BLOW-UP RINGS AND F-RATIONALITY

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ABSTRACT. In this paper, we prove some sufficient conditions for Cohen-Macaulay normal Rees algebras to be F-rational. Let $(R, \mathfrak{m})$ be a Gorenstein normal local domain of dimension $d \geq 2$ and of characteristic $p > 0$. Let I be an ideal generated by a system of parameters. Our first set of results give conditions on the test ideals $\tau(I^n)$, $n \geq 1$ which would imply that the normalization of the Rees algebra $R[It]$ is F-rational. Another sufficient condition is that the socle of $\mathrm{H}^d_{\overline{G}_+}(\overline{G})$ (where $\overline{G}$ is the associated graded ring for the integral closure filtration) is entirely in degree -1, if R is F-rational (but not necessarily Gorenstein). Then we show that if R is a hypersurface of degree 2 or is three-dimensional and F-rational and $\mathrm{Proj}(R[\mathfrak{m}t])$ is F-rational, then $R[\mathfrak{m}t]$ is F-rational.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a d-dimensional excellent local domain of prime characteristic $p > 0$, where $d \geq 2$. Let I be an $\mathfrak{m}$-primary ideal. Write $\mathscr{R}(I)$ for the Rees algebra $\oplus_{n\in\mathbb{N}} I^n$ and $\overline{\mathscr{R}(I)}$ for its normalization. In this paper, we prove some sufficient conditions for $\overline{\mathscr{R}(I)}$ to be F-rational. These are motivated by results of N. Hara, K.-i. Watanabe and K.-i. Yoshida [HWY02], Hara and Yoshida [HY03], M. Koley and the second author [KK21] and the analogous results of E. Hyry [Hyr99] in characteristic zero.

Our first result is the following converse to [HY03, Theorem 5.1]. For an R-ideal $\mathfrak{a}$, $\tau(\mathfrak{a})$ is the *test ideal* of $\mathfrak{a}$ [HY03, 1.9].

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a Gorenstein normal local domain of dimension $d \geq 2$ and of characteristic $p > 0$. Let I be an R-ideal generated by a system of parameters such that $\overline{\mathscr{R}(I)}$ is Cohen-Macaulay. If $\tau(I^n) = I^n : \overline{I^{d-1}}$ for all integers $1 \leq n \leq d-1$, then $\overline{\mathscr{R}(I)}$ is F-rational.*

Using Lemma 3.15, we immediately obtain the following:

Corollary 1.2. *Write $\overline{\mathfrak{M}}$ for the unique homogeneous maximal ideal of $\overline{\mathscr{R}(I)}$. If*

$$\left[0^*_{\mathrm{H}^{d+1}_{\overline{\mathfrak{M}}}(\overline{\mathscr{R}(I)})}\right]_{-n} = 0$$

for all $1 \leq n \leq d-1$, then $\overline{\mathscr{R}(I)}$ is F-rational.

As a corollary of some of the arguments that go into the proof of Theorem 1.1, we get the following proposition.

Proposition 1.3. *Let $(R, \mathfrak{m})$ be Gorenstein normal local domain of dimension $d \geq 2$ and of characteristic $p > 0$. Let I be an R-ideal generated by a system of parameters such that $\overline{\mathscr{R}(I)}$ is Cohen-Macaulay and the associated graded ring $\overline{G} := \oplus_{n\geq 0} \frac{\overline{I^n}}{\overline{I^{n+1}}}$ is Gorenstein. Write a for the a-invariant of $\overline{G}$.*

(1) *If $\tau(\overline{I^{-a-1}}) = R$, then $\overline{\mathscr{R}(I)}$ is F-rational.*
(2) *If $\overline{\mathscr{R}(I)}$ is F-rational and $a \leq -2$, then $\tau(I^n) = R$ for all $0 \leq n \leq -a-1$.*

The next proposition overlaps with Proposition 1.3 (1) when $\overline{G}$ is Gorenstein with $a(\overline{G}) = -1$.

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Proposition 1.4. *Let $(R, \mathfrak{m})$ be a d-dimensional F-rational domain of characteristic $p > 0$ and I an $\mathfrak{m}$-primary ideal such that $\overline{\mathscr{R}(I)}$ is Cohen-Macaulay. Let $\overline{G} := \oplus_{n\geq 0}\frac{\overline{I^n}}{\overline{I^{n+1}}}$. Assume that*

$$\operatorname{soc}\left(\mathrm{H}^d_{\overline{G}_+}(\overline{G})\right) \subseteq \left[\mathrm{H}^d_{\overline{G}_+}(\overline{G})\right]_{-1}.$$

Then $\overline{\mathscr{R}(I)}$ is F-rational.

A consequence of [Hyr99, Theorem 1.5] is the following: Let $(A, \mathfrak{a})$ be an excellent local normal domain of characteristic zero and $\mathfrak{b}$ an $\mathfrak{a}$-primary ideal of A. Assume that the the normalized Rees algebra $B := \oplus_{n\in\mathbb{N}}\overline{\mathfrak{b}^n}$ is Cohen-Macaulay. Then B has rational singularities if and only if Proj B has rational singularities. We wonder whether the following prime-characteristic analogue is true. Assume, with notation as in the top of this section, that R is additionally F-rational; if Proj $\overline{\mathscr{R}(I)}$ is F-rational, is $\overline{\mathscr{R}(I)}$ F-rational? In this context, we have the following.

Theorem 1.5. *Let $(R, \mathfrak{m})$ be a complete F-finite normal domain of characteristic $p \geq 7$ and with an infinite residue field, satisfying one of the following:*

1. *R is a hypersurface of dimension $d \geq 2$ and of multiplicity 2;*
2. *R is a three-dimensional Gorenstein F-rational ring.*

Suppose that $\mathscr{R}(\mathfrak{m})$ is Cohen-Macaulay and that Proj $\mathscr{R}(\mathfrak{m})$ is F-rational. Then $\mathscr{R}(\mathfrak{m})$ is F-rational. Additionally, in (1), R is F-rational.

This paper is organized as follows. In Section 2 we collect relevant definitions and facts about Rees algebras, tight closure and test ideals. Theorem 1.1 is proved in Section 3. Proposition 1.3 and other corollaries of Theorem 1.1 are proved in Section 4. Section 5 contains some background information for the proof of Theorem 1.5, which is given in Section 6.

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2. Preliminaries

All the rings we consider in this article are excellent. The letter p denotes a prime number. When used in the context of the Frobenius map and singularities in prime characteristic, q denotes an arbitrary power of p.

We now collect results on Rees algebras and singularities in prime characteristic that will be required in the proofs.

2.1. **Local cohomology.** Let R be a ring and $I = (r_1, \ldots, r_m)$ be an R-ideal. Then the elements of $\mathrm{H}^m_I(R)$ are the residue classes of the fractions $\frac{a}{(r_1\cdots r_m)^l}$, $a \in R$ and $l \geq 1$, modulo the m-boundaries in the extended Čech complex $\check{\mathrm{C}}^\bullet(r_1, \ldots, r_m)$. When $r_1, \ldots, r_m$ is a regular sequence, the class of $\frac{a}{(r_1\cdots r_m)^l}$ is zero if and only if $a \in (r_1^l, \ldots, r_m^l)$; see [LT81, Proof of Theorem 2.1, p. 104–105].

2.2. **Rees algebras.** Let R be a ring. A *filtration* of R is a sequence $\mathcal{I} := (I_n)_{n\in\mathbb{N}}$ of R-ideals I_n such that

$$I_0 = R, I_{n+1} \subseteq I_n \text{ and } I_nI_m \subseteq I_{n+m} \text{ for all } n, m \in \mathbb{N}.$$

The *Rees algebra* of $\mathcal{I}$ is the graded subring

$$\mathscr{R}(\mathcal{I}) := \bigoplus_{n\in\mathbb{N}} I_n t^n$$

of $R[t]$ (t being an indeterminate) with $\deg r = 0$ for each $r \in R$ and $\deg t = 1$. Let I be an ideal of R. We write $\mathscr{R}(I)$ for the Rees algebra for the I-adic filtration, i.e., the one with $I_n = I^n$ for each n. We say that a filtration $\mathcal{I}$ is *I-admissible* if $I \subseteq I_1$ and $\mathscr{R}(\mathcal{I})$ is a finitely generated $\mathscr{R}(I)$-module. By $\overline{\mathscr{R}(I)}$ we mean the integral closure of $\mathscr{R}(I)$ in $R[t]$. It is the Rees algebra of the filtration $(\overline{I^n})_{n\in\mathbb{N}}$, where for every R-ideal J, $\overline{J}$ denotes its integral closure. Since R is excellent, the filtration $(\overline{I^n})_{n\in\mathbb{N}}$ is I-admissible.

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A *reduction* of a filtration $\mathcal{I}$ is an R-ideal $I \subseteq I_1$ such that $\mathcal{I}$ is I-admissible. A *reduction* of an ideal I is a reduction for the I-adic filtration of R. A reduction I of $\mathcal{I}$ is *minimal* if it is minimal with respect to inclusion. If R is a local ring with maximal ideal $\mathfrak{m}$ and $R/\mathfrak{m}$ is infinite, then every minimal reduction of $\mathcal{I}$ is minimally generated by $\dim(\mathscr{R}(\mathcal{I}) \otimes_R R/\mathfrak{m})$ elements. In particular, if I_1 is $\mathfrak{m}$-primary, then every reduction of $\mathcal{I}$ is minimally generated by $\dim R$ elements. For a reduction I of $\mathcal{I}$, the *reduction number* $r_I(\mathcal{I})$ is the minimum of the integers r such that $I_{n+1} = II_n$ for all $n \geq r$. When $\mathcal{I}$ is the I-adic filtration (for some R-ideal I) and J is a reduction of I, we write $r_J(I) = r_J(\mathcal{I})$.

The *associated graded ring* of a filtration $\mathcal{I} = (I_n)$ is $G(\mathcal{I}) := \bigoplus_{n\geq 0} \frac{I_n}{I_{n+1}}$. When $\mathcal{I}$ is the I-adic filtration, we write $G(I) = G(\mathcal{I})$; when $\mathcal{I} = (\overline{I^n})_{n\geq 0}$, we write $\overline{G}(I) = G(\mathcal{I})$.

We now collect some statements about the local cohomology modules of Rees algebras. Recall that for a noetherian graded ring $S := \oplus_{n\geq 0} S_n$ with S_0 local, the *a-invariant* $a(S)$ of S is

$$\max\{j \mid \left[\mathrm{H}^{\dim S}_{\mathfrak{n}}(S)\right]_j \neq 0\}$$

where $\mathfrak{n}$ is the unique homogeneous maximal ideal of S. If S is a $\mathbb{Z}$-graded ring with unique homogeneous maximal ideal $\mathfrak{n}$ and M is a graded S-module, the *socle* of M, denoted $\operatorname{soc} M$ is $(0 :_M \mathfrak{n})$.

Theorem 2.2.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and I an $\mathfrak{m}$-primary ideal. Let $\mathcal{I}$ be an I-admissible filtration.*

(1) [GN94, Lemma 3.3] $a(\mathscr{R}(\mathcal{I})) = -1$.
(2) [GN94, Corollary 1.2] *$\mathscr{R}(\mathcal{I})$ is Cohen-Macaulay if and only if $G(\mathcal{I})$ is Cohen-Macaulay and $a(G(\mathcal{I})) < 0$.*

In some cases, there is the following relation between the a-invariant and the reduction number, which can be proved similar to [Tru87, Proposition 3.2]. (The second assertion follows from the first and Theorem 2.2.1 (2).)

Proposition 2.2.2. *With notation as in Theorem 2.2.1, assume that $G(\mathcal{I})$ is Cohen-Macaulay. Let I be a minimal reduction of $\mathcal{I}$. Then $r_I(\mathcal{I}) = a(G(\mathcal{I})) + d$. In particular, if $\mathscr{R}(\mathcal{I})$ is Cohen-Macaulay, then $r_I(\mathcal{I}) \leq d-1$.*

Write $\mathscr{R}'(\mathcal{I})$ for the extended Rees algebra $\oplus_{n\in\mathbb{Z}} I_n t^n$ (where $I_n := R$ when $n \leq 0$). W. Heinzer, M.-K. Kim and B. Ulrich [HKU11, Theorem 6.1] gave a description of $\omega_{\mathscr{R}'(\mathcal{I})}$. Together with [KK21, Discussion 3.1], we get the following proposition.

Proposition 2.2.3. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring, of dimension $d \geq 2$. Let I be an ideal generated by a system of parameters and $\mathcal{I}$ be an I-admissible filtration. Write $r = r_I(\mathcal{I})$. If $\mathscr{R}(\mathcal{I})$ is Cohen-Macaulay, then $\left[\omega_{\mathscr{R}(\mathcal{I})}\right]_n = I^n : I_{d-1}$ for each $n \geq 1$. If, further, $G(\mathcal{I})$ is Gorenstein, then $\left[\omega_{\mathscr{R}(\mathcal{I})}\right]_n = I_{n+a+1}$ where $a = a(G(\mathcal{I}))$.*

Proof. By [HKU11, Theorem 6.1(1)],

$$\left[\omega_{\mathscr{R}'(\mathcal{I})}\right]_n = I^{n-(d-1)+r} : I_r$$

for each n. Since $\mathscr{R}(\mathcal{I})$ is Cohen-Macaulay, $r \leq d-1$ (Proposition 2.2.2), so $I^n : I_{d-1} = I^n : I^{(d-1)-r} I_r = I^{n-(d-1)+r} : I_r$. By [KK21, Discussion 3.1], for each $n < 0$,

$$\left[\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathscr{R}(\mathcal{I}))\right]_n = \left[\mathrm{H}^{d+1}_{\mathfrak{M}'}(\mathscr{R}'(\mathcal{I}))\right]_n$$

where $\mathfrak{M}$ and $\mathfrak{M}'$ are the maximal homogeneous ideals of $\mathscr{R}(\mathcal{I})$ and $\mathscr{R}'(\mathcal{I})$ respectively. (In each negative degree, the middle vertical map in the commutative diagram in [KK21, Discussion 3.1] is the identity map, and, therefore, so is the map γ.) Applying the graded Matlis duality functor $\operatorname{Hom}_R(-, E)$ (where E is the injective hull of the R-module $R/\mathfrak{m}$) we see that for each $n > 0$,

$$\left[\omega_{\mathscr{R}(\mathcal{I})}\right]_n = \left[\omega_{\mathscr{R}'(\mathcal{I})}\right]_n = I^n : I_{d-1}.$$

Assume further that $G(\mathcal{I})$ is Gorenstein. Then $\omega_{G(\mathcal{I})} = G(\mathcal{I})(a)$. Since $G(\mathcal{I}) = \frac{\mathscr{R}'(\mathcal{I})}{t^{-1}\mathscr{R}'(\mathcal{I})}$, it follows that $\omega_{\mathscr{R}'(\mathcal{I})} = \mathscr{R}'(\mathcal{I})(a+1)$. ◻

2.3. **Prime characteristic notions.** Let R be a ring of prime characteristic $p > 0$. For an ideal $I \subseteq R$, $I^{[q]}$ denotes the ideal generated by q-th power of elements of I. Let $F : R \longrightarrow R$ be the *Frobenius map* $r \mapsto r^p$. The ring R viewed as an R-module via $F^e : R \longrightarrow R$ is denoted as ${}^eR$. Let M be an R-module and write $F^e(M) = {}^eR \otimes_R M$. For $x \in M$, its image in $F^e(M)$ under the natural map $M \longrightarrow {}^eR \otimes_R M$ is written as x^{p^e}. For a submodule $N \subseteq M$, the image of $F^e(N)$ in $F^e(M)$ is denoted by $N_M^{[p^e]}$. Write

$$R^{\circ} = R \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}.$$

Let $\mathfrak{a}$ be an R-ideal and $N \subseteq M$ R-modules. The $\mathfrak{a}$*-tight closure* of N in M (introduced by N. Hara and K.-i. Yoshida [HY03, Definition 1.1]) is

$$N_M^{*\mathfrak{a}} := \{z \in M \mid \text{there exists } c \in R^{\circ} \text{ such that } c\mathfrak{a}^q z^q \subseteq N_M^{[q]} \text{ for all } q \gg 1\}.$$

When $\mathfrak{a} = R$, this is the same as the tight closure defined by M. Hochster and C. Huneke[HH90]; we then write N_M^* for N_M^{*R}. When $M = R$ and $N = I$ an R-ideal, we write $I^{*\mathfrak{a}}$ and I^* respectively. We say that N is *tightly closed* in M if $N_M^* = N$.

Definition 2.3.1. We say that a local ring R is F*-rational* if an ideal generated by a system of parameters is tightly closed. We say that a ring R is F*-rational* if $R_{\mathfrak{m}}$ is F-rational for each maximal ideal $\mathfrak{m}$ of R. Say that a scheme X is F*-rational* if it has an open cover by the spectra of F-rational rings.

We have used [HH94, Theorem (4.2)] to give the above definition of F-rationality. Moreover, the same theorem also shows that all F-rational rings are normal, and those that we consider in this paper are Cohen-Macaulay. A Gorenstein local F-rational ring is weakly F-regular, i.e., all ideals are tightly closed. It is known that a Cohen-Macaulay ring R is F-rational if and only if $0^*_{H^d_{\mathfrak{m}}(R)} = 0$ [Smi97, Theorem 2.6]. Suppose that $\mathscr{R}(\mathcal{I})$ is Cohen-Macaulay. Then by a similar argument, $\mathscr{R}(\mathcal{I})$ is F-rational if and only if $0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R}(\mathcal{I}))} = 0$.

Remark 2.3.2. A Cohen-Macaulay domain of dimension at least two that is regular in codimension one is normal. Suppose that $\mathscr{R}(\mathcal{I})$ is Cohen-Macaulay and that $\operatorname{Proj} \mathscr{R}(\mathcal{I})$ is F-rational. As discussed in [KK21, Lemma 3.5], $\operatorname{Spec} \mathscr{R}(\mathcal{I}) \setminus \{\mathfrak{M}\}$ (where $\mathfrak{M}$ is the homogeneous maximal ideal of $\mathscr{R}(\mathcal{I})$) is F-rational. Therefore $\mathscr{R}(\mathcal{I})$ is normal. □

An element $c \in R^{\circ}$ is called $\mathfrak{a}$*-test element* if for every ideal I of R and for all $z \in R$, $z \in I^{*\mathfrak{a}}$ if and only if $cz^q\mathfrak{a}^q \subseteq I^{[q]}$ for all $q \geq 1$. An R-test element is a *test element*. An element $c \in R^{\circ}$ is called a *parameter test element* if for every R-ideal I generated by a system of parameters and for all $z \in R$, $z \in I^*$ if and only if $cz^q \subseteq I^{[q]}$ for all $q \geq 1$. The *test ideal* $\tau(\mathfrak{a})$ is $\bigcap_{I \subseteq R}(I : I^{*\mathfrak{a}})$. If $(R, \mathfrak{m})$ is a d-dimension normal and Gorenstein local ring, then

$$\tau(\mathfrak{a}) = \operatorname{Ann}_R(0_E^{*\mathfrak{a}}) = \bigcap_{t \geq 1} (x_1^t, \dots, x_d^t) :_R (x_1^t, \dots, x_d^t)^{*\mathfrak{a}}. \tag{2.3.3}$$

where $x_1, \dots, x_d$ is a system of parameters and E is the *injective hull* of $R/\mathfrak{m}$. Recall that $E \simeq \mathrm{H}^d_{\mathfrak{m}}(R) \simeq \varinjlim_t \frac{R}{(x_1^t, \dots, x_d^t)}$.

We summarize some results from [HY03, Proposition 1.11 and Theorem 2.1] that are relevant in this paper.

Theorem 2.3.4. *Assume the above notation. Let $\mathfrak{a}, \mathfrak{b}$ be R-ideals. Then:*

(1) $\mathfrak{a}\tau(\mathfrak{b}) \subseteq \tau(\mathfrak{a}\mathfrak{b})$.
(2) *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\tau(\mathfrak{b}) \subseteq \tau(\mathfrak{a})$. Moreover, if $\mathfrak{a} \cap R^{\circ} \neq \emptyset$ and $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then $\tau(\mathfrak{b}) = \tau(\mathfrak{a})$.*
(3) *If R is a weakly F-regular ring, then $\mathfrak{a} \subseteq \tau(\mathfrak{a})$.*

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3. Proof of Theorem 1.1

Setup 3.1. Let $(R, \mathfrak{m})$ be an excellent, Cohen-Macaulay, normal local domain of dimension $d \geq 2$ and of characteristic $p > 0$. Let $I = (f_1, \cdots, f_d)$ be a parameter ideal, that is $f_1, \cdots, f_d$ is a system of parameters (and hence a regular sequence). Let $f = f_1 \cdots f_d$. Set $I^{[l]} := (f_1^l, \cdots, f_d^l)$ for $l \geq 1$. Denote the Rees algebra $R[It]$ (respectively $\overline{R[It]}$) by $\mathscr{R}$ (respectively $\overline{\mathscr{R}}$). $\mathfrak{M}$ (respectively $\overline{\mathfrak{M}}$) to be the unique homogeneous maximal ideal of $\mathscr{R}$ (respectively $\overline{\mathscr{R}}$). $\overline{G}$ is the associated graded ring $\bigoplus_{n\geq 0} \frac{\overline{I^n}}{\overline{I^{n+1}}}$.

The following proposition is similar to those in [VV78]. We could not find a proof of the statement as we need it below, so we give a proof.

Proposition 3.2. *Assume Setup 3.1 and that $\overline{G}$ is Cohen-Macaulay. Then for all $k \geq 0$ and $l \geq 0$, $\overline{I^{k+l}} \cap I^{[l]} = \overline{I^k} I^{[l]}$.*

Proof. For brevity of notation, we write x_i for the image of f_i in $\overline{G}_1$. Let $a_i, 1 \leq i \leq d$ be elements of R such that $\sum_i^d a_i f_i^l \in \overline{I^{k+l}}$. Let k_1 be such that $(a_1, \dots, a_d) \in \overline{I^{k_1}} \smallsetminus \overline{I^{k_1+1}}$. Let $\bar{a}_i, 1 \leq i \leq d$ be the images of the a_i in $\overline{G}_{k_1}$. (We write $\bar{a}_i$ instead of a_i^* since not all the a_i might be in $\overline{I^{k_1}} \smallsetminus \overline{I^{k_1+1}}$.) We may assume that $k_1 < k$.

Let $(K_\bullet, \partial_\bullet)$ (respectively, $(K'_\bullet, \delta_\bullet)$) be the Koszul complex on the f_i^l on R (respectively, the x_i^l on $\overline{G}$). Note that $x_1, \dots, x_d$ is a regular sequence on $\overline{G}$. Therefore

$$\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_d \end{bmatrix} \in \ker \delta_1 = \operatorname{Im} \delta_2.$$

Hence there exist $b_1, \dots, b_{\binom{d}{2}} \in \overline{I^{k_1-l}}$ such that

$$\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_d \end{bmatrix} = \delta_2\left(\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_{\binom{d}{2}} \end{bmatrix}\right)$$

Define $a'_i, 1 \leq i \leq d$ by

$$\begin{bmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_d \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} - \partial_2\left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{\binom{d}{2}} \end{bmatrix}\right)$$

Note that $a'_i \in \overline{I^{k_1+1}}$ for all i. Moreover,

$$\sum_i a'_i f_i^l = \partial_1\left(\begin{bmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_d \end{bmatrix}\right) = \partial_1\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}\right) = \sum_i a_i f_i^l.$$

Repeating this argument, we can find $\tilde{a}_i \in \overline{I^k}$ such that $\sum_i a_i f_i^l = \sum_i \tilde{a}_i f_i^l$. □

Discussion 3.3. Assume Setup 3.1. Then $I^{[l]} :_R I^n = I^{dl-n-d+1} + I^{[l]}$ for all $l, n \geq 1$; see, e.g., [HY03, Lemma 2.11]. Using this, Hara and Yoshida [HY03, Theorem 2.7] showed that

$$I^{[l]^{*I^n}} = \{z \in R : \text{ there exists a nonzero } c \in R \text{ such that } cz^q \in I^{(dl-n)q} + I^{[lq]} \text{ for all } q \gg 1\}. \tag{3.4}$$

Let $z \in R$, $c \in R \smallsetminus \{0\}$ and q be such that $cz^q \in \overline{I^{(dl-n)q}} + I^{[lq]}$. Let $r = r_I((\overline{I^n})_{n \geq 0})$. Then for all $n' \geq 0$, $\overline{I^{n'+r}} \subseteq I^{n'}$. Let $c_1 \in I^r \smallsetminus \{0\}$. Then

$$c_1 c z^q \in I^r \overline{I^{(dl-n)q}} + I^{[lq]} \subset I^{(dl-n)q} + I^{[lq]}.$$

Therefore,

$$I^{[l]^{*I^n}} = \{z \in R : \text{ there exists a nonzero } c \in R \text{ such that } cz^q \in \overline{I^{(dl-n)q}} + I^{[lq]} \text{ for all } q \gg 1\}, \tag{3.5}$$

where $l, n \geq 1$ such that $dl - n \geq 1$. In particular,

$$\overline{I^{dl-n}} + I^{[l]} \subseteq I^{[l]^{*I^n}} \tag{3.6}$$

for all $l, n \geq 1$ such that $dl - n \geq 1$.

Discussion 3.7. We now make some observations about $\mathrm{H}^d_{\overline{\mathscr{R}}_+}(\overline{\mathscr{R}})$ that are analogous to those in [HWY02, Subsection 1.3, Lemma 2.8, Corollary 2.9] about $\mathrm{H}^d_{\mathscr{R}_+}(\mathscr{R})$. Assume that $\overline{\mathscr{R}}$ is Cohen-Macaulay. Let $n \in \mathbb{Z}$. Then

$$\left[\mathrm{H}^d_{\overline{\mathscr{R}}_+}(\overline{\mathscr{R}})\right]_n = \left\{\left[\frac{a}{f^l} t^n\right] : l \geq 0, dl + n \geq 0, a \in \overline{I^{dl+n}}\right\}.$$

Assume now that $n \geq 1$ and consider the exact sequence

$$0 \longrightarrow \left[\mathrm{H}^d_{\overline{\mathscr{R}}_+}(\overline{\mathscr{R}})\right]_{-n} \xrightarrow{\phi_{-n}} \mathrm{H}^d_{\mathfrak{m}}(R)t^{-n} \longrightarrow \left[\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})\right]_{-n} \longrightarrow 0. \tag{3.8}$$

(The existence of such an exact sequence can be proved in a way similar to [HWY02, Lemma 2.7].) The map ϕ_{-n} in (3.8) is $\left[\frac{a}{f^l} t^{-n}\right] \mapsto \left[\frac{a}{f^l}\right] t^{-n}$. □

Proposition 3.9. *Assume Setup 3.1 and that $\overline{\mathscr{R}}$ is Cohen-Macaulay. Suppose that l, n are positive integers such that $dl - n \geq 1$. Then for all $a \in R$, $\left[\frac{a}{f^l}\right] t^{-n} \in \operatorname{Im}(\phi_{-n})$ if and only if $a \in \overline{I^{dl-n}} + I^{[l]}$.*

Proof. 'If' follows from Discussion 3.7. For 'only if', let $a \in R$ be such that

$$\left[\frac{a}{f^l}\right] t^{-n} \in \operatorname{Im}(\phi_{-n}).$$

Then there exist $m \geq 0$ and $b \in \overline{I^{dm-n}}$ such that $dm - n \geq 0$ and

$$\left[\frac{a}{f^l}\right] t^{-n} = \left[\frac{b}{f^m}\right] t^{-n} = \phi_{-n}\left(\left[\frac{b}{f^m} t^{-n}\right]\right).$$

We see that

$$\frac{a}{f^l} - \frac{b}{f^m}$$

is a boundary in $\check{C}^\bullet(f_1, \ldots, f_d; R)$. Without loss of generality, $m \geq l$. Therefore there exists $N \geq m$ such that

$$\frac{a}{f^l} - \frac{b}{f^m} = \frac{\sum_i a_i f_i^N}{f^N}.$$

Hence $f^{N-m}(af^{m-l} - b) \in I^{[N]}$ so $af^{m-l} \in \overline{I^{dm-n}} + I^{[m]}$. From this we want to conclude that

$$a \in \overline{I^{dl-n}} + I^{[l]}.$$

We may apply induction on $m - l$ and assume that $m = l + 1$.

Let k be such that $a \in \overline{I^k} \smallsetminus \overline{I^{k+1}}$. We may assume that $k < dl - n$. Further applying induction on $dl - n - k$, we may further assume that

$$((\overline{I^{dl+1-n}} + I^{[l+1]}) : f) \cap \overline{I^{k'}} \subseteq \overline{I^{dl-n}} + I^{[l]}. \tag{3.10}$$

for all k' with $k < k' \leq dl - n$.

Write

$$af = b + \sum_{i=1}^{d} a_i f_i^{l+1} \tag{3.11}$$

where $b \in \overline{I^{d(l+1)-n}}$ and $a_i \in R$ for all $1 \le i \le d$. By Proposition 3.2, we may assume that $a_i \in \overline{I^{k+d-l-1}}$ for all $1 \le i \le d$.

Write x_i for the image of f_i in $\overline{G}_1$ and $\mathbf{x} = x_1 \cdots x_d$. Since $\overline{G}$ is Cohen-Macaulay, the x_i form a regular sequence on it. Then we get a map between the Koszul co-complexes $K^\bullet(x_1^l, \dots, x_d^l; \overline{G}) \longrightarrow K^\bullet(x_1^{l+1}, \dots, x_d^{l+1}; \overline{G})$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \overline{G}((d-2)l)^{\binom{d}{2}} & \longrightarrow & \overline{G}((d-1)l)^d & \xrightarrow{\partial^{(l)}} & \overline{G}(dl) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\cdot \mathbf{x}} & & \\
\cdots & \longrightarrow & \overline{G}((d-2)(l+1))^{\binom{d}{2}} & \longrightarrow & \overline{G}((d-1)(l+1))^d & \xrightarrow{\partial^{(l+1)}} & \overline{G}(d(l+1)) & \longrightarrow & 0
\end{array}$$

Note that $\deg a^*\mathbf{x} = k + d < d(l+1) - n = \deg b^*$, so from (3.11) we see that $a^*\mathbf{x} \in \operatorname{Im} \partial^{(l+1)}$, i.e. it is boundary. Hence it gives the zero element in $\mathrm{H}^d_{\overline{G}_+}(\overline{G})$. Since $\overline{G}$ is Cohen-Macaulay, all the maps in the directed system

$$\mathrm{H}^d(K^\bullet(x_1^l, \dots, x_d^l; \overline{G})) \longrightarrow \mathrm{H}^d(K^\bullet(x_1^{l+1}, \dots, x_d^{l+1}; \overline{G}))$$

are injective, so a^* too is boundary, i.e., $a^* \in \operatorname{Im} \partial^{(l)}$. I.e., there exist $\alpha_1, \dots, \alpha_d \in R$ such that

$$a' := a - \sum_i \alpha_i f_i^l \in \overline{I^{k+1}}$$

Note that $a'f - af \in I^{[l+1]}$ so by (3.10), $a' \in \overline{I^{dl-n}} + I^{[l]}$. Hence $a \in \overline{I^{dl-n}} + I^{[l]}$. □

Discussion 3.12. Since R is an excellent normal domain, its singular locus is a proper closed subset of $\operatorname{Spec} R$, so there exists $c \in R^\circ$ such that R_c is regular, and, *a fortiori*, F-rational. Then $(\mathscr{R})_c \simeq \left(\overline{\mathscr{R}}\right)_c \simeq R_c[t]$ is F-rational; use [Vél95, Proposition 1.2]. Hence by [Vél95, Theorem 3.9], we see that there exists N such that c^N is a parameter test element for R, $\mathscr{R}$ and $\overline{\mathscr{R}}$. In particular, I contains parameter test elements for R, $\mathscr{R}$ and $\overline{\mathscr{R}}$.

Let $c \in I$ a parameter test element for R and $\overline{\mathscr{R}}$. Then for all $n \ge 1$, we have, from (3.8), the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \left[\mathrm{H}^d_{\overline{\mathscr{R}}_+}(\overline{\mathscr{R}})\right]_{-n} & \xrightarrow{\phi_{-n}} & \mathrm{H}^d_{\mathfrak{m}}(R)t^{-n} & \longrightarrow & \left[\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})\right]_{-n} & \longrightarrow & 0 \\
 & & \downarrow{cF^e} & & \downarrow{cF^e} & & \downarrow{cF^e} & & \\
0 & \longrightarrow & \left[\mathrm{H}^d_{\overline{\mathscr{R}}_+}(\overline{\mathscr{R}})\right]_{-nq} & \xrightarrow{\phi_{-nq}} & \mathrm{H}^d_{\mathfrak{m}}(R)t^{-nq} & \longrightarrow & \left[\mathrm{H}^{d+1}_{\mathfrak{m}}(\overline{\mathscr{R}})\right]_{-nq} & \longrightarrow & 0.
\end{array} \tag{3.13}$$

□

Lemma 3.14. *Let $(R, \mathfrak{m})$ and I be as in the Setup 3.1. Then for each $n \ge 1$ and $l \ge 1$, $I^{[l+1]} : I^{[l+1]^{*I^n}} \subseteq I^{[l]} : I^{[l]^{*I^n}}$.*

Proof. Let $n \ge 1$ and $l \ge 1$. Note that $I^{[l+1]} : f = I^{[l]}$ since $f_1, f_2, \cdots, f_d$ is a regular sequence. Hence we have the following sequence of inclusions:

$$fI^{[l]^{*I^n}} \subseteq I^{[l+1]^{*I^n}};$$

$$fI^{[l]^{*I^n}}(I^{[l+1]} : I^{[l+1]^{*I^n}}) \subseteq I^{[l+1]^{*I^n}}(I^{[l+1]} : I^{[l+1]^{*I^n}}) \subseteq I^{[l+1]};$$

$$I^{[l]^{*I^n}}(I^{[l+1]} : I^{[l+1]^{*I^n}}) \subseteq I^{[l+1]} : f = I^{[l]}.$$

Therefore $I^{[l+1]} : I^{[l+1]^{*I^n}} \subseteq I^{[l]} : I^{[l]^{*I^n}}$. □

Lemma 3.15. *Let $(R, \mathfrak{m})$ and I as in the Setup 3.1. Suppose that $\overline{\mathscr{R}}$ is Cohen-Macaulay. Fix $n \geq 1$. Consider the following statements:*

(1) $\left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}\right]_{-n} = 0$.

(2) $I^{[l]^{*I^n}} = \overline{I^{dl-n}} + I^{[l]}$ *for all l such that $dl - n \geq 1$.*

(3) $I^{[l]^{*I^n}} = \overline{I^{dl-n}} + I^{[l]}$ *for all $l \gg 0$.*

(4) $\tau(I^n) = I^n : \overline{I^{d-1}}$.

Then (1) $\iff$ (2) $\iff$ (3)*. If, additionally, R is Gorenstein, then* (3) $\iff$ (4).

Proof. (1) $\implies$ (2): Let l be such that $dl-n \geq 1$. In view of (3.6), we need to show that $I^{[l]^{*I^n}} \subseteq \overline{I^{dl-n}}+I^{[l]}$. Let $a \in I^{[l]^{*I^n}}$. Then there exists a non-zero $c \in R$ such that $ca^q \in I^{(dl-n)q} + I^{[lq]} \subseteq \overline{I^{(dl-n)q}} + I^{[lq]}$ for all $q \gg 1$, by (3.4). Hence by Proposition 3.9,

$$cF^e\left(\left[\frac{a}{f^l}\right]t^{-n}\right) = \left[\frac{ca^{p^e}}{f^{lp^e}}\right]t^{-np^e} \in \operatorname{Im}\phi_{-np^e} \tag{3.16}$$

for all $e \gg 0$. Multiplying c by a parameter test element in I for R and $\overline{\mathscr{R}}$ (Discussion 3.12), we may assume that c is a parameter test element. Consider the element $\left[\frac{a}{f^l}\right]t^{-n} \in \mathrm{H}^d_{\mathfrak{m}}(R)t^{-n}$ and its image ξ in $[\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})]_{-n}$. From (3.16) and the commutative diagram (3.13), we see that $cF^e(\xi) = 0$ for all $e \gg 0$. Hence $\xi \in \left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}\right]_{-n}$ and thus $\xi = 0$. So $[\frac{a}{f^l}]t^{-n} \in \operatorname{Im}(\phi_{-n})$. Therefore $a \in \overline{I^{(dl-n)}} + I^{[l]}$.

(2) $\implies$ (3): Immediate.

(3) $\implies$ (1): Let $\xi \in \left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}\right]_{-n}$. Let $a \in R$ and $l \in \mathbb{N}$ be such that ξ is the image of the element

$$\left[\frac{a}{f^l}\right]t^{-n} \in \mathrm{H}^d_{\mathfrak{m}}(R)t^{-n}.$$

Since $\left[\frac{a}{f^l}\right] = \left[\frac{af^{l'}}{f^{l+l'}}\right]$ in $\mathrm{H}^d_{\mathfrak{m}}(R)$ for all $l' \geq 0$, we may assume that l is sufficiently large. Now $cF^e(\xi) = 0$ for all $e \gg 0$. From the commutative diagram (3.13), we see that

$$\left[\frac{ca^q}{f^{lq}}\right]t^{-nq} \in \operatorname{Im}(\phi_{-nq})$$

so by Proposition 3.9 and (3.5), $a \in I^{[l]^{*I^n}}$. However, $I^{[l]^{*I^n}} = \overline{I^{dl-n}} + I^{[l]}$, by hypothesis. Therefore $\left[\frac{a}{f^l}\right] \in \operatorname{Im}(\phi_{-n})$ and so $\xi = 0$.

We now prove (3) $\iff$ (4) assuming that R is Gorenstein. Note that

$$I^{[l]} : (\overline{I^{dl-n}} + I^{[l]}) = I^{[l]} : \overline{I^{dl-n}} = (I^{[l]} : I^{dl-n-d+1}) : \overline{I^{d-1}} = (I^{[l]} + I^n) : \overline{I^{d-1}} = I^n : \overline{I^{d-1}}. \tag{3.17}$$

for all $l \geq n$. (We have used the fact that $\overline{I^{dl-n}} = I^{dl-n-d+1}\overline{I^{d-1}}$, which, in turn, follows from the fact that $r_I(\overline{I}) \leq d-1$, by Proposition 2.2.2.) It immediately follows that if we assume (3), then, using (2.3.3) and Lemma 3.14,

$$\tau(I^n) = \bigcap_{l \geq 1} I^{[l]} : I^{[l]^{*I^n}} = I^n : \overline{I^{d-1}}.$$

Conversely assume (4). In view of (3.6), we need to show that $I^{[l]^{*I^n}} \subseteq \overline{I^{dl-n}} + I^{[l]}$ for all $l \gg 0$. Or, equivalently, using (3.17), that $I^{[l]} : I^{[l]^{*I^n}} \supseteq I^n : \overline{I^{d-1}}$ for all $l \gg 0$. This follows from (2.3.3). ◻

Remark 3.18. Adopt the notation and hypotheses of Theorem 1.1. By [HY03, Theorem 2.1], $\overline{I^{d-1}}\tau(I^n) \subseteq \tau(I^{n+d-1}) \subseteq I^n$. Hence $\tau(I^n) \subseteq \left[\omega_{\overline{\mathscr{R}}}\right]_n$, using Proposition 2.2.3. Thus, together with [HY03, Theorem 5.1], Theorem 1.1 implies that $\overline{\mathscr{R}}$ is F-rational if and only if $\tau(I^n) = \left[\omega_{\overline{\mathscr{R}}}\right]_n$ for all $n \geq 1$.

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Proof of Theorem 1.1. We first show that $\tau(I^n) = I^n : \overline{I^{d-1}}$ for each $n \geq 1$. By hypothesis this holds for all $1 \leq n \leq d-1$. Now assume that $n \geq d$, by induction, that $\tau(I^n) = I^n : \overline{I^{d-1}}$, which, in turn, equals $\left[\omega_{\overline{\mathscr{R}}}\right]_n$. Therefore

$$\tau(I^{n+1}) \subseteq \left[\omega_{\overline{\mathscr{R}}}\right]_{n+1} = I\left[\omega_{\overline{\mathscr{R}}}\right]_n = I\tau(I^n) \subseteq \tau(I^{n+1}),$$

using Remark 3.18, [Hyr01, Proposition 3.2], the induction hypothesis and Theorem 2.3.4(1).

We now see from Lemma 3.15 that

$$\left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}\right]_{-n} = 0$$

for all $n \geq 1$. Since the a-invariant $a(\overline{\mathscr{R}})$ is -1 (Theorem 2.2.1 (1)), it follows that

$$0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})} = 0.$$

Hence $\overline{\mathscr{R}}$ is F-rational. □

Remark 3.19. We now give a minor generalization of Theorem 1.1. Suppose, more generally, that I has a reduction J generated by a system of parameters. Assume the remaining hypothesis on I from Theorem 1.1. Let $1 \leq n \leq d-1$. Then

$$\tau(I^n) = \tau(J^n) \subseteq J^n :_R \overline{J^{d-1}} \subseteq I^n :_R \overline{I^{d-1}} = \tau(I^n).$$

(The first equality and third inclusion hold since J^n is a reduction of I^n; the second is from Remark 3.18; the last equality is by hypothesis.) Therefore $\tau(J^n) = J^n :_R \overline{J^{d-1}}$. Hence $\overline{\mathscr{R}(I)} = \overline{\mathscr{R}(J)}$ is F-rational. □

4. Corollaries

In this section, we prove some corollaries of the results of the previous section. Throughout this section, we will assume that $(R, \mathfrak{m})$ and I are as in Setup 3.1. We start with the proof of Proposition 1.3.

Proof of Proposition 1.3. (1): Assume that $\tau(I^{-a-1}) = R$. By Remark 3.18, we need to show that $\left[\omega_{\overline{\mathscr{R}}}\right]_n \subseteq \tau(I^n)$ for each $n \geq 1$. Since $\overline{G}$ is Gorenstein and $a < 0$, we see that $\overline{\mathscr{R}}$ is Cohen-Macaulay (Theorem 2.2.1 (2)), and that $\left[\omega_{\overline{\mathscr{R}}}\right]_n = \overline{I^{n+a+1}}$ for all $n \geq 1$ (Proposition 2.2.3). Therefore we will show that $\overline{I^{n+a+1}} \subseteq \tau(I^n)$ for all $n \geq 1$. Since

$$\tau(I^{-a-1}) \subseteq \tau(I^{-a-2}) \subseteq \cdots \subseteq \tau(R) \tag{4.1}$$

we may assume that $n \geq -a$. In view of (2.3.3) and Lemma 3.14, we will take $l \gg 0$ and show that $\overline{I^{n+a+1}} \subseteq I^{[l]} : I^{[l]^{*I^n}}$.

Let $z \in I^{[l]^{*I^n}}$ for some $n > -a-1$ and $l \geq 1$ with $dl-n \geq 1$. Then there exists a nonzero $c \in R$ such that $cz^q \in \overline{I^{(dl-n)q}} + I^{[lq]}$ for all $q \gg 1$. Multiplying by $(\overline{I^{n+a+1}})^{[q]}$, we see that $c(z\overline{I^{n+a+1}})^{[q]} \subseteq \overline{I^{(dl+a+1)q}} + I^{[lq]}$ for all $q \gg 1$. This implies that $z\overline{I^{n+a+1}} \subseteq I^{[l]^{*I^{-a-1}}} = \overline{I^{dl+a+1}} + I^{[l]}$, where the last equality is due to Lemma 3.15. (Note that statement (1) of Lemma 3.15 holds with $n = -(a+1)$.) By Proposition 2.2.2 $\overline{I^{dl+a+1}} = I^{dl-d+1}\overline{I^{a+d}}$. Hence $z\overline{I^{n+a+1}} \subseteq I^{dl-d+1} + I^{[l]} = I^{[l]}$. In other words, $\overline{I^{n+a+1}} \subseteq I^{[l]} : I^{[l]^{*I^n}}$. This completes the proof of the first assertion of the proposition.

(2): Assume that $\overline{\mathscr{R}}$ is F-rational and that $a \leq -2$. By (4.1) it suffices to show that $\tau(I^{-a-1}) = R$. Since $\overline{\mathscr{R}}$ is F-rational, we see from Remark 3.18 and Proposition 2.2.3 that $\tau(I^{-a-1}) = \left[\omega_{\overline{\mathscr{R}}}\right]_{-a-1} = \overline{I^{-a-1+a+1}} = R$. (Note that, by hypothesis, $\overline{G}$ is Gorenstein and $-a-1 \geq 1$.) □

Corollary 4.2. *Suppose that R is a three-dimensional Gorenstein F-rational ring and that I is a reduction of $\mathfrak{m}$ generated by a system of parameters. Let $\mathcal{I} = (\overline{\mathfrak{m}^n})_{n\in\mathbb{N}}$. Assume that $r_I(\mathcal{I}) = 2$. Then $\overline{\mathscr{R}}$ is F-rational.*

Proof. By [HKU11, Corollary 4.4] $\overline{G}$ is Gorenstein. Further, $a(\overline{G}) = \dim R - r_I(\mathcal{I}) = -1$. Now use Proposition 1.3. □

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Proof of Proposition 1.4. Write $\mathscr{R}' = R[It, t^{-1}]$ and $\overline{\mathscr{R}'}$ for its normalization. By [KK21, Theorem 1.1], We need to show that $\overline{\mathscr{R}'}$ is F-rational. Write $\mathfrak{M}'$ for the homogeneous maximal ideal of $\overline{\mathscr{R}'}$. From the exact sequence

$$0 \longrightarrow \overline{\mathscr{R}'}(1) \xrightarrow{t^{-1}} \overline{\mathscr{R}'} \longrightarrow \overline{G} \longrightarrow 0$$

we see that for each $n \in \mathbb{Z}$,

$$\left[\operatorname{soc}\left(\mathrm{H}^d_{\overline{G}_+}(\overline{G})\right)\right]_n = \left[\operatorname{soc}\left(\mathrm{H}^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})\right)\right]_{n+1}.$$

(Note that $\sqrt{\mathfrak{M}'\overline{G}} = \sqrt{\overline{G}_+}$ and this equals the homogeneous maximal ideal of $\overline{G}$.) It follows that

$$\operatorname{soc}\left(\mathrm{H}^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})\right) \subseteq \left[\mathrm{H}^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})\right]_0.$$

Arguing as in [KK21, Discussion 3.1] and using the fact that $a(\overline{G}) < 0$, we see that

$$\left[\mathrm{H}^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})\right]_0 = \mathrm{H}^d_{\mathfrak{m}}(R).$$

We need to show that

$$0^*_{H^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})} = 0.$$

By way of contradiction, assume that this does not hold. Then there exists a homogeneous

$$\xi \in 0^*_{H^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})} \bigcap \operatorname{soc}\left(\mathrm{H}^{d+1}_{\mathfrak{M}'}(\overline{\mathscr{R}'})\right), \ \xi \neq 0.$$

By above, $\deg \xi = 0$. Let $c \in R$ be such that $cF^e(\xi) = 0$ for all $q \gg 1$. Then ξ gives a non-zero element of $0^*_{H^d_{\mathfrak{m}}(R)}$, which is a contradiction. □

When R is F-rational and $a(\overline{G}) < -1$, then it is not necessarily true that $\tau(I^{-a-1}) = R$. Indeed, if R and $\overline{G}$ are Gorenstein and $a(\overline{G}) = -2$, then by Proposition 1.3, $\tau(I) = R$ if and only if $\overline{\mathscr{R}}$ is F-rational. For a specific example, see [Sin00, Example 6.3].

We now have the following corollary of Lemma 3.15, relating the F-rationality of Veronese subrings of $\overline{\mathscr{R}}(I)$ to I^n-tight closure. The implication (1) $\implies$ (3) can also be proved by [KK21, Proposition 5.3].

Proposition 4.3. *Assume Setup 3.1 and that $\overline{\mathscr{R}}$ is Cohen-Macaulay. Then the following are equivalent:*

(1) *$\operatorname{Proj} \overline{\mathscr{R}}$ is F-rational.*
(2) *For all $n \gg 0$, $I^{[l]^{*I^n}} = \overline{I^{dl-n}} + I^{[l]}$ for all l such that $dl - n \geq 1$.*
(3) *For all $n \gg 0$, $\overline{\mathscr{R}(I^n)}$ is F-rational.*

Proof. (1) $\implies$ (2): Since $\operatorname{Proj} \overline{\mathscr{R}}$ is F-rational, $0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}$ is of finite length. Hence $\left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}\right]_{-n} = 0$ for all $n \gg 0$. Then by Lemma 3.15, for all $n \gg 0$, $I^{[l]^{*I^n}} = \overline{I^{dl-n}} + I^{[l]}$ for all l such that $dl - n \geq 1$.

(2) $\implies$ (3): Since $\overline{\mathscr{R}}$ is Cohen-Macaulay and normal, so is $\overline{\mathscr{R}(I^n)}$ for all $n \geq 1$. Choose an n_0 such that for all $n \geq n_0$, $I^{[l]^{*I^n}} = \overline{I^{dl-n}} + I^{[l]}$ for all l such that $dl - n \geq 1$. By Lemma 3.15, $\left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\overline{\mathscr{R}})}\right]_{-n} = 0$ for all $n \geq n_0$. Since

$$\left[\mathrm{H}^{d+1}_{\mathfrak{M}_{\overline{\mathscr{R}(I^{n_0})}}}(\overline{\mathscr{R}(I^{n_0})})\right]_{-n} = \left[\mathrm{H}^{d+1}_{\mathfrak{M}_{\overline{\mathscr{R}}}}(\overline{\mathscr{R}})\right]_{-nn_0}$$

for all $n \geq 1$, it follows that

$$\left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}_{\overline{\mathscr{R}(I^{n_0})}}}(\overline{\mathscr{R}(I^{n_0})})}\right]_{-n} = 0$$

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for all $n \geq 1$. Hence

$$0^{*}_{\mathrm{H}^{d+1}_{\mathfrak{M}_{\overline{\mathscr{R}(I^{n_0})}}}(\overline{\mathscr{R}(I^{n_0})})} = 0.$$

Consequently, $\overline{\mathscr{R}(I^{n_0})}$ is F-rational. Therefore $\overline{\mathscr{R}(I^{n})}$ is F-rational, for all $n \gg 0$.

(3) $\implies$ (1): Note that $\operatorname{Proj} \overline{\mathscr{R}(I^{n})} = \operatorname{Proj} \overline{\mathscr{R}}$. □

5. Preliminaries, II

5.1. **Hypersurface rings.** We now collect some facts about quotients rings of power series rings by a non-zero power series.

For an element f in a noetherian local ring $(R, \mathfrak{m})$, the *order* of f, written $\operatorname{ord}(f)$, is the (unique) integer k such that $f \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$. We need the following corollary of the Weierstrass Preparation Theorem; see [LW12, Corollary 9.6].

Theorem 5.1.1. *Let $\Bbbk$ be an infinite field and let f be a non-zero power series in $S = \Bbbk[[X_0, \cdots X_n]]$, $n \geq 1$. Assume that $\operatorname{ord}(f) = e \geq 2$ and that $e \neq 0 \in \Bbbk$. Then, after a change of coordinates, we have $f = u(X_0^e + b_2X_0^{e-2} + b_3X_0^{e-3} + \cdots + b_{e-1}X_0 + b_e)$, where u is a unit of S and $b_2, \cdots, b_e$ are non-units of $\Bbbk[[X_1, \cdots, X_n]]$.*

Let I be an R-ideal. An element $r \in I$ is said to be a *superficial element* of I if there exists $c \in \mathbb{N}$ such that for all $n \geq c$, $(I^{n+1} : r) \cap I^c = I^n$. A sequence of elements $r_1, r_2, \cdots r_n \in I$ is said to be a *superficial sequence* for I if for all $i = 1, 2, \cdots, n$ the image of x_i in $I/(r_1, \cdots, r_{i-1})$ is a superficial element of $I/(r_1, \cdots, r_{i-1})$. If $r \in I \setminus I^2$ is such that $r^* \in G(I)_1$ is a non-zero-divisor on $G(I)$, then r is a superficial element of I.

Lemma 5.1.2. *Let $d \geq 1$. Let $g \in (X_1, X_2, \cdots, X_d)^2 \subseteq \Bbbk[[X_1, X_2, \cdots, X_d]]$ and $f = X_0^2 + g \in S := \Bbbk[[X_0, X_1, \cdots, X_d]]$. Write $(R, \mathfrak{m}) = S/(f)$. Then for each $1 \leq i \leq d$, the image of X_i in R is a superficial element of $\mathfrak{m}$. In particular, the images of $X_1, \ldots, X_d$ is a superficial sequence for $\mathfrak{m}$.*

Proof. Write $x_0, \ldots, x_d$ for the images of $X_0, \ldots, X_d$ in R and $Y_0, \ldots, Y_d$ for the images of $x_0, \ldots, x_d \in \mathfrak{m}/\mathfrak{m}^2$. Write $g(X_1, \ldots, X_d) = g_2(X_1, \ldots, X_d) + g'(X_1, \ldots, X_d)$ with $g'(X_1, \ldots, X_d) \in (X_1, X_2, \cdots, X_d)^3$. Then $G(\mathfrak{m}) \simeq \Bbbk[Y_0, \ldots, Y_d]/(Y_0^2 + g_2(Y_1, \ldots, Y_d))$. Therefore for all $1 \leq i \leq d$, Y_i is a non-zero-divisor on $G(\mathfrak{m})$, so x_i is a superficial element of $\mathfrak{m}$.

The second assertion now follows by induction on d. □

5.2. **Binomial coefficient modulo a prime.** Let m, n be two positive integers and p a prime number. Assume that $m = m_kp^k + m_{k-1}p^{k-1} + \cdots + m_1p + m_0$ and $n = n_kp^k + n_{k-1}p^{k-1} + \cdots + n_1p + n_0$ are the base p expansions of m and n respectively, i.e., m_i, n_i are integers such that $0 \leq m_i, n_i \leq p - 1$ for all $0 \leq i \leq k$. Then by Lucas's theorem [Fin47, Theorem 1],

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p}. \tag{5.2.1}$$

This uses the convention that $\binom{m}{n} = 0$ if $m < n$ and that $\binom{0}{0} = 1$. Thus $\binom{m}{n} \not\equiv 0 \pmod{p}$ if and only if $m_i \geq n_i$ for all i.

Lemma 5.2.2. *Let p be a prime number. The following quantities are not divisible by p:*

(1) $\binom{p^e-1}{r}$ *for each integer $e \geq 1$ and $0 \leq r \leq p^e - 1$.*

(2) $\binom{\frac{p^e+1}{2}}{1}$ *for all integers $e \geq 1$ and $p > 2$.*

(3) $\binom{\frac{p^2+1}{2}}{\beta}$*, where $p > 3$ and*

$$\beta = \begin{cases} \frac{p^2-1}{3} & \text{if } p \equiv 1 \pmod{3} \\ \frac{2p^2-p+3}{6} & \text{if } p \equiv 2 \pmod{3} \end{cases}.$$

Proof. (1): Since $r \leq p^e - 1$, we can write the base-p representation of r as $r = r_{e-1}p^{e-1} + r_{e-2}p^{e-2} + \cdots r_0$. The base-$p$ representation of $p^e - 1$is $p^e - 1 = (p-1)p^{e-1} + (p-1)p^{e-2} + \cdots + (p-1)$. Hence, by (5.2.1),

$$\binom{p^e-1}{r} \equiv \prod_{i=0}^{e-1} \binom{p-1}{r_i} \not\equiv 0 \pmod p.$$

(2) Note that $\frac{p^e+1}{2} = \frac{p^e-1}{2} + 1 = \frac{p-1}{2}p^{e-1} + \frac{p-1}{2}p^{e-2} + \cdots + \frac{p-1}{2} + 1$. So $\frac{p^e+1}{2} \equiv \frac{p+1}{2} \pmod p$. So $\binom{\frac{p^e+1}{2}}{1} \not\equiv 0 \pmod p$.

(3): Note that the base-p representation of $\frac{p^2+1}{2}$ is $\frac{p^2+1}{2} = \frac{p-1}{2}p + \frac{p+1}{2}$. If $p \equiv 1 \pmod 3$, then the base-p representation of $\frac{p^2-1}{3}$ is $\frac{p^2-1}{3} = \frac{p-1}{3}p + \frac{p-1}{3}$. Since $\frac{p+1}{2} > \frac{p-1}{2} > \frac{p-1}{3}$, we see from (5.2.1) that

$$\binom{\frac{p^2+1}{2}}{\frac{p^2-1}{3}} \not\equiv 0 \pmod p.$$

Now suppose that $p \equiv 2 \pmod 3$. The base-p representation of $\frac{2p^2-p+3}{6}$ is $\frac{2p^2-p+3}{6} = \frac{p-2}{3}p + \frac{p+1}{2}$. As $\frac{p-1}{2} > \frac{p-2}{3}$, we see by the above discussion that $\binom{\frac{p^2+1}{2}}{\frac{2p^2-p+3}{6}} \not\equiv 0 \pmod p$. □

6. Proof of Theorem 1.5

We now list some sufficient conditions for two-dimensional hypersurface rings to be F-rational. They are proved using [Gla96, Theorem 2.3].

Lemma 6.1. *Let $S = \Bbbk[[X_0, X_1, X_2]]$ and $R = S/(U_0X_0^2 + U_1X_1^2 + U_2X_2^m)$ where U_0, U_1, U_2 are units in S and $m \geq 2$. Assume that R is F-finite and that $p \geq 3$. Then R is strongly F-regular. In particular it is F-rational.*

Proof. Write $f = U_0X_0^2 + U_1X_1^2 + U_2X_2^m$. Since X_0 is in the jacobian ideal of f, R_{X_0} is regular. We want to show that $X_0f^{p^e-1} \notin (X_0, X_1, X_2)^{[p^e]}$ for some e, in order to apply [Gla96, Theorem 2.3]. Since $(X_0, X_1, X_2)^{[p^e]}$ is a monomial ideal, it suffices to exhibit a term of $X_0f^{p^e-1}$ that does not belong to $(X_0, X_1, X_2)^{[p^e]}$.

Consider the monomial $g := X_0(X_0^2)^{\frac{p^e-3}{2}}(X_1^2)^{\frac{p^e-1}{2}}X_2^m$. Then for all $e > m$, $g \notin (X_0, X_1, X_2)^{[p^e]}$. Note that the coefficient of g in $X_0f^{p^e-1}$ is

$$\binom{p^e-1}{\frac{p^e-3}{2}} \cdot \frac{p^e+1}{2}$$

times a non-zero element of $\Bbbk$. This is non-zero by Lemma 5.2.2. Hence $X_0f^{p^e-1} \notin (X_0, X_1, X_2)^{[p^e]}$. □

Lemma 6.2. *Let $S = \Bbbk[[X_0, X_1, X_2]]$ and $R = S/(U_0X_0^2 + U_1X_1^3 + U_2X_1X_2^m + U_3X_2^n)$. Assume the following:*

1. *R is F-finite, $p \geq 7$ and U_0 and U_1 are invertible elements of S.*
2. *For $i = 2, 3$, if U_i is non-zero, then it is invertible. At least one of $\{U_2, U_3\}$ is non-zero.*
3. *($U_2 \neq 0$ and $2 \leq m \leq 3$) or ($U_3 \neq 0$ and $3 \leq n \leq 5$).*

Then R is strongly F-regular. In particular it is F-rational.

Proof. Write $f = U_0X_0^2 + U_1X_1^3 + U_2X_1X_2^m + U_3X_2^n$. Since X_0 is in the jacobian ideal of f, R_{X_0} is regular. We will show that $X_0f^{p^2-1} \notin (X_0, X_1, X_2)^{[p^2]}$ and apply [Gla96, Theorem 2.3]. As in the proof of the previous lemma, it suffices to exhibit a term of $X_0f^{p^2-1}$ that does not belong to $(X_0, X_1, X_2)^{[p^2]}$. Let

$$\alpha = \begin{cases} \frac{p^2-1}{3} & \text{if } p \equiv 1 \pmod 3 \\ \frac{2p^2-p+3}{6} & \text{if } p \equiv 2 \pmod 3 \end{cases}$$

and $\beta = \frac{p^2+1}{2} - \alpha$. Note that, by Lemma 5.2.2,

$$\binom{p^2-1}{\frac{p^2-3}{2}}\binom{\frac{p^2+1}{2}}{\alpha} \in \Bbbk^\times. \tag{6.3}$$

Case I: $U_2 \neq 0$ and $2 \leq m \leq 3$. Consider the monomial $g := X_0(X_0^2)^{\frac{p^2-3}{2}}(X_1^3)^\beta(X_1X_2^m)^\alpha = X_0^{p^2-2}X_1^{3\beta+\alpha}X_2^{m\alpha}$. Then $g \notin (X_0, X_1, X_2)^{[p^2]}$. The coefficient of g in $X_0 f^{p^2-1}$ is non-zero by (6.3). Hence $X_0 f^{p^2-1} \notin (X_0, X_1, X_2)^{[p^2]}$.

Case II: $U_3 \neq 0$ and $3 \leq n \leq 5$. Similar argument as above, with $g := X_0(X_0^2)^{\frac{p^2-3}{2}}(X_1^3)^\alpha(X_2^n)^\beta = X_0^{p^2-2}X_1^{3\alpha}X_2^{n\beta}$. □

We now prove a minor modification of [HWY02, Corollary 5.7].

Lemma 6.4. *Let $(R, \mathfrak{m})$ and I be as in Setup 3.1. Assume that $\overline{G}$ is Cohen-Macaulay. Let $x \in \overline{I} \smallsetminus \overline{I^2}$ be such that its image in $\overline{G}$ is a non-zero-divisor on $\overline{G}$. Write $S = R/(x)$. If S and the Rees algebra $\overline{S[ISt]}$ are F-rational, then R and $\overline{\mathscr{R}}$ are F-rational.*

Proof. By [KK21, Theorem 1.1], the extended Rees algebra $\overline{S[ISt, t^{-1}]}$ is F-rational. Since $\overline{G}$ is Cohen-Macaulay, $\overline{R[It, t^{-1}]}$ is Cohen-Macaulay, and, therefore,

$$\overline{R[It, t^{-1}]}/(xt) \simeq \overline{S[ISt, t^{-1}]}.$$

(It follows from [Ito92, Theorem 1] that, in general, $\overline{R[It, t^{-1}]}/(xt)$ and $\overline{S[ISt, t^{-1}]}$ agree except in finitely many positive degrees. Since, in our situation, $\operatorname{depth}(\overline{R[It, t^{-1}]}/(xt)) \geq 2$ and $\operatorname{depth}(\overline{S[ISt, t^{-1}]}) \geq 1$, we get the above isomorphism.) Hence $\overline{R[It, t^{-1}]}$ is F-rational, by [HWY02, Corollary 5.7]). Hence R and $\overline{\mathscr{R}}$ are F-rational [KK21, Theorem 1.1]. □

Proposition 6.5. *Let $S = \Bbbk[[X_0, \cdots, X_d]]$ where $d \geq 2$ and $\Bbbk$ is an infinite F-finite field of characteristic $p \geq 7$. Let $f \in S$ be irreducible of order 2; write $R = S/(f)$ and $\mathfrak{m}$ for the maximal ideal of R. Assume that* $\operatorname{Proj}(\mathscr{R}(\mathfrak{m}))$ *is F-rational. Then R and $\mathscr{R}(\mathfrak{m})$ are F-rational.*

Proof. The strategy of the proof is to show (using the F-rationality of $\operatorname{Proj}\mathscr{R}(\mathfrak{m})$) that after a suitable change of variables, f is of the form $\hat{f} + g$ where $\hat{f}$ the power series in Lemmas 6.1 or 6.2 and $g \in (X_3, \dots, X_d)S$. Hence $R/(X_3, \dots, X_d)R \simeq \Bbbk[[X_0, X_1, X_2]]/(\hat{f})$ is a two-dimensional F-rational ring. By [HWY02, Theorem 3.1], the Rees algebra of an integrally closed ideal in a two-dimensional F-rational ring is F-rational. Now apply Lemma 6.4 repeatedly.

We first observe that $\mathscr{R}(\mathfrak{m})$ is Cohen-Macaulay and normal. Since $G(\mathfrak{m})$ is Gorenstein with a-invariant $2-(d+1)$, $\mathscr{R}(\mathfrak{m})$ is Cohen-Macaulay. By Remark 2.3.2, it is normal.

By Theorem 5.1.1, we may assume that $f = X_0^2 + g$, where $g \in \Bbbk[[X_1, \dots, X_d]]$ and $\operatorname{ord}(g) \geq 2$. (We are concerned only about the ideal fS, not the element f, *per se*.) Since R is a domain, $g \neq 0$.

Let $I = (X_1, \cdots, X_d)R$. Then I is a minimal reduction of $\mathfrak{m}$ and $I\mathfrak{m} = \mathfrak{m}^2$. By Lemma 5.1.2, $x_1, \cdots, x_d$ is a sequence of superficial elements.

We now show that $\operatorname{ord}(g) \leq 3$. By way of contradiction, assume that $\operatorname{ord}(g) \geq 4$. Then for all $q \geq 1$, $X_0^{2q} \equiv g^q \mod f$, so in R, $X_0^{2q} \in I^{4q}$. Hence $X_0 \in \overline{I^2} = \overline{\mathfrak{m}^2} = \mathfrak{m}^2$, by [HS06, 6.8.3] and the normality of $\mathscr{R}(\mathfrak{m})$. This a contradiction. Hence $\operatorname{ord}(g) \leq 3$.

Case I: $\operatorname{ord}(g) = 2$. By Theorem 5.1.1, we may assume that $g = U_1(X_1^2 + h)$, where U_1 is a unit in $\Bbbk[[X_1, \cdots, X_d]]$ and $h \in \Bbbk[[X_2, \cdots, X_d]]$ with $\operatorname{ord}(h) \geq 2$.

We now show that $h \neq 0$. By way of contradiction, assume that $h = 0$. Then $f = X_0^2 + U_1X_1^2 \in (X_0, X_1)S$. The ring $R_{(X_0,X_1)}$ is not normal, since it is one-dimensional, but not regular. (f is in the square of the maximal ideal of $S_{(X_0,X_1)}$.) Hence $\operatorname{Spec} R \smallsetminus \{\mathfrak{m}\}$ is not normal. In particular, $\operatorname{Proj}\mathscr{R}(\mathfrak{m})$ is not normal, which contradicts the hypothesis that $\operatorname{Proj}\mathscr{R}(\mathfrak{m})$ is F-rational. Hence $h \neq 0$.

Let $m = \operatorname{ord}(h)$. Again by Theorem 5.1.1, write $h = U_2(X_2^m + h')$, where U_2 is a unit in $\Bbbk[[X_2, \cdots, X_d]]$ and $h' \in (X_3, \cdots, X_d)\Bbbk[[X_2, \cdots, X_d]]$ with $\operatorname{ord}(h') \geq \operatorname{ord}(h)$. Let

$$\overline{R} := \frac{R}{(X_3, \cdots, X_d)} \simeq \frac{\Bbbk[[X_0, X_1, X_2]]}{(X_0^2 + U_1X_1^2 + U_2X_2^m)}$$

for some units $U_1, U_2 \in \Bbbk[[X_0, X_1, X_2]]$. By Lemma 6.1, $\overline{R}$ is F-rational. By [HWY02, Theorem 3.1], the Rees algebra $\overline{R}[X_0t, X_1t, X_2t]$ is F-rational. By Lemmas 5.1.2 and 6.4, $\mathscr{R}(\mathfrak{m})$ is F-rational.

Case II: $\operatorname{ord}(g) = 3$. By Theorem 5.1.1, we may assume that $g = U_1(X_1^3 + X_1h_1 + h_2)$, where U_1 is a unit in $\Bbbk[[X_1, \cdots, X_d]]$ and h_1, h_2 are non-units in $\Bbbk[[X_2, \cdots, X_d]]$ such that $m := \operatorname{ord}(h_1) \geq 2$ and $n := \operatorname{ord}(h_2) \geq 3$. As in the above case, if $h_1 = h_2 = 0$, then f is in the square of the maximal ideal of $S_{(X_0,X_1)}$, so $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$ would not be normal, contradicting the F-rationality of $\operatorname{Proj} \mathscr{R}(\mathfrak{m})$. Hence $h_1 \neq 0$ or $h_2 \neq 0$.

Note that $(X_0)^{q+1}X_1^{lq} = (-g)^{\frac{q+1}{2}}X_1^{lq} = \pm(U_1X_1^3 + h_1X_1 + h_2)^{\frac{q+1}{2}}X_1^{lq}$ and that a general term of the above expression is of the form $X_1^{3\alpha+\beta+lq}h_1^{\beta}h_2^{\gamma}$, with some coefficient, where $\alpha + \beta + \gamma = \frac{q+1}{2}$.

Now suppose that $m \geq 4$ and $n \geq 6$. We claim that

$$X_1^{3\alpha+\beta+lq}h_1^{\beta}h_2^{\gamma} \in I^{(l+2)q} + I^{[(l+1)q]} \tag{6.6}$$

for all $q \gg 1$ and $l \gg 0$. Assume the claim. Then

$$X_0(X_0X_1^l)^q \in I^{(l+2)q} + I^{[(l+1)q]}$$

for all $q \gg 1$ and $l \gg 0$. By (3.5) $X_0X_1^l \in I^{[l+1]*^{I^{(d-1)(l+1)-1}}}$ for all $l \gg 0$. Since $\operatorname{Proj} \mathscr{R}(\mathfrak{m})$ is F-rational,

$$\left[0^*_{\mathrm{H}^{d+1}_{\mathfrak{M}}(\mathscr{R})}\right]_{-((d-1)(l+1)-1)} = 0$$

for all $l \gg 0$, so by Lemma 3.15, we see that $X_0X_1^l \in \mathfrak{m}^{l+2} + I^{[(l+1)]}$ for all $l \gg 0$. Then $x_0(x_1x_2\cdots x_d)^l \in \mathfrak{m}^{dl+2} + I^{[l+1]}$. So $x_0 \in (\mathfrak{m}^{dl+2} + I^{[l+1]}) : (x_1x_2\cdots x_d)^l \subseteq \mathfrak{m}^2 + I \subseteq I$, which is a contradiction. (Use an argument as in the proof of Proposition 3.9.) This is a contradiction. Hence $m \leq 3$ or $n \leq 5$. Using Theorem 5.1.1, we may write $g = U_1(X_1^3 + U_2X_1X_2^m + U_3X_2^n + H_3)$ for some $H_3 \in (X_3, \cdots, X_d)\Bbbk[[X_1, \cdots, X_d]]$ and U_2, U_3 that are either invertible or zero but not zero simultaneously. Let

$$\overline{R} := \frac{R}{(X_3, \cdots, X_d)} \simeq \frac{\Bbbk[[X_0, X_1, X_2]]}{(X_0^2 + U_1X_1^3 + U_1U_2X_1X_2^m + U_1U_3X_2^n)}.$$

By Lemma 6.2, $\overline{R}$ is F-rational. As in the earlier case, use [HWY02, Theorem 3.1] and Lemmas 5.1.2 and 6.4.

It remains to prove the claim (6.6). If $3\alpha+\beta \leq q-1$ and $3\alpha+\beta+m\beta+n\gamma \leq 2q-1$, then $(q+4-(3\alpha+\beta)) + (m-4)\beta + (n-6)\gamma \leq 0$; which gives a contradiction. Hence either $3\alpha+\beta \geq q$ or $3\alpha+\beta+m\beta+n\gamma \geq 2q$. In either case, $X_1^{3\alpha+\beta+lq}h_1^{\beta}h_2^{\gamma} \in I^{(l+2)q} + I^{[(l+1)q]}$. ◻

Proposition 6.7. *Let $(R, \mathfrak{m})$ be a three-dimensional Gorenstein F-finite F-rational complete local domain of characteristic $p \geq 7$ with an infinite residue field. Suppose that $\operatorname{Proj} \mathscr{R}(\mathfrak{m})$ is F-rational and that $\mathscr{R}(\mathfrak{m})$ is Cohen-Macaulay. Then $\mathscr{R}(\mathfrak{m})$ is F-rational.*

Proof. Since $\mathscr{R}(\mathfrak{m})$ is Cohen-Macaulay, the reduction number $r(\mathfrak{m})$ is at most 2.

Case I: $r(\mathfrak{m}) = 0$. Then R is a regular local ring; hence $\mathscr{R}(\mathfrak{m})$ is F-rational.

Case II: $r(\mathfrak{m}) = 1$. Then $\mathscr{R}(\mathfrak{m})$ is Gorenstein [HRZ94, Theorem 4.4]. Consequently R is a hypersurface with multiplicity $e(R) = 2$ [HRZ94, Corollary 4.5]. Now use Proposition 6.5.

Case III: $r(\mathfrak{m}) = 2$. As we observed in Remark 2.3.2 $\mathscr{R}(\mathfrak{m})$ is normal. Now apply Corollary 4.2. ◻

Proof of Theorem 1.5. (1): Follows from Proposition 6.5.

(2): Follows from Proposition 6.7. ◻

It is not difficult to find examples of non-F-pure (even non-F-injective) hypersurfaces $(R, \mathfrak{m})$ such that $\operatorname{Proj} \mathscr{R}(\mathfrak{m})$ is F-rational but $\mathscr{R}(\mathfrak{m})$ is not. E.g., look at generic homogeneous polynomials in d variables with degree at least d. Below, we give an example of an F-pure hypersurface $(R, \mathfrak{m})$ such that $\operatorname{Proj} \mathscr{R}(\mathfrak{m})$ is F-rational but $\mathscr{R}(\mathfrak{m})$ is Cohen-Macaulay, normal but not F-rational or F-injective. It also shows that some hypothesis on the characteristic is necessary in Proposition 6.5.

Example 6.8. Let $\Bbbk$ is an algebraically closed field of characteristic 2, $S = \Bbbk[[X, Y, Z, W]]$, $f = X^2 + XYZW + Y^3 + Z^3 + W^3$ and $R = \frac{S}{(f)}$. Write $\mathfrak{m} = (X, Y, Z, W)R$ and $I = (Y, Z, W)R$. Then we have the following:

(1) The singular locus of R is $\{\mathfrak{m}\}$. Since $f \notin (X^2, Y^2, Z^2, W^2)S$, R is F-pure by [Fed83, Proposition 1.7]. On the other hand, $X^q \in (XYZW)^{\frac{q}{2}} R + I^{[q]}$ for all $q \geq 2$. Apply induction on q to show that
$$XX^q \in XX^{\frac{q}{2}}(YZW)^{\frac{q}{2}} R + I^{[q]} \subseteq (YZW)^{[\frac{q}{2}]} I^{[\frac{q}{2}]} + I^{[q]} \subseteq I^{[q]}$$
for all $q \geq 2$. (The second inclusion is by inductive hypothesis.) Hence $X \in I^* \smallsetminus I$, so R is not F-rational.

(2) The associated graded ring $G := G(\mathfrak{m})$ is Gorenstein; its a-invariant is -2. Hence $\mathscr{R} := \mathscr{R}(\mathfrak{m})$ is Gorenstein, and, in particular, Cohen Macaulay. The reduction number $r_I(\mathfrak{m})$ is 1, i.e., $I\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for each $n \geq 1$.

(3) We now show that $\operatorname{Proj} \mathscr{R}$ is F-rational. For this, it suffices to show that $\mathscr{R}$ is F-rational except at its homogeneous maximal ideal $\mathfrak{M}$. Since $\mathscr{R}$ is Cohen-Macaulay, it would then follow that $\mathscr{R}$ is normal.
   (a) For all $a \in \mathfrak{m}$, $\mathscr{R}_a \simeq R_a[t]$ is regular.
   (b) $\mathscr{R}_{yt} \cong \frac{\Bbbk[X,Y,Z,W]}{(X^2+Y(XYZW+1+Z^3+W^3))}$. Its non-regular locus is defined by $(X, Y, 1+Z^3+W^3)$. Choose $\alpha, \beta \in \Bbbk$ be such that $1 + \alpha^3 + \beta^3 = 0$. It is enough to show localization of $\mathscr{R}_{yt}$ at $(X, Y, Z - \alpha, W - \beta)$ is F-rational.
   Replacing Z by $Z + \alpha$ and W by $W + \beta$, it is enough to show that $A := \frac{\Bbbk[X,Y,Z,W]}{g}$ is F-rational, where
$$\begin{aligned} g &= X^2 + Y(XY(Z+\alpha)(W+\beta) + 1 + (Z+\alpha)^3 + (W+\beta)^3) \\ &= X^2 + Y(\alpha^2 Z + \beta^2 W) + XY^2(Z+\alpha)(W+\beta) + YZ^2(Z+\alpha) + YW^2(W+\beta). \end{aligned}$$
   Let $I = (Y, Z, W)A$; it is a minimal reduction of $(X, Y, Z, W)A$, with $I(X, Y, Z, W)^n A = (X, Y, Z, W)^{n+1} A$ for each $n \geq 1$. Hence it is enough to show $X \notin I^*$. By way of contradiction, assume that $X \in I^*$. Then there exist a nonzero element c such that $cX^q \in I^{[q]}$ for all $q \gg 1$. Note that $cX^q = c(X^2)^{\frac{q}{2}} \equiv cY^{\frac{q}{2}}(\alpha^q Z^{q/2} + \beta^q W^{q/2}) \mod I^{[q]}$ for all $q \gg 1$. Hence $cY^{\frac{q}{2}}(\alpha^q Z^{q/2} + \beta^q W^{q/2}) \in I^{[q]}$. Since Y, Z, W is a regular sequence, we see that $c(\alpha^q Z^{q/2} + \beta^q W^{q/2}) \in (Y^{\frac{q}{2}}, Z^q, W^q)$ for all $q \gg 1$. Since $\alpha^3 + \beta^3 = 1$, assume, without loss of generality, that $\alpha \neq 0$. Use a similar argument to see that $c\alpha^q \in I^{[\frac{q}{2}]}$ for all $q \gg 1$, which is a contradiction. Hence $\mathscr{R}_{yt}$ is F-rational.
   (c) Since $\mathscr{R}_{yt} \cong \mathscr{R}_{zt} \cong \mathscr{R}_{wt}$, all these rings are F-rational. Since $\mathfrak{M} = \sqrt{\mathfrak{m}\mathscr{R} + It\mathscr{R}}$ it follows that $\operatorname{Spec} \mathscr{R} \smallsetminus \mathfrak{M}$ is F-rational.

(4) Since R is not F-rational, $\mathscr{R}$ is not F-rational by [HWY02, Corollary 2.13]. One can also see it using Remark 3.18 and the inclusions
$$\tau(\mathfrak{m}) \subseteq \tau(R) \subsetneq R = [\omega_{\mathscr{R}}]_1.$$
(The first one is a standard property of test ideals, the second holds since R is not F-regular, the final one holds since $\mathscr{R}$ is Cohen-Macaulay and $a(G) = -2$.)

(5) We now observe that $\mathscr{R}$ is not F-injective, and, therefore, not F-pure. By way of contradiction, suppose that $\mathscr{R}$ is F-injective. Since $\operatorname{Proj} \mathscr{R}$ is F-rational and $\mathscr{R}$ is not F-rational, $M := 0^*_{H^{d+1}_{\mathfrak{M}}(\mathscr{R})}$ is a non-zero module of finite length. Let $\xi \in M$ be a non-zero element of minimum degree. Since $\mathscr{R}$ is Cohen-Macaulay, $\deg \xi < 0$. By the F-injectivity of $\mathscr{R}$, we see that $\xi^p \neq 0$. Since M is

closed under the application of the Frobenius map, we get a contradiction of the minimality of $\deg \xi$. Hence $\mathscr{R}$ is not F-injective.

References

[Fed83] R. Fedder. F-purity and rational singularity. *Trans. Amer. Math. Soc.*, 278(2):461–480, 1983. 15

[Fin47] N. J. Fine. Binomial coefficients modulo a prime. *Amer. Math. Monthly*, 54:589–592, 1947. 11

[Gla96] D. Glassbrenner. Strong F-regularity in images of regular rings. *Proc. Amer. Math. Soc.*, 124(2):345–353, 1996. 12

[GN94] S. Goto and K. Nishida. Filtrations and the Gorenstein property of the associated Rees algebras. In *The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations*, volume 110, pages 69–134. American Mathematical Society, Providence, RI, 1994. 3

[HH90] M. Hochster and C. Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990. 4

[HH94] M. Hochster and C. Huneke. F-regularity, test elements, and smooth base change. *Trans. Amer. Math. Soc.*, 346(1):1–62, 1994. 4

[HKU11] W. Heinzer, M.-K. Kim, and B. Ulrich. The Cohen-Macaulay and Gorenstein properties of rings associated to filtrations. *Comm. Algebra*, 39(10):3547–3580, 2011. 3, 9

[HRZ94] M. Herrmann, J. Ribbe, and S. Zarzuela. On the Gorenstein property of Rees and form rings of powers of ideals. *Trans. Amer. Math. Soc.*, 342(2):631–643, 1994. 14

[HS06] C. Huneke and I. Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006. 13

[HWY02] N. Hara, K.-i. Watanabe, and K.-i. Yoshida. F-rationality of Rees algebras. *J. Algebra*, 247(1):153–190, 2002. 1, 6, 13, 14, 15

[HY03] N. Hara and K.-I. Yoshida. A generalization of tight closure and multiplier ideals. *Trans. Amer. Math. Soc.*, 355(8):3143–3174, 2003. 1, 4, 5, 8

[Hyr99] E. Hyry. Blow-up rings and rational singularities. *Manuscripta Math.*, 98(3):377–390, 1999. 1, 2

[Hyr01] E. Hyry. Coefficient ideals and the Cohen-Macaulay property of Rees algebras. *Proc. Amer. Math. Soc.*, 129(5):1299–1308, 2001. 9

[Ito92] S. Itoh. Coefficients of normal Hilbert polynomials. *J. Algebra*, 150(1):101–117, 1992. 13

[KK21] M. Koley and M. Kummini. F-rationality of Rees algebras. *J. Algebra*, 571:151–167, 2021. arXiv:1803.05382 [math.AC]. 1, 3, 4, 10, 13

[LT81] J. Lipman and B. Teissier. Pseudorational local rings and a theorem of Briançon-Skoda about integral closures of ideals. *Michigan Math. J.*, 28(1):97–116, 1981. 2

[LW12] G. J. Leuschke and R. Wiegand. *Cohen-Macaulay representations*, volume 181 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2012. 11

[Sin00] A. K. Singh. Multi-symbolic Rees algebras and strong F-regularity. *Math. Z.*, 235(2):335–344, 2000. 10

[Smi97] K. E. Smith. F-rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997. 4

[Tru87] N. V. Trung. Reduction exponent and degree bound for the defining equations of graded rings. *Proc. Amer. Math. Soc.*, 101(2):229–236, 1987. 3

[Vél95] J. D. Vélez. Openness of the F-rational locus and smooth base change. *J. Algebra*, 172(2):425–453, 1995. 7

[VV78] P. Valabrega and G. Valla. Form rings and regular sequences. *Nagoya Math. J.*, 72:93–101, 1978. 5

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