a linear resolution if and only if  $\text{reg}(I) = d$ .  
 42

43 **Definition 3.3.** A simple graph  $G$  is called *chordal* if there are no induced cycles of length four or  
 44 more. The complement of a chordal graph is called *co-chordal*.

45 Fröberg [7] classified which quadratic square-free monomial ideals have a linear resolution.



**Theorem 3.4** (Fröberg's Theorem). *Let  $G$  be a finite simple graph. Then the edge ideal  $I(G) = I_{\text{Ind}_1(G)}$  has a linear resolution (equivalently regularity is 2) if and only if  $G$  is co-chordal.*

We will prove the “if” direction of Fröberg's theorem in the context of higher independence complexes; i.e., the Stanley-Reisner ideal of  $\text{Ind}_r(G)$  has a linear free resolution whenever  $G$  is the complement of a chordal graph.

Let  $I$  be a square-free monomial ideal in a polynomial ring  $R = \mathbb{K}[x_1, \dots, x_n]$  and let  $\mathcal{G}(I)$  denote the unique set of minimal generators of the ideal  $I$ . We require the notion of a vertex splittable ideal, first introduced by Moradi and Khosh-Ahang [15].

**Definition 3.5.** We say that a monomial ideal  $I$  is *vertex splittable* if  $I$  can be obtained by the following recursive procedure:

- (i) If either  $I = \langle m \rangle$  where  $m$  is a monomial, or  $I = \langle 0 \rangle$ , or  $I = R$ .
- (ii) If there exists a variable  $x_i$  and two vertex splittable ideals  $I_1$  and  $I_2$  of the polynomial ring  $\mathbb{K}[x_1, \dots, \widehat{x}_i, \dots, x_n]$  such that  $I = x_i I_1 + I_2$  with  $I_2 \subseteq I_1$  and the minimal generators of  $I$  is the disjoint union of the minimal generators of  $x_i I_1$  and  $I_2$ .

**Lemma 3.6.** *Let  $I$  be an ideal of  $R$  such that  $I$  is generated by variables. Then  $I$  is vertex splittable.*

*Proof.* This follows by applying induction on the number of generators of  $I$ .  $\square$

**Lemma 3.7.** *Let  $I$  be a vertex splittable ideal of  $R$ . Then the ideal  $x_{n+1}I \subseteq R[x_{n+1}]$  is also vertex splittable.*

*Proof.* We prove this by induction on  $n$ . If  $n = 1$ , then  $I = \langle 0 \rangle$ , or  $\langle 1 \rangle$ , or  $\langle x_1 \rangle$ . Hence  $x_2 I$  is a vertex splittable ideal.

Let us assume that  $n \geq 2$ . If  $I = \langle 0 \rangle$ , or  $\langle 1 \rangle$ , or  $\langle m \rangle$ , where  $m$  is a monomial, then we can see that  $x_{n+1}I$  is a vertex splittable ideal. Now, suppose that

$$I = x_i I_1 + I_2,$$

where  $I_1$  and  $I_2$  are vertex splittable ideals of  $\mathbb{K}[x_1, \dots, \widehat{x}_i, \dots, x_n]$  with  $\mathcal{G}(I) = \mathcal{G}(x_i I_1) \sqcup \mathcal{G}(I_2)$  and  $I_2 \subseteq I_1$ . Then

$$x_{n+1}I = x_i(x_{n+1}I_1) + x_{n+1}I_2.$$

By induction  $x_{n+1}I_1$  and  $x_{n+1}I_2$  are vertex splittable ideals of  $\mathbb{K}[x_1, \dots, \widehat{x}_i, \dots, x_n, x_{n+1}]$  with  $\mathcal{G}(x_{n+1}I) = \mathcal{G}(x_i x_{n+1}I_1) \sqcup \mathcal{G}(x_{n+1}I_2)$  and  $x_{n+1}I_2 \subseteq x_{n+1}I_1$ . Therefore,  $x_{n+1}I$  is a vertex splittable ideal.  $\square$

There is another characterization of chordal graphs that we will use frequently. A vertex  $v$  of a graph is called a *simplicial vertex* if the induced subgraph on  $v$  and its neighbours  $N_G(v)$  is a clique. A graph  $G$  is chordal if and only if there is a subset of vertices  $\{v_1, \dots, v_n\}$  such that  $v_i$  is simplicial in the graph induced on  $V(G) \setminus \{v_1, \dots, v_{i-1}\}$  for  $i = 1, \dots, n$ .

Moradi and Khosh-Ahang in [15] showed that the edge ideal of a co-chordal graph  $G$ , i.e., the Stanley-Reisner ideal of the 1-independence complex of  $G$  is a vertex splittable ideal.

**Theorem 3.8.** [15, Theorem 3.6] *If  $G$  is a co-chordal graph on the vertex set  $V(G)$ , then  $I_1(G) = I_{\text{Ind}_1(G)}$  is a vertex splittable ideal of  $R$ .*

Our goal is to extend the above theorem to  $I_{\text{Ind}_r(G)}$  for  $r > 1$ . In order to do this, we construct a new graph  $\tilde{G}$  from the given co-chordal graph  $G$  with  $x_1$  the simplicial vertex of  $G^c$  as follows:

$$V(\tilde{G}) = V(G) \setminus \{x_1\};$$

$$E(\tilde{G}) = E(G \setminus x_1) \cup \{\{x_i, x_j\} \mid \{x_i, x_j\} \in E(G^c \setminus x_1) \text{ and } N_{G^c}(x_1) \cap \{x_i, x_j\} = \emptyset\}.$$

Note that if  $N_{G^c}(x_1) = Y$ , then the induced subgraph of  $G^c$  on the vertex set  $\{x_1\} \cup Y$  is a clique since  $x_1$  is a simplicial vertex of  $G^c$ . Thus we can rename the vertices of  $G$  as  $V(G) = V(G^c) = \{x_1\} \cup Y \cup W$  such that  $\{x_1\} \cup Y$  is an independent set in  $G$  and,  $x_1$  is connected by an edge in  $G$  to all the vertices in  $W$ . Informally, to construct  $\tilde{G}$  from  $G$  we first remove the vertex  $x_1$  and all its adjacent edges. Then we add edges to the remaining vertices of  $G$  according to the following rules. Among the vertices in  $Y \sqcup W$ , if  $\{w_i, w_j\} \notin E(G \setminus x_1)$  for some  $w_i, w_j \in W$ , then we add it to  $G$  since the induced subgraph on  $\{x_1, w_i, w_j\}$  is connected in  $G$ . However, if  $\{y_i, w_j\} \notin E(G \setminus x_1)$  for some  $y_i \in Y$  and  $w_j \in W$ , then we don't add it to  $\tilde{G}$  since the induced subgraph on  $\{x_1, y_i, w_j\}$  is disconnected in  $G$ . Moreover,  $\{y_i, y_j\} \notin E(G \setminus x_1)$  for each  $y_i, y_j \in Y$ , and we also don't add it to  $\tilde{G}$  since  $\{x_1, y_i, y_j\}$  is an independent set in  $G$ . Thus in  $\tilde{G}$ , the vertices in  $Y$  form an independent set and the induced subgraph on  $W$  is a clique. In other words,  $\tilde{G}^c$  is the graph on the vertex set  $Y \sqcup W$  such that the induced graph on  $Y$  forms a clique, whereas the set  $W$  forms an independent set.

We illustrate the construction of  $\tilde{G}$  using an example.

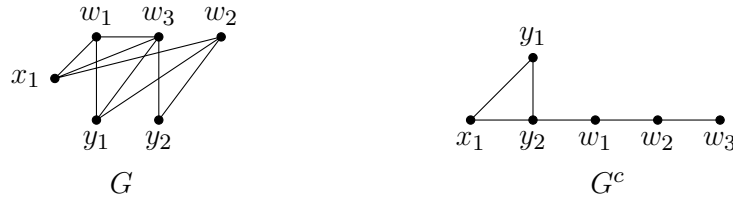


FIGURE 1. A co-chordal graph and its complement.

**Example 3.9.** Let  $G$  be the co-chordal graph in Figure 1. Then  $V(G) = \{x_1\} \cup Y \cup W$ , where  $Y = \{y_1, y_2\}$  and  $W = \{w_1, w_2, w_3\}$ . We see that  $\{x_1, y_1, y_2\}$  forms an independent set in  $G$  and  $\{x_1, w_i\} \in E(G)$  for  $1 \leq i \leq 3$ . The graph  $\tilde{G}$  is constructed from  $G$  by first removing the vertex  $x_1$  and its adjacent edges and then adding the edges  $\{w_1, w_2\}$  and  $\{w_2, w_3\}$ . Thus in  $\tilde{G}^c$ ,  $W = \{w_1, w_2, w_3\}$  forms an independent set and the induced subgraph on  $Y = \{y_1, y_2\}$  is a clique.

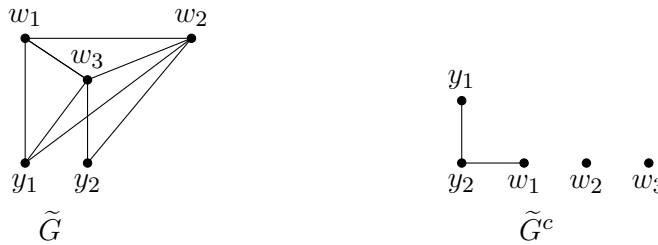


FIGURE 2. The graph  $\tilde{G}$  and its complement.

1 Note that the graph  $\tilde{G}$  in the example is again co-chordal. It is in fact true that for any co-chordal  
2 graph  $G$ , the graph  $\tilde{G}$  is co-chordal, which we now prove.

3 **Proposition 3.10.** *Let  $G$  be a graph such that  $V(G) = W \sqcup Y$ , where  $W$  forms an independent  
4 set in  $G$ , and the induced subgraph  $G[Y]$  is a clique. Then both  $G$  and  $G^c$  are chordal.*

5 *Proof.* Note that, in  $G^c$ ,  $Y$  forms an independent set and  $G^c[W]$  is a clique. Thus it is enough  
6 to show that  $G$  is a chordal graph. Suppose  $z_1 \cdots z_p$  is a minimal cycle in  $G$  of length at least  
7 four. Let  $Z = \{z_1, \dots, z_p\}$ . If  $|Z \cap Y| \geq 3$ , then for any  $y_{i_1}, y_{i_2}, y_{i_3} \in Z \cap Y$  the induced subgraph  
8  $G[\{y_{i_1}, y_{i_2}, y_{i_3}\}]$  would be a smaller cycle, a contradiction. Thus  $|Z \cap Y| \leq 2$ . In that case, as  $W$   
9 forms an independent set in  $G$  we must have  $p \leq 4$ . Consequently,  $p = 4$ . If  $|Z \cap Y| = 2$ , then there  
10 exists some  $y_{l_1}, y_{l_2} \in Z \cap Y$  and  $w_{m_1}, w_{m_2} \in Z \cap W$  such that  $G[Z]$  is the cycle  $y_{l_1} w_{m_1} y_{l_2} w_{m_2}$ . Note  
11 here that  $y_{l_2} w_{m_2} y_{l_1}$  is a smaller cycle in  $G[Z]$ , a contradiction. If  $|Z \cap Y| \leq 1$ , then  $|Z \cap W| \geq 3$ .  
12 Hence there exists  $w_{j_1}, w_{j_2} \in Z \cap W$  such that  $\{w_{j_1}, w_{j_2}\} \in E(G)$ , again a contradiction. Thus  $G$   
13 contains induced cycles of length at most 3. This completes the proof.  $\square$

14 **Lemma 3.11.** *Given a co-chordal graph  $G$ , let  $\tilde{G}$  be the graph constructed above. Then  $\tilde{G}$  is also  
15 a co-chordal graph.*

16 *Proof.* In  $\tilde{G}^c$ , we have  $V(\tilde{G}^c) = Y \sqcup W$  such that the vertices in  $W$  form an independent set and  
17 the induced subgraph on  $Y$  is a clique. Therefore, by Proposition 3.10,  $\tilde{G}^c$  is a chordal graph.  $\square$

18 Now we are ready to prove the main theorem of this section.

19 **Theorem 3.12.** *If  $G$  is co-chordal, then the ideal  $I_r(G) = I_{\text{Ind}_r(G)}$  is vertex splittable for all  $r \geq 1$ .*

20 *Proof.* We prove this by induction on  $n$  and  $r$ . For a fixed  $r$ , if  $n \in \{1, 2, \dots, r\}$  then  $I_r(G) = \langle 0 \rangle$   
21 and hence a vertex splittable ideal. If  $n = r + 1$ , then  $I_r(G) = \langle 0 \rangle$  if  $G$  is a disconnected graph.  
22 Otherwise,  $I_r(G) = \langle x_1 \cdots x_{r+1} \rangle$ . In both cases  $I_r(G)$  is a vertex splittable ideal. Also, for  $n \geq 1$ ,  
23 the case  $r = 1$  follows from Theorem 3.8. Now take any  $n \geq 3$ . Since  $G^c$  is a chordal graph, without  
24 loss of generality we can assume that  $x_1$  is a simplicial vertex of  $G^c$ . Then we can write

$$25 \quad I_r(G) = x_1 J_1 + J_2,$$

26 where

$$27 \quad J_1 = \left\langle \prod_{s=1}^r x_{i_s} \mid G[\{x_1, x_{i_1}, \dots, x_{i_r}\}] \text{ is a connected subgraph of } G \right\rangle,$$

28 and the ideal

$$29 \quad J_2 = \left\langle \prod_{s=1}^{r+1} x_{j_s} \mid x_1 \notin \{x_{j_1}, \dots, x_{j_{r+1}}\} \text{ and } G[\{x_{j_1}, \dots, x_{j_{r+1}}\}] \text{ is connected} \right\rangle.$$

30 Now construct the graph  $\tilde{G}$  from  $G$  as described above, i.e.,

$$31 \quad V(\tilde{G}) = V(G) \setminus \{x_1\};$$

$$32 \quad E(\tilde{G}) = E(G \setminus x_1) \cup \{\{x_i, x_j\} \mid \{x_i, x_j\} \in E(G^c \setminus x_1) \text{ and } N_{G^c}(x_1) \cap \{x_i, x_j\} = \emptyset\}.$$

33 Rename the vertices of  $G$  as before:  $V(G) = V(G^c) = \{x_1\} \cup Y \cup W$ , where  $Y = N_{G^c}(x_1)$  and  
34  $W = N_G(x_1)$ . Then  $\{x_1\} \cup Y$  is an independent set in  $G$ . Moreover,  $x_1$  is connected by an edge in  
35  $G$  to all the vertices in  $W$ . In  $\tilde{G}$  the vertices in  $Y$  forms an independent set and the the induced  
36 subgraph on  $W$  is a clique. Thus in  $\tilde{G}^c$  the vertices in  $Y$  forms a clique, whereas the vertices in  $W$   
37



1 forms an independent set. By Lemma 3.11,  $\tilde{G}$  is a co-chordal graph. Now our aim is to prove the  
2 following.

3 (i)  $J_2 = I_r(G \setminus x_1)$ ,

4 (ii)  $J_2 \subseteq J_1$ , and

5 (iii)  $J_1 = I_{r-1}(\tilde{G})$ .

6  
7 *Proof of (i).* If  $\mathbf{x}_S \in J_2$  is a minimal generator, then  $x_1 \notin S$  and  $G[S]$  is connected, where  $|S| = r + 1$ .  
8 Therefore,  $\mathbf{x}_S \in I_r(G \setminus x_1)$ . Similarly, if  $\mathbf{x}_S \in I_r(G \setminus x_1)$  is a minimal generator, then  $x_1 \notin S$  and  
9 hence  $\mathbf{x}_S \in J_2$ . Thus  $J_2 = I_r(G \setminus x_1)$ .  $\square$

10  
11 *Proof of (ii).* Let  $\mathbf{x}_S \in J_2$ . Then  $x_1 \notin S$  and  $G[S]$  is connected, where  $|S| = r + 1$ . If  $Y \cap S = \emptyset$ ,  
12 then for any  $w \in W \cap S$ , the induced subgraph on  $\{x_1\} \cup (S \setminus \{w\})$  is a connected subgraph of  
13  $G$  since  $N_G(x_1) = W$ . Note here that  $|S \setminus \{w\}| = r$  and  $\mathbf{x}_{S \setminus \{w\}} \in J_1$ . Since  $\mathbf{x}_{S \setminus \{w\}} | \mathbf{x}_S$ , we have  
14  $\mathbf{x}_S \in J_1$ . If  $Y \cap S \neq \emptyset$ , then for any  $y \in Y$ , the induced subgraph on  $\{x_1\} \cup (S \setminus \{y\})$  is a connected  
15 subgraph of  $G$ . Indeed, if  $z \in (S \setminus \{y\}) \cap W$ , then  $z$  is adjacent to  $x_1$ . If  $y' \in (S \setminus \{y\}) \cap Y$ , then  
16 there is  $w \in W$  such that  $\{y', w\} \in E(G)$  since  $y'$  is connected to some element of  $S$ , but it cannot  
17 be  $y$  since the set  $Y$  is an independent set. But because  $\{w, x_1\} \in E(G)$ , there is a path for all  
18 vertices  $(S \setminus \{y\})$  to  $x_1$ . Consequently,  $\mathbf{x}_{S \setminus \{y\}} \in J_1$ , where  $|S \setminus \{y\}| = r$ . Hence  $\mathbf{x}_S \in J_1$ . Thus we  
19 have  $J_2 \subseteq J_1$ .  $\square$

20  
21 *Proof of (iii).* Let  $\mathbf{x}_S \in J_1$ . Then  $|S| = r$  and  $G[S \cup \{x_1\}]$  is a connected subgraph of  $G$ . Since  
22  $\{x_1\} \cup Y$  forms an independent set in  $G$ , we have  $S \cap W \neq \emptyset$ . Moreover, for each  $y \in Y \cap S$ , there  
23 exists some  $w \in S \cap W$  such that  $\{y, w\} \in E(G)$ . Therefore,  $\tilde{G}[S]$  is connected as  $W$  forms a clique  
24 in  $\tilde{G}$ . Thus  $\mathbf{x}_S \in I_{r-1}(\tilde{G})$  and hence  $J_1 \subseteq I_{r-1}(\tilde{G})$ . Now let  $\mathbf{x}_S \in I_{r-1}(\tilde{G})$ . Then  $|S| = r$  and  $\tilde{G}[S]$   
25 is connected. Since  $Y$  forms an independent set in  $\tilde{G}$ , we see that for each  $y \in Y \cap S$ , there exists  
26 some  $w \in S \cap W$  such that  $\{y, w\} \in E(\tilde{G})$ . Now  $x_1$  is connected by an edge in  $G$  to all the vertices  
27 in  $W$ . Hence  $G[S \cup \{x_1\}]$  is connected and consequently,  $I_{r-1}(\tilde{G}) \subseteq J_1$ . This completes the proof  
28 of (iii).  $\square$

29 Now by induction on  $n$ ,  $J_2$  is a vertex splittable ideal since  $G \setminus x_1$  is co-chordal on fewer vertices.  
30 Also, since  $J_1 = I_{r-1}(\tilde{G})$  and  $\tilde{G}$  is a co-chordal graph (by Lemma 3.11), we see that  $J_1$  is a vertex  
31 splittable ideal by induction on  $r$ . If  $J_2 = \langle 0 \rangle$ , then by Lemma 3.7,  $I_r(G) = x_1 J_1$  is a vertex  
32 splittable ideal. Otherwise, by Definition 3.5, we see that  $I_r(G) = x_1 J_1 + J_2$  is a vertex splittable  
33 ideal.  $\square$

34  
35 A *claw* is a graph on four vertices  $\{x, y_1, y_2, y_3\}$  with edge set  $\{\{x, y_1\}, \{x, y_2\}, \{x, y_3\}\}$ . We say  
36 that a graph is *claw free* if it does not contain any claw as an induced subgraph. A graph  $G$  is said  
37 to be *gap free* if  $G^c$  does not contain  $C_4$  as an induced subgraph.

38 **Corollary 3.13.** *Let  $G$  be a gap free and claw free graph such that it contains a leaf. Then both*  
39  *$I_r(G)$  and  $I_r(G^c)$  are vertex splittable ideals for any integer  $r \geq 1$ .*

40  
41 *Proof.* Let  $x \in V(G)$  be a leaf of  $G$  and let  $N_G(x) = \{y\}$ . If  $N_G(y) = \{x\}$ , then for any  $u, v \in$   
42  $V(G) \setminus \{y, x\}$ ,  $\{u, v\} \notin E(G)$ . Because if  $\{u, v\} \in E(G)$ , then  $\{y, x, u, v\}$  forms a cycle of length  
43 four in  $G^c$ , a contradiction. Thus  $V(G) \setminus \{x, y\}$  forms an independent set in  $G$ . Therefore, by  
44 Proposition 3.10, both  $G$  and  $G^c$  are chordal. Hence by Theorem 3.12,  $I_r(G)$  and  $I_r(G^c)$  are vertex  
45 splittable ideals for  $r \geq 1$ .

1 Now let  $N_G(y) = \{x, y_1, \dots, y_t\}$  for some  $t \geq 1$ . For  $i \neq j$ , if  $\{y_i, y_j\} \notin E(G)$  then  $\{x, y, y_i, y_j\}$   
 2 forms a claw in  $G$ , which is not possible. Hence  $\{y, y_1, \dots, y_t\}$  forms a clique in  $G$ . Now let  
 3  $a, b \in V(G) \setminus N_G[y]$ . If  $\{a, b\} \in E(G)$ , then  $\{a, b, x, y\}$  forms a cycle of length 4 in  $G^c$ , a contradiction.  
 4 Hence  $V(G) \setminus N_G[y]$  forms an independent set in  $G$ . Consequently,  $\{x\} \cup (V(G) \setminus N_G[y])$  forms an  
 5 independent set in  $G$ . Let  $Y = N_G[y] \setminus \{x\}$  and  $W = \{x\} \cup (V(G) \setminus N_G[y])$ . Then  $V(G) = Y \sqcup W$   
 6 such that  $G[Y]$  is a clique and  $W$  forms an independent set in  $G$ . Therefore, by Proposition 3.10,  
 7 both  $I_r(G)$  and  $I_r(G^c)$  are vertex splittable ideals for  $r \geq 1$ .  $\square$

8  
 9 A vertex splittable ideal has a linear resolution (by [15, Theorem 2.4]). Thus, as an application of  
 10 Theorem 3.12, we see that one direction of Fröberg's theorem is true in the context of Stanley-Reisner  
 11 ideals of  $r$ -independence complexes of graphs:

12 **Corollary 3.14.** *If  $G$  is a co-chordal graph, then  $I_r(G)$  has a  $(r+1)$ -linear resolution.*

13  
 14 **Remark 3.15.** The converse of Corollary 3.14 is not true in general. For example, if  $r \geq |V(G)|$   
 15 and  $G$  is any graph, then the ideal  $I_{\text{Ind}_r(G)}$  is the zero ideal and hence has a linear resolution.  
 16 Moreover, if  $r = |V(G)| - 1$  then  $I_{\text{Ind}_r(G)}$  is the zero ideal if  $G$  is not connected and is generated by  
 17 the monomial  $\prod_{x_i \in V(G)} x_i$  if  $G$  is connected. Thus in both cases  $I_{\text{Ind}_r(G)}$  has a linear resolution.

18 For another example, let  $C_n = x_1 x_2 \cdots x_n$  be a cycle of length  $n$ . Consider the graph  $H =$   
 19  $C_n * \{x_{n+1}\}$ , the graph formed by adding a new vertex  $x_{n+1}$  and joining this vertex by an edge  
 20 to all vertices in the cycle. Then  $H$  is not chordal but for  $n \geq 5$ ,  $I_{\text{Ind}_{n-1}(G)} = \langle x_1 x_2 \cdots x_n \rangle$  has a  
 21 linear resolution, where  $G = H^c$ . In general, if  $G$  is any graph such that its connected components  
 22 have cardinality at most  $s$ , then for each  $r \geq s$ ,  $I_{\text{Ind}_r(G)}$  is the zero ideal and hence has a linear  
 23 resolution. Moreover, if  $\mathcal{C}_1, \dots, \mathcal{C}_k$  are connected components of  $G$  such that  $|V(\mathcal{C}_i)| = s$  for some  $i$   
 24 and  $|V(\mathcal{C}_j)| < s$  for each  $j \neq i$ , then  $I_{\text{Ind}_{s-1}(G)} = \langle \prod_{x_i \in V(\mathcal{C}_i)} x_i \rangle$  and hence has a linear resolution.

#### 25 26 4. The graded Betti numbers

27 The vertex splittability property of a square-free monomial ideal gives important information about  
 28 the graded Betti numbers. Namely, the Betti numbers of the ideal can be expressed as a sum of the  
 29 Betti numbers of some 'smaller' ideals. More precisely, we have the following theorem by Moradi  
 30 and Khosh-Ahang.  
 31

32 **Theorem 4.1.** [15, Theorem 2.8] *Let  $I = xJ_1 + J_2 \subseteq R$  be a vertex splitting for the monomial ideal*  
 33  *$I$ . Then the graded Betti numbers of  $R/I$  can be computed by the following recursive formula:*

$$34 \quad \beta_{i,j}(R/I) = \beta_{i,j-1}(R/J_1) + \beta_{i,j}(R/J_2) + \beta_{i-1,j-1}(R/J_2) \text{ for all } i, j \geq 0.$$

35  
 36 Using Theorem 4.1, we can compute all the graded Betti numbers of Stanley-Reisner ideals of  
 37 higher independence complexes of some well-known families of graphs, namely, the complete graphs,  
 38 star graphs and the complement of path graphs. But before proceeding to derive the formulas we  
 39 first make the following remark.  
 40

41 **Remark 4.2.** In the calculations below, we make extensive use of Pascal's identity:

$$42 \quad \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

43  
 44  
 45 The following theorem is well-known using the properties of complete intersections.

**Theorem 4.3.** Let  $I = \langle x_{i_1}, \dots, x_{i_k} \rangle \subset R$  be a monomial ideal generated by  $k$  variables. Then the  $\mathbb{N}$ -graded Betti numbers of  $R/I$  are given by the following formula:  $\beta_{i,j}(R/I) = 0$  if  $i \neq j$ , and

$$\beta_{i,i}(R/I) = \begin{cases} \binom{k}{i} & \text{for } 0 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

We now give explicit formulas for all the Betti numbers of the  $r$ -independence complexes of complete graphs. Note that the formula is well-known in the case of  $r = 1$  and it was first derived in the thesis of Jacques [11, Theorem 5.1.1]. We provide two different proofs; one using the formula in Theorem 4.1 and another using the Hilbert series of the ideal.

**Theorem 4.4.** Let  $I_r(K_n)$  denote the Stanley-Reisner ideal of the  $r$ -independence complex of  $K_n$  for  $r \geq 1$ . Then the  $\mathbb{N}$ -graded Betti numbers of  $R/I_r(K_n)$  can be expressed as follows:  $\beta_{i,j}(R/I_r(K_n)) = 0$  if  $j \neq i+r$ , and

$$\beta_{i,i+r}(R/I_r(K_n)) = \begin{cases} \binom{i+r-1}{r} \binom{n}{i+r} & \text{for } 1 \leq i \leq n-r \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We prove this by induction on  $n$  and  $r$ . For a fixed  $r$ , if  $n \in \{1, 2, \dots, r\}$  then  $I_r(K_n) = \langle 0 \rangle$ . Hence we have the above formula. Now fix an  $n$ . We first prove the above formula for  $r = 1$ . By Theorem 3.8 we have

$$I_1(K_n) = x_1 J_1 + J_2,$$

where  $J_1 = \langle x_2, \dots, x_n \rangle$  and  $J_2 = I_1(K_n \setminus \{x_1\})$ . Note that  $K_n \setminus \{x_1\} = K_{n-1}$ . Therefore, using Theorem 4.1 and by the induction on  $n$ , we have, for  $1 \leq i \leq n-1$ ,

$$\begin{aligned} \beta_{i,i+1}(R/I_1(K_n)) &= \beta_{i,i}(R/J_1) + \beta_{i,i+1}(R/I_1(K_{n-1})) + \beta_{i-1,i}(R/I_1(K_{n-1})) \\ &= \binom{n-1}{i} + i \binom{n-1}{i+1} + (i-1) \binom{n-1}{i} \quad (\text{by Theorem 4.3}) \\ &= \binom{i}{1} \binom{n}{i+1}. \end{aligned}$$

Now we take any  $r \geq 2$ . Then by the construction in Theorem 3.12,

$$I_r(K_n) = x_1 J_1 + J_2,$$

where  $J_1 = I_{r-1}(\widetilde{K}_n)$  and  $J_2 = I_r(K_{n-1})$ . Observe that  $\widetilde{K}_n = K_{n-1}$ . Hence using Theorem 4.1 and by the induction on  $n$  and  $r$ , we have, for  $1 \leq i \leq n-r$ ,

$$\begin{aligned} \beta_{i,i+r}(R/I_r(K_n)) &= \beta_{i,i+r-1}(R/I_{r-1}(K_{n-1})) + \beta_{i,i+r}(R/I_r(K_{n-1})) + \beta_{i-1,i+r-1}(R/I_r(K_{n-1})) \\ &= \binom{i+r-2}{r-1} \binom{n-1}{i+r-1} + \binom{i+r-1}{r} \binom{n-1}{i+r} + \binom{i+r-2}{r} \binom{n-1}{i+r-1} \\ &= \binom{i+r-1}{r} \binom{n}{i+r} \end{aligned}$$

□

1 **Remark 4.5.** The ideal  $I_r(K_n)$  is same as the edge ideal of the  $(r+1)$ -complete hypergraph  $K_n^{r+1}$   
 2 defined by Emtander [6]. Thus Theorem 4.4 shows that the edge ideal of the complete hypergraph  
 3  $K_n^{r+1}$  is vertex splittable.

4 **Example 4.6.** Let  $G = K_7$  and  $r = 3$ . Then the Betti table of  $R/I_3(K_7)$  computed using Theorem  
 5 4.4 as follows.  
 6

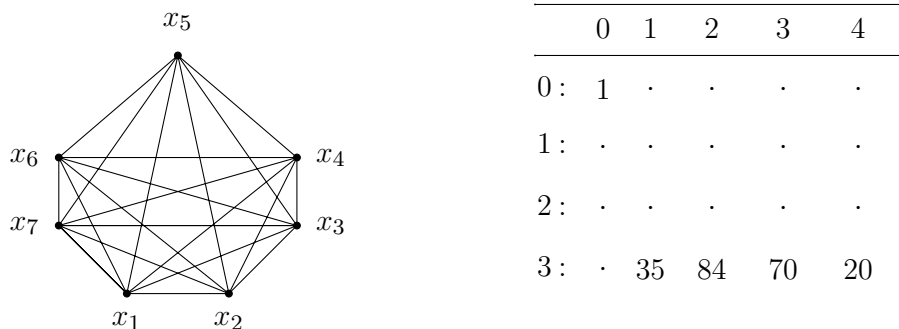


FIGURE 3. The graph  $K_7$  and the Betti table of  $R/I_3(K_7)$

22 Recall that, if  $M$  is an  $\mathbb{N}$ -graded  $R$  module, then the Hilbert series  $H_M(t)$  measures the  $\mathbb{K}$ -vector  
 23 space dimensions of the graded pieces  $M_i$  of  $M$ . More specifically, if each graded piece  $M_i$  has  
 24 finite  $\mathbb{K}$ -vector space dimension, then  $H_M(t)$  is the formal power series

$$H_M(t) = \sum_{i \in \mathbb{N}} \dim_{\mathbb{K}}(M_i) t^i.$$

27 The following result is well-known. See, for example, [9, Section 6.1.3].

28 **Lemma 4.7.** Let  $R$  be the polynomial ring  $\mathbb{K}[x_1, \dots, x_n]$ , and consider a finitely generated  $\mathbb{N}$ -graded  
 29  $R$ -module  $M$ . Then

$$H_M(t) = \frac{R_M(t)}{(1-t)^n},$$

32 where  $R_M(t) = \sum_i (-1)^i \sum_j \beta_{i,j}(M) t^j$ .

33 Note that, if  $M$  is the Stanley–Reisner ring of a simplicial complex  $\Delta$ , then the Hilbert series of  
 34  $M$  can be calculated from combinatorial data of  $\Delta$ . In particular, we have the following formula  
 35 (see [17, Chapter 6]):

$$H_{R/I_\Delta} = \frac{1}{(1-t)^n} \sum_{s=0}^l f_{s-1} t^s (1-t)^{n-s},$$

36 where  $f_s$  equals the number of  $s$ -dimensional faces of  $\Delta$  and  $l = \dim \Delta + 1$ .

37 If  $R/I_\Delta$  has a linear minimal free resolution, then the Betti numbers of  $R/I_\Delta$  can be easily  
 38 deduced from the Hilbert series. Let  $G$  be a co-chordal graph, then  $R/I_r(G)$  has a linear minimal  
 39 free resolution (by Corollary 3.14). Thus by [6, Lemma 3.11] we get the following formula.  
 40  
 41  
 42  
 43  
 44  
 45

**Lemma 4.8.** *If  $G^c$  is a chordal graph, then the Betti numbers  $\beta_{i,i+r}(R/I_r(G))$  are given by the formula*

$$(2) \quad \beta_{i,i+r}(R/I_r(G)) = \sum_{s=0}^l (-1)^{r-s} f_{s-1} \binom{n-s}{i+r-s},$$

where  $f_s$  equals the number of  $s$ -dimensional faces of  $\text{Ind}_r(G)$  and  $l = \dim \text{Ind}_r(G) + 1$ .

Lemma 4.8, like Theorem 4.1, provides us with a tool to compute the graded Betti numbers of  $R/I_r(G)$  when  $G$  is co-chordal. By playing the two results off each other, we may be able to derive some combinatorial identities.

As a specific example, if we take  $G = K_n$ , then  $f_s(\text{Ind}_r(K_n)) = \binom{n}{s+1}$  and  $l = r$ . Thus

$$\beta_{i,i+r}(R/I_r(G)) = \sum_{s=0}^r (-1)^{r-s} \binom{n}{s} \binom{n-s}{i+r-s}.$$

We also computed these graded Betti numbers in Theorem 4.4 by using Theorem 4.1. Thus, since these two expressions are equal, we can derive the combinatorial identity:

$$\binom{i+r-1}{r} \binom{n}{i+r} = \sum_{s=0}^r (-1)^{r-s} \binom{n}{s} \binom{n-s}{i+r-s}.$$

See also [6, Section 3.3] for a combinatorial proof.

If  $G = K_{n_1, \dots, n_s}$ , the complete multipartite graph on  $N_s := \sum_{t=1}^s n_t$  number of vertices, then the ideal  $I_r(G)$  is same as the edge ideal of the  $(r+1)$ -complete multipartite hypergraph  $K_{n_1, \dots, n_s}^{r+1}$  as defined by Emtander [6, Section 3.1]. Hence, by Corollary 3.14 we see that the edge ideal of the  $(r+1)$ -complete multipartite hypergraph  $K_{n_1, \dots, n_s}^{r+1}$  is vertex splittable. Also, we have the formula for the Betti numbers of  $I_r(K_{n_1, \dots, n_s})$ .

**Theorem 4.9.** [6, cf. Theorem 3.5] *Let  $G = K_{n_1, n_2, \dots, n_s}$  denote the complete multipartite graph on  $N_s$  number of vertices. Then for all positive integers  $r$ , the  $\mathbb{N}$ -graded Betti numbers of the ideal  $I_r(G)$  can be expressed as follows:*

$$\beta_{i,i+d}(R_G/I_r(G)) = \begin{cases} \sum_{t=1}^s \binom{n_t}{i+r} \binom{i+r-1}{r} - \sum_{J(r,s)} \left[ \prod_{t=1}^s \binom{n_t}{j_t} \right] \sum_{t=1}^s \binom{j_t-1}{r} & \text{for } d = r, \\ 0 & \text{otherwise.} \end{cases}$$

For brevity we let  $J(r, s)$  denote all those tuples  $(j_1, j_2, \dots, j_s) \in \mathbb{N}^s$  such that  $\sum_{t=1}^s j_t = i + r$ .

Let  $P_n^c$  denote the complement of path graph on  $n$  vertices. We now proceed to give formulas for the Betti numbers of  $R/I_r(P_n^c)$  using the vertex splittability of the ideal  $I_r(P_n^c)$ . Note that if  $r = 1$  then  $I_1(P_n^c)$  is the edge ideal of  $P_n^c$ . To the best of our knowledge, the formula for  $\beta_{i,j}(R/I_1(P_n^c))$  in Theorem 4.12 is new.

In order to find the Betti numbers of  $R/I_r(P_n^c)$ , we first analyze the graph  $K_n^x$  defined as follows:

$$\begin{aligned} V(K_n^x) &= \{x, x_1, x_2, \dots, x_n\} \\ E(K_n^x) &= \{\{\{x_i, x_j\} \mid 1 \leq i < j \leq n\} \cup \{\{x, x_i\} \mid i \in [n-1]\}\}. \end{aligned}$$

Note that the graph  $K_n^x$  is the graph  $K_{n+1}$  on the vertex set  $\{x, x_1, \dots, x_n\}$  with the edge  $\{x, x_n\}$  removed. For  $r \geq 1$ , consider the ideal  $I_r(K_n^x)$  in the polynomial ring  $R_x = \mathbb{K}[x, x_1, x_2, \dots, x_n]$ .

**Lemma 4.10.** *The  $\mathbb{N}$ -graded Betti numbers of  $R_x/I_r(K_n^x)$  can be expressed as follows:*

$$\beta_{i,i+r}(R_x/I_r(K_n^x)) = \begin{cases} i \binom{n+1}{i+1} - \binom{n-1}{i-1}, & \text{for } r = 1 \text{ and } 1 \leq i \leq n, \\ \binom{i+r-1}{r} \binom{n+1}{i+r}, & \text{for } r \geq 2 \text{ and } 1 \leq i \leq n-r+1, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Note that  $K_n^x$  is a co-chordal graph with  $x$  as a simplicial vertex of the complement of  $K_n^x$ . Thus by Theorem 3.12,  $I_r(K_n^x)$  is a vertex splittable ideal. Let  $r = 1$ . Then

$$I_1(K_n^x) = xJ_1 + J_2,$$

where  $J_1 = \langle x_1, \dots, x_{n-1} \rangle$  and  $J_2 = I_1(K_n^x \setminus \{x\})$ . Note that  $K_n^x \setminus \{x\} = K_n$ . Therefore, by Theorem 4.1,

$$\begin{aligned} \beta_{i,i+1}(R_x/I_1(K_n^x)) &= \beta_{i,i}(R_x/J_1) + \beta_{i,i+1}(R_x/I_1(K_n)) + \beta_{i-1,i}(R_x/I_1(K_n)) \\ &= \binom{n-1}{i} + i \binom{n}{i+1} + (i-1) \binom{n}{i} \\ &= i \binom{n+1}{i+1} - \binom{n-1}{i-1}. \end{aligned}$$

Now let  $r \geq 2$ . Then by the construction in Theorem 3.12,

$$I_r(K_n^x) = xJ_1 + J_2,$$

where  $J_1 = I_{r-1}(\widetilde{K_n^x})$  and  $J_2 = I_r(K_n^x \setminus \{x\})$ . Note that  $\widetilde{K_n^x} = K_n^x \setminus \{x\} = K_n$ . Therefore, by Theorem 4.1,

$$\begin{aligned} \beta_{i,i+r}(R_x/I_r(K_n^x)) &= \beta_{i,i+r-1}(R_x/I_{r-1}(K_n)) + \beta_{i,i+r}(R_x/I_r(K_n)) + \beta_{i-1,i+r-1}(R_x/I_r(K_n)) \\ &= \binom{i+r-2}{r-1} \binom{n}{i+r-1} + \binom{i+r-1}{r} \binom{n}{i+r} + \binom{i+r-2}{r} \binom{n}{i+r-1} \\ &= \binom{i+r-1}{r} \binom{n+1}{i+r}. \end{aligned}$$

□

**Remark 4.11.** Note that  $\beta_{i,i+r}(R_x/I_r(K_n^x)) = \beta_{i,i+r}(R/I_r(K_{n+1}))$  for  $r \geq 2$ .

Now we are ready to give explicit formulas for the Betti numbers  $\beta_{i,i+r}(R/I_r(P_n^c))$ .

**Theorem 4.12.** *Let  $I_r(P_n^c)$  denote the Stanley-Reisner ideal of the  $r$ -independence complex of the complement of the path graph  $P_n$  on  $n$  vertices  $\{x_1, \dots, x_n\}$ . Then the  $\mathbb{N}$ -graded Betti numbers of  $R/I_r(P_n^c)$  can be expressed as follows:  $\beta_{i,j}(R/I_r(P_n^c)) = 0$  if  $j \neq i+r$ , and*

$$\beta_{i,i+r}(R/I_r(P_n^c)) = \begin{cases} i \binom{n-1}{i+1}, & \text{for } r = 1 \text{ and } 1 \leq i \leq n-2, \\ \binom{i+1}{2} \binom{n}{i+2} - i \binom{n-2}{i}, & \text{for } r = 2 \text{ and } 1 \leq i \leq n-2, \\ \binom{i+r-1}{r} \binom{n}{i+r}, & \text{for } r \geq 3 \text{ and } 1 \leq i \leq n-r, \\ 0, & \text{otherwise.} \end{cases}$$



*Proof.* The graph  $P_n^c$  is a co-chordal graph with  $x_1$  as a simplicial vertex of  $P_n$ . We first consider the case  $r = 1$ . By Theorem 3.8,

$$I_1(P_n^c) = x_1 J_1 + J_2,$$

where  $J_1 = \langle x_3, x_4, \dots, x_n \rangle$  and  $J_2 = I_1(P_n^c \setminus \{x_1\})$ . Thus  $J_2 = I_1(P_{n-1}^c)$  since  $P_n^c \setminus \{x_1\} = P_{n-1}^c$ . We now proceed to prove the formula by induction on  $n$ . For  $n = 1, 2$ ,  $I_1(P_n^c) = \langle 0 \rangle$  and thus we have the above formula. For  $n = 3$ ,  $I_1(P_n^c) = \langle x_1 x_3 \rangle = I_1(K_2)$  and hence using Theorem 4.4 we have our desired formula. Now let  $n \geq 4$ . Then by Theorem 4.1 and using the induction hypothesis we have,

$$\begin{aligned} \beta_{i,i+1}(R/I_r(P_n^c)) &= \beta_{i,i}(R/J_1) + \beta_{i,i+1}(R/I_1(P_{n-1}^c)) + \beta_{i-1,i}(R/I_1(P_{n-1}^c)) \\ &= \binom{n-2}{i} + i \binom{n-2}{i+1} + (i-1) \binom{n-2}{i} \\ &= i \binom{n-1}{i+1}. \end{aligned}$$

Now we consider the case  $r = 2$ . By Theorem 3.12 we have,

$$I_2(P_n^c) = x_1 J_1 + J_2,$$

where  $J_1 = I_1(\widetilde{P}_n^c)$  and  $J_2 = I_2(P_{n-1}^c)$ . Note that  $\widetilde{P}_n^c \cong K_{n-2}^x$ . Now, for  $n = 1, 2, 3$ , we have  $I_1(\widetilde{P}_n^c) = I_2(P_{n-1}^c) = \langle 0 \rangle$ . For  $n = 4$ , we have  $J_1 = I_1(K_2^x)$  and  $J_2 = I_2(P_3^c) = \langle 0 \rangle$ . Therefore, by Theorem 4.1 and Lemma 4.10,

$$\begin{aligned} \beta_{i,i+2}(R/I_2(P_n^c)) &= \beta_{i,i+1}(R/I_1(K_2^x)) + \beta_{i,i+1}(R/I_2(P_3^c)) + \beta_{i-1,i}(R/I_2(P_3^c)) \\ &= i \binom{3}{i+1} - \binom{1}{i-1}. \end{aligned}$$

It is not difficult to see that for all non-negative integers  $i$ ,  $i \binom{3}{i+1} - \binom{1}{i-1} = \binom{i+1}{2} \binom{4}{i+2} - i \binom{2}{i}$ . Thus we have  $\beta_{i,i+2}(R/I_2(P_n^c)) = \binom{i+1}{2} \binom{4}{i+2} - i \binom{2}{i}$ . Now let  $n \geq 5$ . Then by Theorem 4.1 and by the induction hypothesis,

$$\begin{aligned} \beta_{i,i+2}(R/I_2(P_n^c)) &= \beta_{i,i+1}(R/I_1(K_{n-2}^x)) + \beta_{i,i+1}(R/I_2(P_{n-1}^c)) + \beta_{i-1,i}(R/I_2(P_{n-1}^c)) \\ &= i \binom{n-1}{i+1} - \binom{n-3}{i-1} + \binom{i+1}{2} \binom{n-1}{i+2} - i \binom{n-3}{i} + \binom{i}{2} \binom{n-1}{i+1} - (i-1) \binom{n-3}{i-1} \\ &= \binom{i+1}{2} \binom{n}{i+2} - i \binom{n-2}{i}. \end{aligned}$$

We now take  $r \geq 3$  and prove the formula by induction on  $n$  and  $r$ . By Theorem 3.12,

$$I_r(P_n^c) = x_1 J_1 + J_2,$$

where  $J_1 = I_{r-1}(K_{n-2}^x)$  and  $J_2 = I_r(P_{n-1}^c)$ . Note that for a fixed  $r$ ,  $I_r(P_n^c) = \langle 0 \rangle$  for  $n = 1, 2, \dots, r$ . If  $n = r + 1$ , then  $I_r(P_n^c) = \langle \prod_{i=1}^{r+1} x_i \rangle$ . Hence  $\beta_{i,i+r}(R/I_r(P_n^c)) = \binom{i+r-1}{r} \binom{r+1}{i+r}$  since

$$\binom{i+r-1}{r} \binom{r+1}{i+r} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

1 Now let  $n \geq r + 2$ . Then by Theorem 4.1, Lemma 4.10 and by the induction on  $n$  and  $r$ ,

$$\begin{aligned}
 & \beta_{i,i+r}(R/I_r(P_n^c)) \\
 &= \beta_{i,i+r-1}(R/I_{r-1}(K_{n-2}^x)) + \beta_{i,i+r}(R/I_r(P_{n-1}^c)) + \beta_{i-1,i+r-1}(R/I_r(P_{n-1}^c)) \\
 &= \binom{i+r-2}{r-1} \binom{n-1}{i+r-1} + \binom{i+r-1}{r} \binom{n-1}{i+r} + \binom{i+r-2}{r} \binom{n-1}{i+r-1} \\
 &= \binom{i+r-1}{r} \binom{n}{i+r}.
 \end{aligned}$$

□

12 **Example 4.13.** Let  $G = P_7^c$ . Then the Betti tables of  $R/I_2(P_7^c)$  and  $R/I_3(P_7^c)$  computed using  
 13 Theorem 4.4 are given as in Figure 4.

	0	1	2	3	4	5		0	1	2	3	4
0:	1	·	·	·	·	·	0:	1	·	·	·	·
1:	·	·	·	·	·	·	1:	·	·	·	·	·
2:	·	30	85	96	50	10	2:	·	·	·	·	·
							3:	·	35	84	70	20

FIGURE 4. Betti table of  $R/I_2(P_7^c)$  and  $R/I_3(P_7^c)$ , respectively

27 **Remark 4.14.** Note that if we compare the formulas for the graphs  $P_n^c$ ,  $K_n$  and  $K_{1,n}$ , for  $r \geq 3$ ,  
 28 we have  $\beta_{i,i+r}(R/I_r(P_n^c)) = \beta_{i,i+r}(R/I_r(K_n)) = \beta_{i,i+r+1}(R/I_r(K_{1,n}))$  for all  $i \geq 0$ .

30 **Remark 4.15.** Note that for  $r \geq 3$ ,  $I_r(P_n^c) = I_r(K_n)$ . However, for  $1 \leq r < 3$  they are not the same  
 31 ideal. Thus for  $r = 1, 2$ , we can calculate the Betti numbers  $\beta_{i,i+r}(R/I_r(P_n^c))$  using Equation (2)  
 32 and obtain the following two combinatorial identities.

$$\begin{aligned}
 & i \binom{n-1}{i+1} = n \binom{n-1}{i} - \binom{n}{i+1} - (n-1) \binom{n-2}{i-1} \\
 & \binom{i+1}{2} \binom{n}{i+2} - i \binom{n-2}{i} = \binom{n}{i+2} - n \binom{n-1}{i+1} + \binom{n}{2} \binom{n-2}{i} - (n-2) \binom{n-3}{i-1}.
 \end{aligned}$$

### 5. Fröberg's theorem via collapsibility

40 In this section we use the concept of  $d$ -collapsibility from topological combinatorics to give an  
 41 alternative proof of Theorem 1.1. The notion of  $d$ -collapsibility was introduced by Wegner in [19].  
 42

43 **Definition 5.1.** Let  $\Delta$  be a simplicial complex. A face  $\sigma \in \Delta$  is called  $d$ -collapsible if there is  
 44 only one facet  $\tau = \tau(\sigma)$  in  $\Delta$  containing  $\sigma$  (possibly  $\tau = \sigma$ ), and moreover  $\dim(\sigma) \leq d - 1$ . In this  
 45 case  $(\sigma, \tau)$  is called a *free pair* and we say that the complex  $\Delta$  *elementary  $d$ -collapses* onto the

1 subcomplex  $\Delta' = \Delta \setminus \{\gamma \in \Delta : \sigma \subseteq \gamma\}$ . We denote this collapsing by  $\Delta \searrow \Delta'$ . Complex  $\Delta$  is called  
 2 *d-collapsible* if there is a sequence

$$3 \quad \Delta \searrow \Delta_1 \searrow \Delta_2 \searrow \cdots \searrow \emptyset,$$

4  
 5 of elementary d-collapses ending with the empty complex. Note that if a complex  $\Delta$  is *d-collapsible*  
 6 for some integer  $d \geq 1$ , then it is *r-collapsible* for any integer  $r \geq d$ .

7 A simplicial complex  $\Delta$  is called *d-Leray* if  $\tilde{H}_i(Y; \mathbb{Z}) = 0$  for all induced subcomplexes  $Y \subseteq \Delta$   
 8 and for all  $i \geq d$ . The *Leray number* of  $\Delta$ , denoted  $L(\Delta)$ , is the minimal  $d$  such that  $\Delta$  is *d-Leray*.  
 9 From Hochster's formula [10], we know that

$$10 \quad (3) \quad \text{reg}(I_\Delta) = L(\Delta) + 1.$$

11  
 12 We know that the ideal  $I_r(G)$  has a linear resolution if and only if  $\text{reg}(I_r(G)) = r + 1$  [21, Lemma  
 13 49]). Thus, to prove Theorem 1.1, it is enough to show that  $\text{reg}(I_r(G)) = r + 1$ . We do this by  
 14 showing that the complex  $\text{Ind}_r(G)$  is *r-Leray* (recall that  $I_r(G) = I_{\text{Ind}_r(G)}$ ) for any  $r \geq 1$  and for any  
 15 co-chordal graph  $G$ , which along with Equation (3), will prove Theorem 1.1. For more discussion  
 16 on the Leray number of a complex  $\Delta$  and the regularity of its Stanley-Reisner ideal  $I_\Delta$ , the reader  
 17 is referred to [12].

18 Wegner [19] proved that every *d-collapsible* complex is *d-Leray*. However, the converse is not  
 19 true (see [13] for examples). Hence, the following result along with the discussion above gives an  
 20 alternate proof of Theorem 1.1.

21 **Theorem 5.2.** *Let  $G$  be a finite simple graph and let  $r \geq 1$  be an integer. If  $G$  is the complement of*  
 22 *a chordal graph, then  $\text{Ind}_r(G)$  is *r-collapsible*. In particular,  $I_r(G)$  has a  $(r + 1)$ -linear resolution.*  
 23

24 *Proof.* It is known [19] that the complex  $\text{Ind}_1(G)$  is 1-collapsible for any co-chordal graph  $G$  and  
 25 hence  $\text{Ind}_1(G)$  is *r-collapsible* for any  $r \geq 1$ . Therefore, we may assume that  $r \geq 2$ .

26 Let  $\Delta = \text{Ind}_r(G)$  and  $F$  be a face of  $\Delta$ . By definition, each connected component of the induced  
 27 subgraph  $G[F]$  has at most  $r$  vertices. Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two connected components of  $G[F]$ . We first  
 28 show that either  $|V(\mathcal{C}_1)| = 1$  or  $|V(\mathcal{C}_2)| = 1$ . On the contrary, let  $v_1, v_2 \in V(\mathcal{C}_1)$  and  $v_3, v_4 \in V(\mathcal{C}_2)$   
 29 such that  $\{v_1, v_2\}$  and  $\{v_3, v_4\}$  are edges in the graph  $G$ . In that case,  $v_1 v_4 v_2 v_3 v_1$  becomes an  
 30 induced cycle of length four in the chordal graph  $G^c$ , which is a contradiction. Hence, if  $F$  is a face  
 31 of  $\Delta$  such that  $|F| \geq r$ , then  $F = F' \sqcup F''$ , where both  $F'$  and  $F''$  are faces of  $\Delta$  such that  $|F'| \leq r$ ,  
 32 the induced subgraph  $G[F']$  is connected and the induced subgraph  $G[F'']$  consists of only isolated  
 33 vertices. Moreover, no vertex in  $F'$  is connected to any vertex in  $F''$  by an edge. Now we proceed  
 34 to show that  $\Delta$  is *r-collapsible*.

35 It is easy to see that  $\text{Ind}_1(G) \subseteq \Delta$ . Here, we show that the complex  $\Delta$  can be collapsed onto  
 36  $\text{Ind}_1(G)$  using elementary *r-collapses*. We know that if  $F$  is a face of  $\Delta$  such that  $|F| \geq r$ , then  
 37 the induced subgraph  $G[F]$  has at most one connected component of cardinality more than one  
 38 (and at most  $r$ ). To collapse  $\Delta$  onto  $\text{Ind}_1(G)$ , it is enough to collapse all the faces in which the  
 39 cardinality of the connected component is more than one. We do these collapses in the decreasing  
 40 order of cardinality of the connected components, that is, we first collapse those faces in which the  
 41 connected component has vertex cardinality  $r$ .

42 Let  $F$  be a face of  $\Delta$  such that  $G[F]$  is connected and  $|F| = r > 1$ . To show that  $F$  is *r-collapsible*,  
 43 it is enough to show that  $F$  is contained in a unique facet. On the contrary, let  $F_1$  and  $F_2$  be two  
 44 facets of  $\Delta$  such that  $F \subseteq F_1 \cap F_2$ . In that case if  $F'_1 = F_1 \setminus F$  and  $F'_2 = F_2 \setminus F$  then the induced  
 45 subgraphs  $G[F'_1]$  and  $G[F'_2]$  consists of only isolated vertices. Since  $F_1$  and  $F_2$  are different facets,

1  $F'_2 \setminus F'_1 \neq \emptyset$  and  $F'_1 \setminus F'_2 \neq \emptyset$ . If there is no edge in the induced subgraph  $G[F'_1 \cup F'_2]$ , then  $F_1 \cup F_2$  is  
2 a face of  $\Delta$  which contradicts the fact that  $F_1$  and  $F_2$  are facets. Therefore, let  $v \in F'_2 \setminus F'_1$  such  
3 that  $v$  is connected to some  $w \in F'_1$ . Then  $vu_1wu_2v$  forms an induced cycle of length 4 in the  
4 graph  $G^c$ , where  $u_1, u_2 \in F$  such that  $\{u_1, u_2\} \in E(G)$ . Note that such an edge exists since  $G[F]$   
5 is a connected graph on at least two vertices. This gives us a contradiction since  $G^c$  is assumed  
6 to be chordal. Thus if  $F \in \Delta$  such that  $|F| = r$  and  $G[F]$  is connected, then  $F$  is contained in a  
7 unique facet. Moreover, note that if  $F, F'$  are two different faces in  $\Delta$  with  $|F| = |F'| = r$  and the  
8 induced subgraphs  $G[F]$  and  $G[F']$  are both connected such that  $F_1$  and  $F'_1$ , are the unique facets  
9 in  $\Delta$  containing  $F$  and  $F'$ , respectively, then  $F_1 \neq F'_1$ . Indeed, if  $F_1 = F'_1$  then either  $G[F \cup F']$  is  
10 contained in a connected component  $\mathcal{C}$  of  $G[F_1]$ , or there exists two connected components  $\mathcal{C}_1$  and  
11  $\mathcal{C}_2$  of  $G[F_1]$  such that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  contains  $G[F]$  and  $G[F']$ , respectively. In the first case we have a  
12 contradiction since  $|F \cup F'| > r$ . In the second case,  $G[F_1]$  has at least two connected components  
13 of vertex cardinality more than one, again a contradiction.

14 After collapsing all the faces  $F \in \Delta$  such that  $G[F]$  is connected and  $|F| = r$ , we do the same  
15 with faces  $F' \in \Delta$  such that  $|F'| = r - 1 > 1$  and  $G[F']$  is connected. Continuing this way, let  $\Delta_i$   
16 be the simplicial complex obtained from  $\Delta$  by collapsing all  $F \in \Delta$  such that  $|F| > i$  and  $G[F]$  are  
17 connected. Note that if  $\sigma \in \Delta_i$ , then  $\sigma$  is also a face of  $\Delta$  and hence  $\sigma = \sigma' \sqcup \sigma''$ , where both  $\sigma'$   
18 and  $\sigma''$  are faces of  $\Delta_i$ ,  $|\sigma'| \leq i$  where the induced subgraph  $G[\sigma']$  is connected and the induced  
19 subgraph  $G[\sigma'']$  consists of only isolated vertices. Moreover, no vertex in  $\sigma'$  is connected to any  
20 vertex in  $\sigma''$  by an edge. Proceeding as above we see that if  $\sigma$  is a face of  $\Delta_i$  such that  $G[\sigma]$  is  
21 connected and  $|\sigma| = i > 1$ , then  $\sigma$  is contained in a unique facet of  $\Delta_i$ . Moreover, if  $\sigma, \sigma' \in \Delta_i$  with  
22  $|\sigma| = |\sigma'| = i$  and the induced subgraphs  $G[\sigma]$  and  $G[\sigma']$  are both connected such that  $\sigma_1$  and  $\sigma'_1$   
23 are the unique facets containing  $\sigma$  and  $\sigma'$ , respectively, then  $\sigma_1 \neq \sigma'_1$ . Thus each face of  $\Delta_i$  of  
24 cardinality  $i$  whose corresponding induced subgraph is connected, is  $r$ -collapsible. We continue  
25 these collapses till we collapse every face  $\sigma \in \Delta$  such that  $|\sigma| > 1$  and  $G[\sigma]$  are connected. These  
26 collapses are done in the decreasing order of the cardinality of  $\sigma$ . Observe that the remaining  
27 complex is  $\text{Ind}_1(G)$  which is 1-collapsible. This completes the proof of Theorem 5.2.  $\square$

28  
29 The  $r$ -independence complexes ( $r \geq 1$ ) of graphs we encountered in Corollary 3.13 are  $r$ -collapsible  
30 as we now show.

31 **Corollary 5.3.** *Let  $G$  be a gap-free and claw-free graph such that it contains a leaf. Then  $\text{Ind}_r(G)$   
32 and  $\text{Ind}_r(G^c)$  are  $r$ -collapsible for any integer  $r \geq 1$ .*

33  
34 *Proof.* Proceeding as in Corollary 3.13, we have both  $G$  and  $G^c$  chordal. Therefore, by the proof  
35 of Theorem 5.2, both  $\text{Ind}_r(G)$  and  $\text{Ind}_r(G^c)$  are  $r$ -collapsible.  $\square$

36  
37 We know that if  $G = C_n^c$ , where  $n \geq 4$ , then  $I_{\text{Ind}_1(G)}$  does not have a linear resolution by Fröberg's  
38 theorem, since  $G^c = C_n$  is not a chordal graph for  $n \geq 4$ . However, in what follows we show that  
39  $I_{\text{Ind}_r(C_n^c)}$  has a linear resolution for each  $r \geq 2$ . Consequently, the converse of Theorem 1.1 cannot  
40 hold. We prove that  $I_{\text{Ind}_r(G)}$  has a linear resolution as a corollary of the following result.

41 **Proposition 5.4.** *Let  $G$  be a graph so that  $G^c$  does not have an induced cycle of length 4. If  
42  $\dim \text{Ind}_1(G) \leq r - 1$ , then  $I_{\text{Ind}_r(G)}$  has a  $(r + 1)$ -linear resolution.*

43  
44 *Proof.* Following the proof of Theorem 5.2 we see that if  $G^c$  does not have an induced 4-cycle then  
45 for each  $r \geq 2$ ,  $\text{Ind}_r(G)$  is  $r$ -collapsible to  $\text{Ind}_1(G)$ . Since  $\dim \text{Ind}_1(G) \leq r - 1$  we see that  $\text{Ind}_r(G)$

1 is  $r$ -collapsible to the empty complex. Therefore,  $\text{Ind}_r(G)$  is  $r$ -Leray. Consequently,  $I_{\text{Ind}_r(G)}$  has a  
 2 linear resolution.  $\square$

3 **Corollary 5.5.** For  $n \geq 4$ , the ideal  $I_{\text{Ind}_r(C_n^c)}$  has a linear resolution for each  $r \geq 2$ .  
 4

5 *Proof.* For  $n \leq 4$ ,  $I_{\text{Ind}_r(C_n^c)}$  is the zero ideal when  $r \geq 2$  and hence has a linear resolution. For  $n \geq 5$ ,  
 6 note that  $\dim \text{Ind}_1(C_n^c) = 1$  and hence the result follows from Proposition 5.4.  $\square$

## 7 8 6. Concluding Remarks

9 As stated in the Introduction, determining a combinatorial description of higher degree square-free  
 10 monomial ideals that have a linear resolution over all fields is an active area of research. A  
 11 prominent setting to undertake such a study is that of hypergraphs, since they generalize graphs  
 12 and the edge ideal of a hypergraph is a square-free monomial ideal. Inspired by Fröberg's theorem,  
 13 there were attempts to generalize the notion of *chordality* to hypergraphs and then prove that the  
 14 edge ideal of the complement hypergraph has a linear resolution. A recent approach is by Bigdeli,  
 15 Yazdan Pour and Zaare-Nahandi [3]. The authors first introduce the notion of chordal hypergraphs  
 16 (by generalizing the perfect elimination order of chordal graphs) and then show that the edge ideal  
 17 associated with the complement hypergraph has a linear resolution over any field. They further  
 18 show that this particular class of chordal hypergraphs contains several, previously defined classes of  
 19 chordal hypergraphs. More recently, Bigdeli and Faridi [2] have extended this notion of chordality  
 20 to the realm of simplicial complexes.

21 Let  $\mathcal{H}$  be an  $(r+1)$ -uniform hypergraph (i.e., every edge is of cardinality  $r+1$ ) on the vertex  
 22 set  $V$ . A subset  $W \subset V$  is called a *simplicial maximal subcircuit* if  $|W| = r$ , it is contained in  
 23 an edge, and its neighborhood is a clique (see [3, Definition 1.2] for details). Now a *chordal*  
 24 *hypergraph* is recursively defined as the  $(r+1)$ -uniform hypergraph  $\mathcal{H}$  which is either empty or  
 25 contains a simplicial maximal subcircuit  $W$  such that the deletion  $\mathcal{H} \setminus W$  is also chordal. Further,  
 26 a hypergraph is *co-chordal* if its complement is chordal. We have already seen in the Introduction  
 27 that the notion of  $r$ -independence lets us define a hypergraph, which we denote by  $\text{Con}_r(G)$ . Based  
 28 on our calculations we propose the following conjecture:

29 **Conjecture 6.1.** For all  $r \geq 1$ ,  $\text{Con}_r(G)$  is a co-chordal hypergraph if  $G$  is a co-chordal graph.  
 30

31 The reader should note that our work allows us to directly show that  $I_r(G)$  has a linear resolution  
 32 without first checking if  $\text{Con}_r(G)$  is chordal or not. Moving on, because of Corollary 5.5 the following  
 33 question is worth answering:  
 34

35 **Question 6.2.** Are there examples of (non-co-chordal) graphs  $G$  such that  $I_r(G)$  has a linear  
 36 resolution but the hypergraph  $\text{Con}_r(G)$  is not a co-chordal hypergraph?  
 37

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