GORENSTEIN HOMOLOGICAL DIMENSION AND SOME INVARIANTS OF GROUPS

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ABSTRACT. For any group G, the Gorenstein homological dimension $\mathrm{Ghd}_R G$ is defined to be the Gorenstein flat dimension of the coefficient ring R, which is considered as an RG-module with trivial group action. We prove that $\mathrm{Ghd}_R G < \infty$ if and only if the Gorenstein flat dimension of any RG-module is finite, if and only if there exists an R-pure RG-monic $R \to A$ with A being R-flat and $\mathrm{fd}_{RG} A < \infty$, where R is a commutative ring with finite Gorenstein weak global dimension. As applications, properties of Ghd on subgroup, quotient group, extension of groups as well as Weyl group are investigated. Moreover, we compare the relations between some invariants such as $\mathrm{sfli} RG$, $\mathrm{silf} RG$, $\mathrm{spli} RG$, $\mathrm{silp} RG$, and Gorenstein projective, Gorenstein flat and PGF dimensions of RG-modules; a sufficient condition for the Gorenstein projective-flat problem over group rings is given.

1. INTRODUCTION

Let G be any group. Recall that the cohomological dimension $\mathrm{cd}_{\mathbb{Z}} G$ and homological dimension $\mathrm{hd}_{\mathbb{Z}} G$ are defined as the projective and flat dimension of the $\mathbb{Z}G$-module $\mathbb{Z}$ respectively, where G acts trivially. Studying groups through these dimensions arose from both topological and algebraic sources, and has a long history in group theory.

Let R be a commutative ring of coefficients. For any group G, the Gorenstein cohomological dimension $\mathrm{Gcd}_R G$ and the Gorenstein homological dimension $\mathrm{Ghd}_R G$ are defined as the Gorenstein projective and Gorenstein flat dimension of the trivial RG-module R, respectively. The notions of Gorenstein projective, injective and flat modules, which have the origin dating back to the study of G-dimension by Auslander and Bridger [2] in 1960s, now form the basis of a version of relative homological algebra known as Gorenstein homological algebra [18]. The Gorenstein projective (resp. Gorenstein flat) dimension generalizes the projective (resp. flat) dimension, in the sense that whenever the latter is finite, then they coincide.

For any group G, there is an interesting result [3, Theorem 2.7] that $\mathrm{Gcd}_{\mathbb{Z}} G < \infty$ if and only if there is a $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$, where A is a $\mathbb{Z}$-projective ($\mathbb{Z}$-free) $\mathbb{Z}G$-module such that the projective dimension $\mathrm{pd}_{\mathbb{Z}G} A$ of A is finite; in particular, $\mathrm{pd}_{\mathbb{Z}G} A = \mathrm{Gcd}_{\mathbb{Z}} G$. This was generalized in [15, Theorem 1.7] from the coefficient ring of integers $\mathbb{Z}$ to any commutative ring with finite global dimension.

First, we are motivated to characterize the finiteness of $\mathrm{Ghd}_R G$, and its relations with Gorenstein flat dimension of modules over the group ring RG. However, the arguments are not simply analogous to those for Gorenstein cohomological dimension $\mathrm{Gcd}_R G$. There are some evidences for this phenomenon. For example, unlike the well-known fact that every projective module is flat, the relation between

Gorenstein projective and Gorenstein flat modules is not fully understood, and it is still an open problem whether Gorenstein projective modules are Gorenstein flat.

In fact, projectivity is closely connected to the splitness, however, flatness is a bit inseparable from purity. We succeed in getting the following, which extend the aforementioned results in [3, 15] to the case of flatness. Let R be a commutative ring such that the supremum of flat dimension (length) of injective R-modules sfli$R$ is finite. We show in Theorem 3.7 that the following conditions are equivalent:

(1) $\mathrm{Ghd}_R G < \infty$.
(2) There exists an R-pure RG-exact sequence $0 \to R \to A$, where A is an R-flat RG-module of finite flat dimension.
(3) Every RG-module has finite Gorenstein flat dimension.

In this case, we have an equality $\mathrm{Ghd}_R G = \mathrm{fd}_{RG} A$, and the following inequalities

$$\mathrm{sfli}R \leq \mathrm{sfli}RG = \mathrm{G.wgldim}RG \leq \mathrm{Ghd}_R G + \mathrm{sfli}R,$$

where $\mathrm{G.wgldim}RG$ denotes the Gorenstein weak global dimension of the group ring RG. In particular, we show in Corollary 3.6 that if sfli$R < \infty$, then $\mathrm{Gfd}_{RG}M < \infty$ for any RG-module M if and only if $\mathrm{Ghd}_R G = \mathrm{Gfd}_{RG}R < \infty$ for the trivial RG-module R; moreover, $\mathrm{Gfd}_{RG}M$ is bounded by $\mathrm{Ghd}_R G + \mathrm{Gfd}_R M$, i.e. $\mathrm{Gfd}_{RG}M \leq \mathrm{Ghd}_R G + \mathrm{Gfd}_R M$.

It is worth to remark that the assumption sfli$R < \infty$ on the coefficient ring R is not too restrictive. For a commutative ring R, it follows from [13, Theorem 5.3] that sfli$R < \infty$ if and only if every R-module has finite Gorenstein flat dimension. In this case, R is called a ring with finite Gorenstein weak global dimension. Rings with finite weak global dimension are strictly contained in those of finite Gorenstein weak global dimension, including some interesting examples of coefficient rings in studying group rings such as the ring of integers $\mathbb{Z}$, the field of rationals $\mathbb{Q}$ and finite fields.

For further applications, we strengthen the role of Gorenstein homological dimension of groups by proving the following properties. As shown in Section 4, the arguments heavily rely on the above R-pure monomorphism $R \to A$. More precisely, for any commutative ring R with sfli$R < \infty$ and any group G, we show that

(I) $\mathrm{Ghd}_R G \leq \mathrm{Ghd}_{\mathbb{Z}} G$; especially $\mathrm{Ghd}_{\mathbb{Q}} G \leq \mathrm{Ghd}_{\mathbb{Z}} G$ (see Proposition 3.8).
(II) $\mathrm{Ghd}_R H \leq \mathrm{Ghd}_R G$ for any subgroup H of G (see Proposition 4.1).
(III) If $1 \to H \to G \to L \to 1$ is an extension of groups, then $\mathrm{Ghd}_R G \leq \mathrm{Ghd}_R H + \mathrm{Ghd}_R L$ (see Proposition 4.4).
(IV) If R is a noetherian ring, then $\mathrm{Ghd}_R G = \mathrm{Ghd}_R(G/H)$ for any finite normal subgroup H of G (see Proposition 4.5).
(V) If R is a noetherian ring, H is a finite subgroup of G, and $N_G(H)$ its normalizer in G, then $\mathrm{Ghd}_R W \leq \mathrm{Ghd}_R G$ for the Weyl group $W = N_G(H)/H$ (see Corollary 4.6).

In particular, we generalize [1, Proposition 4.11] by (II) immediately: $\mathrm{Ghd}_{\mathbb{Z}} H \leq \mathrm{Ghd}_{\mathbb{Z}} G$, where H was assumed to be a subgroup of G of finite index in [1].

Moreover, we compare Gorenstein homological dimension of groups with the generalized homological dimension of groups introduced by Ikenaga [24, III, Definition]; see details in Proposition 4.7 and 4.8. In particular, we obtain in Corollary 4.10 that the equality $\mathrm{Ghd}_{\mathbb{Z}} G = \underline{\mathrm{hd}}_{\mathbb{Z}} G$ holds for any group G.

In Section 5, we are devoted to study the relation between $\mathrm{Ghd}_R G$ and some invariants such as silpRG, spliRG and sfliRG, which are closely related to the theory of Gorenstein projective, Gorenstein flat and PGF dimensions of modules; see for example [10, 11, 13].

First, we observe that if both $\mathrm{Ghd}_R G$ and $\mathrm{sfli}R$ are finite, then the Gorenstein projective-flat problem admits an affirmative answer over the group ring RG; see Proposition 5.1. If R is a commutative Gorenstein ring with $\mathrm{silf}RG < \infty$, then $\mathrm{Ghd}_R G < \infty$ and the Gorenstein projective-flat problem over RG holds true; moreover, for any Gorenstein flat RG-module M, $\mathrm{PGF\text{-}dim}_{RG} M = \mathrm{Gpd}_{RG} M < \infty$; see Corollary 5.3.

Inspired by [1] and the study of relations between $\mathrm{spli}RG$ and $\mathrm{silp}RG$ ([20, 12]), we compare the relations between $\mathrm{sfli}RG$ and $\mathrm{silf}RG$. In Proposition 5.2 we get that $\mathrm{silf}RG < \infty$ if and only if both $\mathrm{sfli}RG < \infty$ and $\mathrm{splf}RG < \infty$ whenever R is a commutative Gorenstein ring; this extends [1, Corollary 3.5]. It is trivially true that for any finite group G, its integral group ring $\mathbb{Z}G$ is noetherian; it follows from [22] that for any virtually polycyclic group G, $\mathbb{Z}G$ is a noetherian ring. If R is a commutative Gorenstein ring and G is a group such that RG is a left noetherian ring, we show in Proposition 5.6 that $\mathrm{sfli}RG < \infty$ if and only if $\mathrm{silf}RG < \infty$; in this case $\mathrm{sfli}RG = \mathrm{silf}RG$.

2. Preliminaries

In this section, we recall some notions and facts which will be needed in the following.

Gorenstein flat dimension of modules. Let Λ be an associative ring with identity. Recall that $\mathbf{F} = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ is called a *totally acyclic complex of flat modules*, provided it is acyclic with each F_i being a flat left Λ-module, and for any injective right Λ-module I, the complex remains acyclic after applying $I \otimes_\Lambda -$. A left Λ-module M is called *Gorenstein flat* [19], if there exists a totally acyclic complex of flat modules $\mathbf{F}$, such that $M \cong \mathrm{Ker}(F_0 \to F_{-1})$.

Let M be any left Λ-module. The *Gorenstein flat dimension* of M, denoted by $\mathrm{Gfd}_\Lambda M$, is defined by declaring that $\mathrm{Gfd}_\Lambda M \leq n$ if and only if M has a Gorenstein flat resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ of length n, where each P_i is a Gorenstein flat left Λ-module. The *Gorenstein weak global dimension* [5] of Λ, denoted by $\mathrm{G.wgldim}\Lambda$, is defined as the supremum of the Gorenstein flat dimensions of all left Λ-modules.

It was proved in [23, Theorem 3.7] that for a right coherent ring, the class of Gorenstein flat left modules is closed under extensions and direct summands. This result was extended and generalized to any associative ring by [27, Corollary 4.12]. This basic property is so crucial for studying homology of Gorenstein flat modules that we can remove the assumption of coherent rings in many situations when dealing with Gorenstein flat modules and Gorenstein flat dimension of modules, see for example [23, Theorem 3.14].

Modules over group rings. Let G be a group, and R be a commutative ring. A module over the group ring RG is simply an R-module M together with an action of G on M. In particular, R is an RG-module with trivial G-action, that is $gr = r$ for any $g \in G$ and any $r \in R$.

For an RG-module M, the *module of invariants of M*, denoted by M^G, is defined as the largest submodule of M on which G acts trivially, that is, $M^G := \{m \in M | gm = m \text{ for all } g \in G\}$. Analogously, the *module of coinvariants of M*, denoted by M_G, is defined to be the largest quotient of M on which G acts trivially.

Let H be any subgroup of G. There exist simultaneously an induction functor $\mathrm{Ind}_H^G = RG \otimes_{RH} -$ and a coinduction functor $\mathrm{Coind}_H^G = \mathrm{Hom}_{RH}(RG, -)$ from the category of RH-modules $\mathrm{Mod}(RH)$ to the category of RG-modules $\mathrm{Mod}(RG)$. We denote the restriction functor from $\mathrm{Mod}(RG)$ to $\mathrm{Mod}(RH)$ by Res_H^G. In particular, for any RG-module M, $\mathrm{Res}_H^G M$ is exactly the underlying R-module if we

consider the subgroup H formed only by the identity element of G. It is clear that $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ and $(\mathrm{Res}_H^G, \mathrm{Coind}_H^G)$ are adjoint pairs of functors. To simplify the notations, in the following we will denote $\mathrm{Res}_H^G M$ by M if there is no risk of ambiguity.

Invariants silp, spli, silf and sfli. The Gorenstein flat dimension is closely related to some homological invariants. Recall that in connection with the existence of complete cohomological functors in the category of left Λ-modules, Gedrich and Gruenberg have defined in [20] the invariant silpΛ as the supremum of the injective length (dimension) of projective left Λ-modules, and the invariant spliΛ as the supremum of the projective length (dimension) of injective left Λ-modules. Analogously, we use silfΛ to denote the supremum of the injective length (dimension) of flat left Λ-modules, and use sfliΛ to denote the supremum of the flat length (dimension) of injective left Λ-modules.

It follows from [13, Theorem 5.3] and [10, Corollary 2.5] that Gorenstein weak global dimension of rings is symmetric, that is, G.wgldim$\Lambda =$ G.wgldimΛ^{op}. Moreover, G.wgldimΛ is finite if and only if sfli$\Lambda =$ sfliΛ^{op} is finite. Hence, for a commutative ring R, if sfli$R$ is finite, then R is called a *ring with finite Gorenstein weak global dimension*. Note that if sfli$R$ is finite, then any acyclic complex of flat R-modules is totally acyclic.

Gorenstein homological dimension of groups. Recall that for any group G, the *Gorenstein homological dimension* of G is defined to be the Gorenstein flat dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$; see [1, Definition 4.5]. Analogously, we may define the Gorenstein homological dimension of G over any commutative ring R, denoted by $\mathrm{Ghd}_R G$, to be the Gorenstein flat dimension of the trivial RG-module R. It follows from Proposition 3.8 that if the coefficient ring R is a commutative ring with finite Gorenstein weak global dimension, then $\mathrm{Ghd}_R G$ is not more than $\mathrm{Ghd}_{\mathbb{Z}} G$, that is, $\mathrm{Ghd}_R G \leq \mathrm{Ghd}_{\mathbb{Z}} G$.

3. Finiteness of Gorenstein homological dimension of groups

Throughout the paper, R is assumed to be a commutative ring, G is a group. All modules over the group ring RG are left RG-modules unless stated otherwise.

In this section, we intend to characterize the finiteness of $\mathrm{Ghd}_R G$, which is equivalent to that every RG-module has finite Gorenstein flat dimension; see Theorem 3.7. We begin with the following observation.
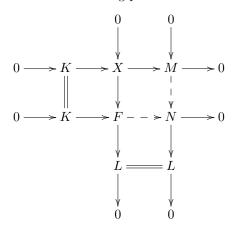
Lemma 3.1. *If* sfli$R < \infty$, *then any Gorenstein flat RG-module is also a Gorenstein flat R-module.*

Proof. Let M be a Gorenstein flat RG-module. There is a totally acyclic complex of flat RG-modules $\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$ such that $M \cong \mathrm{Ker}(F_0 \to F_{-1})$. Since any flat RG-module is also R-flat, by restricting this totally acyclic complex, we get an acyclic complex of flat R-modules. Noting that sfli$R < \infty$, every acyclic complex of flat R-modules is totally acyclic. Hence, M is also Gorenstein flat as an R-module. $\square$

Proposition 3.2. *Assume* sfli$R < \infty$. *Let M be an RG-module with* $\mathrm{Gfd}_{RG} M < \infty$. *If the flat dimension* $\mathrm{fd}_R M$ *of the underlying R-module is finite, then there exists an R-pure RG-exact sequence* $0 \to M \to N \to L \to 0$, *for which L is an R-flat RG-module and* $\mathrm{Gfd}_{RG} M = \mathrm{fd}_{RG} N$.

Proof. Let $\mathrm{Gfd}_{RG} M = n$. Since the class of Gorenstein flat modules is closed under extensions over any associative ring by [27, Corollary 4.12], it follows from [7, Theorems 2, 4] that there exists an exact sequence $0 \to K \to X \to M \to 0$,

where X is a Gorenstein flat RG-module, and $\mathrm{fd}_{RG}K = n - 1$. For X, there is an exact sequence of RG-modules $0 \to X \to F \to L \to 0$, where F is flat and L is Gorenstein flat. We consider the following pushout of $X \to M$ and $X \to F$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \vdots & & \\
0 & \longrightarrow & K & \longrightarrow & F & \dashrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = = = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From the middle row we infer that $\mathrm{fd}_{RG}N = \mathrm{fd}_{RG}K + 1 = n$, which also implies the finiteness of $\mathrm{fd}_R N$. In view of the assumption $\mathrm{fd}_R M < \infty$, we infer from the right column that $\mathrm{fd}_R L < \infty$. For the Gorenstein flat RG-module L, it follows from Lemma 3.1 that L is also Gorenstein flat as an R-module. Furthermore, we imply that L is R-flat since flat dimension of any Gorenstein flat module is either zero or infinity; see [18, Corollary 10.3.4]. Hence, the exact sequence of RG-modules $0 \to M \to N \to L \to 0$ is R-pure exact. This completes the proof. □

Corollary 3.3. *Assume* $\mathrm{sfli}R < \infty$. *If* $\mathrm{Ghd}_R G$ *is finite, then there exists an R-pure RG-exact sequence* $0 \to R \to A$, *where A is an R-flat RG-module such that* $\mathrm{Ghd}_R G = \mathrm{fd}_{RG}A$.

The following interesting result is from [16, Proposition 3.3], and it could be regarded as the converse of Proposition 3.2.

Proposition 3.4. *Assume that* $\mathrm{sfli}R < \infty$ *and there exists an R-pure monomorphism of RG-modules* $\iota : R \to A$, *where A is R-flat with* $\mathrm{fd}_{RG}A < \infty$. *Let M be any left RG-module which is Gorenstein flat as an R-module. Then* $\mathrm{Gfd}_{RG}M \leq \mathrm{fd}_{RG}A$.

Corollary 3.5. *Assume* $\mathrm{sfli}R < \infty$. *If there exists an R-pure monomorphism of RG-modules* $\iota : R \to A$, *where A is R-flat with* $\mathrm{fd}_{RG}A < \infty$, *then for any RG-module M, we have*

$$\mathrm{Gfd}_{RG}M \leq \mathrm{fd}_{RG}A + \mathrm{Gfd}_R M \leq \mathrm{fd}_{RG}A + \mathrm{sfli}R.$$

Proof. Let $\mathrm{fd}_{RG}A = n$. The second inequality is obvious since $\mathrm{G.wgldim}R = \mathrm{sfli}R$; see [13, Theorem 5.3] and [10, Corollary 1.5].

For the first inequality, it suffices to assume that $\mathrm{Gfd}_R M = m$ is finite. In this case, we may prove the result by induction on m. If $m = 0$, that is, M is a Gorenstein flat R-module, then the assertion is immediate from Proposition 3.4.

We now assume $m > 0$, and consider a short exact sequence of RG-modules

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is flat. Since F is restricted to be a flat R-module, we have $\mathrm{Gfd}_R K = m-1$. Invoking the induction hypothesis, we may conclude that $\mathrm{Gfd}_{RG}K \leq n + (m - 1)$, and hence $\mathrm{Gfd}_{RG}M \leq n + m$. This completes the proof. □

The following is immediate from Corollary 3.3 and 3.5.

Corollary 3.6. *Let R be a ring with $\mathrm{sfli}R < \infty$. Then $\mathrm{Ghd}_R G < \infty$ if and only if $\mathrm{Gfd}_{RG}M < \infty$ for any RG-module M. In this case, one has*

$$\mathrm{Gfd}_{RG}M \leq \mathrm{Ghd}_R G + \mathrm{Gfd}_R M.$$

Now, we are in a position to state the main result of this section.

Theorem 3.7. *Let G be a group, R a commutative ring with $\mathrm{sfli}R < \infty$. The following are equivalent:*

 (1) $\mathrm{Ghd}_R G$ *is finite.*
 (2) *There exists an R-pure RG-exact sequence $0 \to R \to A$, where A is an R-flat RG-module of finite flat dimension.*
 (3) *Any RG-module has finite Gorenstein flat dimension.*
 (4) $\mathrm{sfli}RG$ *is finite.*

In this case, there is an equality $\mathrm{Ghd}_R G = \mathrm{fd}_{RG}A$, and moreover, the following inequalities hold:

$$\mathrm{sfli}R \leq \mathrm{sfli}RG = \mathrm{G.wgldim}RG \leq \mathrm{Ghd}_R G + \mathrm{sfli}R.$$

Proof. The implication (1)$\Longrightarrow$(2) follows by Corollary 3.3, and the implication (2)$\Longrightarrow$(3) follows from Corollary 3.5. The implication (3)$\Longrightarrow$(1) is trivial, and (3)$\Longleftrightarrow$(4) follows from [13, Theorem 5.3].

For the first inequality $\mathrm{sfli}R \leq \mathrm{sfli}RG$, it suffices to assume that $\mathrm{sfli}RG = n$ is finite. We consider any injective R-module I. For the subgroup $H = \{1\}$ of G, note that $\mathrm{Coind}_H^G I$ is an injective RG-module. Then $\mathrm{fd}_{RG}\mathrm{Coind}_H^G I \leq n$, which yields $\mathrm{fd}_R \mathrm{Coind}_H^G I \leq n$. Moreover, as R-modules I is a direct summand of $\mathrm{Coind}_H^G I$, and hence $\mathrm{fd}_R I \leq n$. This implies that $\mathrm{sfli}R \leq n$.

By [13, Theorem 5.3] we have $\mathrm{sfli}R = \mathrm{G.wgldim}R$ and $\mathrm{sfli}RG = \mathrm{G.wgldim}RG$. Then, the inequality $\mathrm{G.wgldim}RG \leq \mathrm{Ghd}_R G + \mathrm{G.wgldim}R$ follows immediately from Corollary 3.6. $\qquad\square$

As an application, we can compare $\mathrm{Ghd}_R G$ with $\mathrm{Ghd}_{\mathbb{Z}}G$ introduced in [1].

Proposition 3.8. *If $\mathrm{sfli}R < \infty$, then for any group G, $\mathrm{Ghd}_R G \leq \mathrm{Ghd}_{\mathbb{Z}}G$. In particular, $\mathrm{Ghd}_{\mathbb{Q}}G \leq \mathrm{Ghd}_{\mathbb{Z}}G$.*

Proof. Since the inequality obviously holds if $\mathrm{Ghd}_{\mathbb{Z}}G = \infty$, it only suffices to consider the case where $\mathrm{Ghd}_{\mathbb{Z}}G = n$ is finite. In this case, there exists a $\mathbb{Z}$-pure monomorphism of $\mathbb{Z}G$-modules $\iota : \mathbb{Z} \to \Lambda$, where Λ is $\mathbb{Z}$-torsion-free and $\mathrm{fd}_{\mathbb{Z}G}\Lambda = n$. By applying $R\otimes_{\mathbb{Z}}-$, we have an R-pure monomorphism of RG-modules $R \otimes_{\mathbb{Z}} \iota : R \to R \otimes_{\mathbb{Z}} \Lambda$, where $R \otimes_{\mathbb{Z}} \Lambda$ is R-flat.

Now assume that $0 \to F_n \to \cdots \to F_1 \to F_0 \to \Lambda \to 0$ is a $\mathbb{Z}G$-flat resolution of Λ. Since the modules F_i are all flat over $\mathbb{Z}$, the sequence is pure exact over $\mathbb{Z}$. Then, by applying $R \otimes_{\mathbb{Z}} -$, we have an exact sequence of RG-modules

$$0 \longrightarrow R \otimes_{\mathbb{Z}} F_n \longrightarrow \cdots \longrightarrow R \otimes_{\mathbb{Z}} F_1 \longrightarrow R \otimes_{\mathbb{Z}} F_0 \longrightarrow R \otimes_{\mathbb{Z}} \Lambda \longrightarrow 0.$$

Noting that $R \otimes_{\mathbb{Z}} F_i$ are flat RG-modules, we have $\mathrm{fd}_{RG}R \otimes_{\mathbb{Z}} \Lambda \leq n$. Consequently, we get an R-pure RG-exact sequence $0 \to R \to A$, where $A = R \otimes_{\mathbb{Z}} \Lambda$ is an R-flat RG-module of finite flat dimension. Then, it follows from Proposition 3.4 that $\mathrm{Ghd}_R G \leq \mathrm{fd}_{RG}A \leq n$ holds as expected. $\qquad\square$
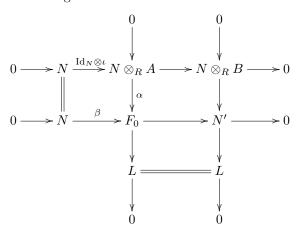
The following might be of independent interest.

Lemma 3.9. *Let $\iota : R \to A$ be an R-pure monomorphism, where A is R-flat. For any RG-module N, N is Gorenstein flat if and only if the induced left RG-module $N \otimes_R A$ is Gorenstein flat.*

Proof. Since A is R-flat, the "only if" part is easy. For the "if" part, we assume that $N \otimes_R A$ is a Gorenstein flat RG-module. Then, there is an exact sequence of RG-modules

$$0 \longrightarrow N \otimes_R A \xrightarrow{\alpha} F_0 \longrightarrow L \longrightarrow 0,$$

where F_0 is flat and L is Gorenstein flat. By applying $N \otimes_R -$ to the R-pure monomorphism $\iota : R \to A$, we get a monomorphism of induced left RG-modules $\mathrm{Id}_N \otimes \iota : N = N \otimes_R R \to N \otimes_R A$. Let $\beta = \alpha(\mathrm{Id}_N \otimes \iota) : N \to F_0$, and consider the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \xrightarrow{\mathrm{Id}_N \otimes \iota} & N \otimes_R A & \longrightarrow & N \otimes_R B & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{\ \beta\ } & F_0 & \longrightarrow & N' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $B = \mathrm{Coker}\,\iota$, and $N' = \mathrm{Coker}\,\beta$.

Let E be any injective right RG-module. It is clear that

$$\mathrm{Id}_E \otimes \alpha : E \otimes_{RG} (N \otimes_R A) \to E \otimes_{RG} F_0$$

is monic. Since $\iota : R \to A$ is an R-pure monomorphism, we infer that

$$\mathrm{Id}_E \otimes (\mathrm{Id}_N \otimes \iota) : E \otimes_{RG} N = (E \otimes_{RG} N) \otimes_R R \to (E \otimes_{RG} N) \otimes_R A \cong E \otimes_{RG} (N \otimes_R A)$$

is monic. Hence, $\mathrm{Id}_E \otimes \beta : E \otimes_{RG} N \to E \otimes_{RG} F_0$ is a monomorphism, and moreover, we infer from the middle row of the above diagram that $\mathrm{Tor}_1^{RG}(E, N') = 0$.

By applying $- \otimes_R A$ to the middle row, we get an exact sequence of induced left RG-modules

$$0 \longrightarrow N \otimes_R A \longrightarrow F_0 \otimes_R A \longrightarrow N' \otimes_R A \longrightarrow 0,$$

where $F_0 \otimes_R A$ is flat and $N \otimes_R A$ is assumed to be Gorenstein flat. Since A is a flat R-module, there is an isomorphism $\mathrm{Tor}_i^{RG}(E, N' \otimes_R A) \cong \mathrm{Tor}_i^{RG}(E, N') \otimes_R A$ for any $i > 0$, and moreover, we get $\mathrm{Tor}_1^{RG}(E, N' \otimes_R A) = 0$ from $\mathrm{Tor}_1^{RG}(E, N') = 0$. Note that by [27, Corollay 4.12], the class of all Gorenstein flat modules is always closed under extensions. Then, we get from [4, Theorem 2.3] that the induced left RG-module $N' \otimes_R A$ is also Gorenstein flat.

Analogous to the above, for N' we obtain an exact sequence of left RG-modules $0 \to N' \to F_{-1} \to N'' \to 0$ with F_{-1} being flat, which remains exact after applying $E \otimes_{RG} -$ for any injective right RG-module E, and for N'' the induced left RG-module $N'' \otimes_R A$ is Gorenstein flat. Proceeding in this manner, we obtain an acyclic complex

$$\mathbf{F}_{\leq 0} = 0 \longrightarrow N \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

with each F_i being a flat left RG-module, which remains acyclic after applying $E \otimes_{RG} -$ for any injective right RG-module E.

Now consider a flat resolution of N:

$$\mathbf{F}_{>0} = \cdots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow N \longrightarrow 0.$$

Since A is R-flat, it is clear that the induced left RG-modules $F_i \otimes_R A$ are flat, and $\mathbf{F}^\bullet_{>0} \otimes_R A \to N \otimes_R A$ is a flat resolution of the induced left RG-module $N \otimes_R A$. Invoking the assumption that $N \otimes_R A$ is a Gorenstein flat left RG-module, we get $\mathrm{Tor}_i^{RG}(E, N \otimes_R A) = 0$ for any injective right RG-module E and any $i \geq 1$, and this implies that the complex $E \otimes_{RG} (\mathbf{F}^\bullet_{>0} \otimes_R A) \to E \otimes_{RG} (N \otimes_R A)$ is acyclic. Thus, we infer from the isomorphisms

$$(A \otimes_R E) \otimes_{RG} \mathbf{F}_{>0} \cong A \otimes_R (E \otimes_{RG} \mathbf{F}_{>0}) \cong (E \otimes_{RG} \mathbf{F}_{>0}) \otimes_R A \cong E \otimes_{RG} (\mathbf{F}_{>0} \otimes_R A)$$

that the complex $(A \otimes_R E) \otimes_{RG} \mathbf{F}_{>0}$ is acyclic.

By applying $- \otimes_R E$ to the R-pure exact sequence $0 \to R \to A \to B \to 0$, we get an exact sequence of induced right RG-modules

$$0 \longrightarrow E = R \otimes_R E \longrightarrow A \otimes_R E \longrightarrow B \otimes_R E \longrightarrow 0.$$

The sequence is split since E is injective, and moreover, E is a direct summand of $A \otimes_R E$. Thus, as a direct summand of $(A \otimes_R E) \otimes_{RG} \mathbf{F}_{>0}$, the complex $E \otimes_{RG} \mathbf{F}_{>0}$ is acyclic.

Consequently, by pasting $\mathbf{F}_{\leq 0}$ and $\mathbf{F}_{>0}$ we will get a totally acyclic complex of flat RG-modules such that $N \cong \mathrm{Ker}(F_0 \to F_{-1})$. This yields that N is a Gorenstein flat RG-module. This completes the proof. $\square$

We can prove that the tensor product of two modules is Gorenstein flat if one factor is a flat RG-module and the other is a Gorenstein flat R-module.

Lemma 3.10. *Let H be any subgroup of G. For any Gorenstein flat RH-module M, the induced RG-module $\mathrm{Ind}_H^G M$ is also Gorenstein flat.*

Proof. Let M be a Gorenstein flat RH-module. There is a totally acyclic complex of flat left RH-modules

$$\mathbf{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots$$

such that $M = \mathrm{Ker}(F_0 \to F_{-1})$. We infer that $\mathrm{Ind}_H^G \mathbf{F} = RG \otimes_{RH} \mathbf{F}$ is an acyclic complex of flat RG-modules such that $\mathrm{Ind}_H^G M = \mathrm{Ker}(\mathrm{Ind}_H^G F_0 \to \mathrm{Ind}_H^G F_{-1})$. For any injective right RG-module E, it is restricted to be an injective RH-module. Then, we infer from the isomorphism

$$E \otimes_{RG} \mathrm{Ind}_H^G \mathbf{F} = E \otimes_{RG} RG \otimes_{RH} \mathbf{F} \cong E \otimes_{RH} \mathbf{F}$$

that $E \otimes_{RG} \mathrm{Ind}_H^G \mathbf{F}$ is acyclic. Hence, $\mathrm{Ind}_H^G \mathbf{F}$ is a totally acyclic complex of flat RG-modules, and this implies that $\mathrm{Ind}_H^G M$ is a Gorenstein flat RG-module. $\square$

Proposition 3.11. *Let M be a Gorenstein flat R-module. For any flat RG-module F, the induced left RG-module $F \otimes_R M$ is also Gorenstein flat.*

Proof. Let $H = \{1\}$ be the subgroup formed by the identity element of G, and M be a Gorenstein flat R-module. Then, the induction functor $\mathrm{Ind}_H^G = RG \otimes_R -$ sends every R-module to be an RG-module. By Lemma 3.10 we infer that the induced RG-module $\mathrm{Ind}_H^G M = RG \otimes_R M$ is Gorenstein flat. Notice that any flat RG-module F is the filtered colimit of free RG-modules. Since the class of Gorenstein flat modules is closed under filtered colimits and direct sums [27, Corollary 4.12], it follows that the induced RG-module $F \otimes_R M$ is Gorenstein flat. $\square$

4. Properties of Gorenstein homological dimension of groups

Using the above characterization on the finiteness of $\mathrm{Ghd}_R G$, we obtain the following properties of the Gorenstein homological dimensions of groups.

Proposition 4.1. *Let H be any subgroup of G. If $\mathrm{sfli} R < \infty$, then $\mathrm{Ghd}_R H \leq \mathrm{Ghd}_R G$.*

Proof. There is nothing to prove if $\mathrm{Ghd}_R G = \infty$ and hence we may assume that $\mathrm{Ghd}_R G = n$ is finite. It follows from Corollary 3.3 that there exists an R-pure RG-exact sequence $0 \to R \to A$ for which A is R-flat such that $\mathrm{fd}_{RG} A = n$. Note that every flat RG-module is restricted to be a flat RH-module, and $0 \to R \to A$ is also an exact sequence of RH-modules. We infer that $\mathrm{fd}_{RH} A \leq \mathrm{fd}_{RG} A$, and moreover, for the trivial RH-module R, by Proposition 3.4 we have $\mathrm{Ghd}_R H = \mathrm{Gfd}_{RH} R \leq \mathrm{fd}_{RH} A \leq n$. This completes the proof. $\square$

The following generalizes [1, Proposition 4.11]: let H be a subgroup of G of finite index, then $\mathrm{Ghd}_{\mathbb{Z}} H \leq \mathrm{Ghd}_{\mathbb{Z}} G$. We remove the assumption of finite index therein.

Corollary 4.2. *Let H be any subgroup of G. Then $\mathrm{Ghd}_{\mathbb{Z}} H \leq \mathrm{Ghd}_{\mathbb{Z}} G$, and $\mathrm{Ghd}_{\mathbb{Q}} H \leq \mathrm{Ghd}_{\mathbb{Q}} G$.*

The following result implies that the Gorenstein flat dimension commutes with direct limits of modules.

Lemma 4.3. *The class of modules with the Gorenstein flat dimension at most m is closed under direct limits, where m is a nonnegative integer.*

Proof. Note that if $f : M_1 \to M_2$ is a homomorphism, where M_i are modules with the Gorenstein flat dimension at most m for $i = 1, 2$, then one has exact sequences

$$0 \longrightarrow K_i^m \longrightarrow P_i^{m-1} \longrightarrow \cdots \longrightarrow P_i^1 \longrightarrow P_i^0 \longrightarrow M_i \longrightarrow 0$$

with P_i^k projective and K_i^m Gorenstein flat for each $k \geq 0$ and $i = 1, 2$. Thus we get the following commutative diagram

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & K_1^m & \longrightarrow & P_1^{m-1} & \longrightarrow & \cdots & \longrightarrow & P_1^1 & \longrightarrow & P_1^0 & \longrightarrow & M_1 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & K_2^m & \longrightarrow & P_2^{m-1} & \longrightarrow & \cdots & \longrightarrow & P_2^1 & \longrightarrow & P_2^0 & \longrightarrow & M_2 & \longrightarrow & 0
\end{array}
$$

with exact rows. Invoking [29, Lemma 3.1], we have the class of Gorenstein flat modules being closed under direct limits. Then the result can be proved similarly as [29, Lemma 3.1], using the convention from [17, Proposition 2.3]. $\square$

Proposition 4.4. *Let $1 \to H \to G \to L \to 1$ be an extension of groups. If $\mathrm{sfli} R < \infty$, then $\mathrm{Ghd}_R G \leq \mathrm{Ghd}_R H + \mathrm{Ghd}_R L$.*

Proof. It suffices to assume that both $\mathrm{Ghd}_R H = m$ and $\mathrm{Ghd}_R L = n$ are finite. Then, it follows from Corollary 3.3 that there exists an R-pure RL-exact sequence $0 \to R \xrightarrow{\alpha} A$ for which A is R-flat such that $\mathrm{fd}_{RL} A = \mathrm{Ghd}_R L = n$.

For the quotient group $L = G/H$, we may consider every RL-module as an RG-module through the natural morphism of group rings $RG \to RL$. We claim that for any flat RL-module F, one has $\mathrm{Gfd}_{RG} F \leq \mathrm{Ghd}_R H$. Note that $RL = R[G/H] = \mathrm{Ind}_H^G R$; see for example [8, Proposition III 5.6]. Since $\mathrm{Ghd}_R H = \mathrm{Gfd}_{RH} R = m$, there is a Gorenstein flat RH-resolution of length m

$$0 \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow R \longrightarrow 0.$$

By applying the induction functor, we get an exact sequence of RG-modules

$$0 \longrightarrow \mathrm{Ind}_H^G M_m \longrightarrow \cdots \longrightarrow \mathrm{Ind}_H^G M_1 \longrightarrow \mathrm{Ind}_H^G M_0 \longrightarrow \mathrm{Ind}_H^G R = RL \longrightarrow 0.$$

By Lemma 3.10 we infer that all $\mathrm{Ind}_H^G M_i$ are Gorenstein flat RG-modules. Then, we get that $\mathrm{Gfd}_{RG} RL \leq m$, and moreover, by [23, Proposition 3.13] we induce that $\mathrm{Gfd}_{RG} P \leq m$ for any free RL-module P. For any flat RL-module F, it follows from the Lazard-Govorov Theorem that $F = \varinjlim P_i$ is a direct limit of

finitely generated free RL-modules P_i. By Lemma 4.3, we prove the claim that $\mathrm{Gfd}_{RG}F = \mathrm{Gfd}_{RG}(\varinjlim P_i) \le m = \mathrm{Ghd}_R H$.

Recall that $\mathrm{fd}_{RL}A = \mathrm{Ghd}_R L = n$. We consider an exact sequence of RL-modules

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

in which all F_i are flat. It follows from the above argument that $\mathrm{Gfd}_{RG}F_i \le m$ for $0 \le i \le n$. By a standard argument, we infer from the above exact sequence that $\mathrm{Gfd}_{RG}A \le m + n$.

Since A is an R-flat module and $\mathrm{Gfd}_{RG}A \le m + n$ is finite, it follows from Proposition 3.2 that there is an R-pure RG-exact sequence $0 \to A \xrightarrow{\beta} B$, where B is an R-flat RG-module such that $\mathrm{fd}_{RG}B = \mathrm{Gfd}_{RG}A \le m + n$. Let $\iota = \beta\alpha$. Then, we obtain an R-pure RG-exact sequence $0 \to R \xrightarrow{\iota} B$, and consequently, it follows from Proposition 3.4 that

$$\mathrm{Ghd}_R G = \mathrm{Gfd}_{RG}R \le \mathrm{fd}_{RG}B \le m + n.$$

This completes the proof. $\square$

Proposition 4.5. *Let H be a finite normal subgroup of G, and R be a commutative noetherian ring such that $\mathrm{sfli}R < \infty$. Then $\mathrm{Ghd}_R G = \mathrm{Ghd}_R(G/H)$.*

Proof. By Proposition 4.4, we have $\mathrm{Ghd}_R G \le \mathrm{Ghd}_R H + \mathrm{Ghd}_R(G/H)$. Since the subgroup H is finite, we infer that $\mathrm{Ghd}_R H = 0$; see for example [1, Proposition 4.12]. This yields $\mathrm{Ghd}_R G \le \mathrm{Ghd}_R(G/H)$.

Conversely, since the inequality $\mathrm{Ghd}_R(G/H) \le \mathrm{Ghd}_R G$ is obvious if $\mathrm{Ghd}_R G = \infty$, it suffices to assume $\mathrm{Ghd}_R G = n$ is finite. In this case, it follows immediately from Corollary 3.3 that there exists an R-pure RG-exact sequence $0 \to R \xrightarrow{\iota} A$ for which A is R-flat such that $\mathrm{fd}_{RG}A = n$. Since the group G acts trivially on R, it implies that $\mathrm{Im}\iota \subseteq A^G \subseteq A^H$. We may therefore consider the $R[G/H]$-module A^H, and the R-pure $R[G/H]$-monomorphism $R \to A^H$.

In the following, we will prove that A^H is R-flat and $\mathrm{fd}_{R[G/H]}A^H \le n$. Then, the desired inequality $\mathrm{Ghd}_R(G/H) = \mathrm{Gfd}_{R[G/H]}R \le n$ will hold by Proposition 3.4.

It follows from $\mathrm{fd}_{RG}A < \infty$ that $\mathrm{fd}_{RH}A < \infty$. Since H is a finite group, analogous to [6, Lemma 2.3] we can prove that for the RH-module A of finite flat dimension whose restriction is R-flat, A itself is a flat RH-module. Then $A = \varinjlim P_i$ for some finitely generated free RH-modules P_i. Since $(RH)^H \cong R$, we infer that P_i^H are finitely generated free R-modules. Note that for any RG-module M, $M^H \cong \mathrm{Hom}_{RH}(R, M)$ as $R[G/H]$-modules; see for example [8, pp.56]. Since H is a finite group, R is a finitely presented RH-module. Hence, we induce from the isomorphisms

$$A^H \cong \mathrm{Hom}_{RH}(R, \varinjlim P_i) \cong \varinjlim \mathrm{Hom}_{RH}(R, P_i) \cong \varinjlim P_i^H$$

that A^H is a flat R-module.

For the finite group H, $R \to RH$ is a Frobenius extension of rings, and then it follows from [26, Theorem 2.2] (see also [9, Theorem 3.2]) that R is a Gorenstein projective RH-module. Hence, $\mathrm{Ext}^i_{RH}(R, P) = 0$ for any projective RH-module P and any $i \ge 1$. Since RH is a noetherian ring (see for example [21, Theorem 7.10]), and R is finitely presented as an RH-module, we infer from [28, Theorem 3.1] that $\mathrm{Ext}^i_{RH}(R, F) = 0$ for any flat RH-module F and any $i \ge 1$. Moreover, for any RH-module D of finite flat dimension and any $i \ge 1$, we obtain by dimension shifting that $\mathrm{Ext}^i_{RH}(R, D) = 0$.

Since $\mathrm{fd}_{RG}A = n$, there is an exact sequence

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

for which F_i are flat RG-modules. Analogous to the above argument, we get that $F_i^H \cong \operatorname{Hom}_{RH}(R, F_i)$ are flat modules over $R[G/H]$. By restriction, F_i are flat RH-modules. By applying $\operatorname{Hom}_{RH}(R, -)$ to the above sequence, we can obtain an exact sequence of $R[G/H]$-modules

$$0 \longrightarrow F_n^H \longrightarrow \cdots \longrightarrow F_1^H \longrightarrow F_0^H \longrightarrow A^H \longrightarrow 0.$$

Hence, $\operatorname{fd}_{R[G/H]} A^H \leq n$, as expected. This completes the proof. $\square$

The Weyl groups of the subgroups of a given group G are useful tools in the study of actions of G on topological spaces, as these Weyl groups act naturally on the fixed point spaces of the actions. Let H be a subgroup of G. We denote by $N_G(H)$ the normalizer of H in G.

Corollary 4.6. *Let H be a finite subgroup of G, and R be a commutative noetherian ring such that $\operatorname{sfli} R < \infty$. Then for the Weyl group $W = N_G(H)/H$, one has $\operatorname{Ghd}_R W \leq \operatorname{Ghd}_R G$.*

Proof. In view of Proposition 4.1, we have $\operatorname{Ghd}_R N_G(H) \leq \operatorname{Ghd}_R G$. It follows from Proposition 4.5 that $\operatorname{Ghd}_R W = \operatorname{Ghd}_R N_G(H)$. Then, we get the inequality $\operatorname{Ghd}_R W \leq \operatorname{Ghd}_R G$. $\square$

For a group G, recall that Ikenaga introduced the *generalized homological dimension* of G over the ring of integers $\mathbb{Z}$; see [24, III, Definition]. Analogously, we may define the generalized homological dimension of G over any coefficient ring R as follows:

$$\underline{\operatorname{hd}}_R G = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i^{RG}(I, M) \neq 0, \ \exists M \ R\text{-flat}, \ \exists \ I \ RG\text{-injective}\}.$$

We conclude this section by comparing the Gorenstein homological dimension to the generalized homological dimension of groups.

Proposition 4.7. *Let R be a commutative ring of coefficients. If $\operatorname{Ghd}_R G < \infty$, then $\operatorname{Ghd}_R G \leq \underline{\operatorname{hd}}_R G$. Moreover, if $\operatorname{sfli} R < \infty$, then $\operatorname{Ghd}_R G = \underline{\operatorname{hd}}_R G$.*

Proof. Assume $\operatorname{Ghd}_R G = \operatorname{Gfd}_{RG} R$ is finite. It follows from [23, Theorem 3.14] that

$$\operatorname{Ghd}_R G = \sup\{i \in \mathbb{N} \mid \operatorname{Tor}_i^{RG}(I, R) \neq 0, \ \exists \ I \ RG\text{-injective}\}.$$

Then, we infer from the definition of $\underline{\operatorname{hd}}_R G$ that $\operatorname{Ghd}_R G \leq \underline{\operatorname{hd}}_R G$.

Moreover, if we assume $\operatorname{sfli} R < \infty$, then every RG-module has finite Gorenstein flat dimension; see Theorem 3.7. Since both $\operatorname{Ghd}_R G$ and $\operatorname{sfli} R$ are assumed to be finite, for any RG-module M, by Corollary 3.6 we have

$$\operatorname{Gfd}_{RG} M \leq \operatorname{Ghd}_R G + \operatorname{Gfd}_R M.$$

Then, $\underline{\operatorname{hd}}_R G = \sup\{\operatorname{Gfd}_{RG} M \mid M \ R\text{-flat}\} \leq \operatorname{Ghd}_R G$. Consequently, we obtain the desired equality $\operatorname{Ghd}_R G = \underline{\operatorname{hd}}_R G$. $\square$

In the following, we denote by $\operatorname{wgldim} R$ the weak global dimension of R.

Proposition 4.8. *For any group G and any commutative ring R, there are inequalities*

$$\underline{\operatorname{hd}}_R G \leq \operatorname{sfli} RG \leq \underline{\operatorname{hd}}_R G + \operatorname{wgldim} R.$$

Proof. If we assume $\operatorname{sfli} RG < \infty$, it follows from [13, Theorem 5.3] that $\operatorname{sfli} RG = \operatorname{G.wgldim} RG$, and then the first inequality is clear.

In order to prove the second inequality, it suffices to assume that both $\underline{\operatorname{hd}}_R G = m$ and $\operatorname{wgldim} R = n$ are finite. Let M be any RG-module, and we consider an exact sequence of RG-modules

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

for which each P_i is projective for $0 \leq i < \max\{n, 1\}$. Since P_i is restricted to be a projective R-module, we get that K is a flat R-module as $\mathrm{wgldim}R = n$. For any injective right RG-module I and any $k > 0$, there are isomorphisms

$$\mathrm{Tor}_k^{RG}(I, M) \cong \mathrm{Tor}_{k-n}^{RG}(I, K).$$

We infer from the assumption $\underline{\mathrm{hd}}_R G = m$ that $\mathrm{Tor}_{k-n}^{RG}(I, K) = 0$ for any $k - n > m$, and then $\mathrm{Tor}_k^{RG}(I, M) = 0$ for any $k > m + n$. Hence, we have $\mathrm{fd}_{(RG)^{op}} I \leq m + n$, and moreover, $\mathrm{sfli}(RG)^{op} \leq m + n$. Note that the group ring RG is isomorphic with its opposite ring RG^{op} and hence there is no need to distinguish between left and right modules. Therefore, $\mathrm{sfli}RG \leq m + n$; see for example [13, 6.2]. This completes the proof. $\qquad\square$

Corollary 4.9. *Let R be a commutative ring with finite weak global dimension. Then $\mathrm{Ghd}_R G = \underline{\mathrm{hd}}_R G$.*

Proof. Under the assumption, if we show that $\underline{\mathrm{hd}}_R G < \infty$ if and only if $\mathrm{Ghd}_R G < \infty$, then the equality $\mathrm{Ghd}_R G = \underline{\mathrm{hd}}_R G$ follows by Proposition 4.7 immediately.

If $\underline{\mathrm{hd}}_R G$ is finite, then we infer from Proposition 4.8 that $\mathrm{sfli}RG \leq \underline{\mathrm{hd}}_R G + \mathrm{wgldim}R < \infty$. By Theorem 3.7, this induces $\mathrm{Ghd}_R G < \infty$. Conversely, if $\mathrm{Ghd}_R G < \infty$, then $\mathrm{sfli}RG$ is finite by Theorem 3.7. Therefore, $\underline{\mathrm{hd}}_R G$ is finite by Proposition 4.8. $\qquad\square$

The following result follows directly from Corollary 4.9.

Corollary 4.10. *The equality $\mathrm{Ghd}_{\mathbb{Z}} G = \underline{\mathrm{hd}}_{\mathbb{Z}} G$ holds for any group G.*

5. Some relative homological dimensions and invariants

Let Λ be any associative ring with identity. Recall that a Λ-module M is called *Gorenstein projective* [18] if there exists an acyclic complex of projective left Λ-modules

$$\mathbf{P} = \cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots$$

which remains acyclic after applying $\mathrm{Hom}_\Lambda(-, Q)$ for any projective module Q, such that $M \cong \mathrm{Ker}(P_0 \to P_{-1})$; M is called a *projectively coresolved Gorenstein flat module*, or a *PGF-module* for short [27], provided that there exists such a complex $\mathbf{P}$ which remains acyclic after applying $I \otimes_\Lambda -$ for any injective right Λ-module I and M is a syzygy of $\mathbf{P}$. It is clear that PGF-modules are Gorenstein flat. As shown in [27, Theorem 4.4], PGF-modules are also Gorenstein projective.

Every projective module is flat, however, it is not at all clear from the definitions that Gorenstein projective modules are Gorenstein flat. It was shown in [23, Proposition 3.4] that every left Gorenstein projective module is Gorenstein flat if the base ring is right coherent and has finite left finitistic dimension. In general, the relation between Gorenstein projective and Gorenstein flat modules remains somehow mysterious, and it is still an open problem if every Gorenstein projective module is Gorenstein flat.

We have the following observation, which gives a sufficient condition for affirmative answer to the Gorenstein projective-flat problem.

Proposition 5.1. *Let G be a group and R be a commutative ring. If $\mathrm{Ghd}_R G < \infty$, then $\mathrm{sfli}R < \infty$ if and only if $\mathrm{sfli}RG < \infty$. Moreover, if both $\mathrm{Ghd}_R G$ and $\mathrm{sfli}R$ are finite, then every Gorenstein projective RG-module is a PGF-module, and furthermore is a Gorenstein flat module.*

Proof. The equivalence between the finiteness of $\mathrm{sfli}R$ and $\mathrm{sfli}RG$ is easy to see. The "if" part follows from the inequality $\mathrm{sfli}R \leq \mathrm{sfli}RG$. The "only if" part follows from Theorem 3.7.

Note that the finiteness of both $\mathrm{Ghd}_R G$ and $\mathrm{sfli} R$ implies $\mathrm{sfli}(RG)^{op} = \mathrm{sfli} RG < \infty$. In this case, by induction on the flat dimension of any injective right RG-module I, we infer that any acyclic complex of projective RG-modules remains acyclic after applying $I \otimes_{RG} -$. Hence, invoking the definitions it is immediate that every Gorenstein projective RG-module is a PGF-module, and moreover, is a Gorenstein flat module; see also [13, Remark 2.3 (ii)]. $\square$

Let Λ be a ring. It follows from [20, 1.6] that the finiteness of both $\mathrm{spli}\Lambda$ and $\mathrm{silp}\Lambda$ implies that $\mathrm{spli}\Lambda = \mathrm{silp}\Lambda$. If Λ is a commutative noetherian ring, by [25, 5.9] one has the equality $\mathrm{spli}\Lambda = \mathrm{silp}\Lambda$. Let R be a *commutative Gorenstein ring* (noetherian ring with finite self-injective dimension), and G be any group. It follows from [12, Theorem 4.4] that $\mathrm{spli} RG = \mathrm{silp} RG$.

Inspired by the above and [1], we will consider $\mathrm{sfli} RG$ and $\mathrm{silf} RG$. The next result extends [1, Corollary 3.5]. Let $\overline{\mathcal{P}}(RG)$ and $\overline{\mathcal{F}}(RG)$ denote the classes of RG-modules with finite projective dimension and finite flat dimension, respectively. The invariant $\mathrm{splf} RG$ is defined as the supremum of projective length (dimension) of flat RG-modules; see [14] and also [10, Proposition 3.2].

Proposition 5.2. *Let R be a commutative Gorenstein ring, G be a group. The following are equivalent:*

(1) $\mathrm{silf} RG < \infty$.
(2) $\mathrm{sfli} RG < \infty$ *and* $\mathrm{splf} RG < \infty$.
(3) $\mathrm{sfli} RG < \infty$ *and* $\overline{\mathcal{P}}(RG) = \overline{\mathcal{F}}(RG)$.

Proof. $(1){\Longrightarrow}(2)$ Since the coefficient ring R is commutative Gorenstein, combining [12, Theorem 4.4] and [14, Proposition 2.1], we have $\mathrm{spli} RG = \mathrm{silp} RG = \mathrm{silf} RG$. Hence, we infer from $\mathrm{silf} RG < \infty$ that $\mathrm{sfli} RG \le \mathrm{spli} RG < \infty$. Moreover, the finiteness of both $\mathrm{silf} RG$ and $\mathrm{spli} RG$ yields that the projective dimension of any flat RG-module is finite, and hence $\mathrm{splf} RG < \infty$.

$(2){\Longrightarrow}(1)$ The finiteness of both $\mathrm{sfli} RG$ and $\mathrm{splf} RG$ implies that the projective dimension of any injective RG-module is finite, that is, $\mathrm{spli} RG < \infty$. Then, it follows from [20, Theorem 2.4] that $\mathrm{silp} RG$ is also finite. Moreover, it follows from [14, Proposition 2.1] that $\mathrm{silf} RG = \mathrm{silp} RG < \infty$.

$(2){\Longleftrightarrow}(3)$ Since projective modules are flat, it is clear that $\overline{\mathcal{P}}(RG) \subseteq \overline{\mathcal{F}}(RG)$. It is easy to see that the projective dimension of any flat RG-module is finite if and only if $\overline{\mathcal{F}}(RG) \subseteq \overline{\mathcal{P}}(RG)$. Hence, the assertion follows. $\square$

For any Λ-module M, the *Gorenstein projective dimension*, denoted by $\mathrm{Gpd}_\Lambda M$, is defined in the standard way via resolutions by Gorenstein projective modules; see [18, 23]. The *Gorenstein global dimension* of Λ [5], denoted by $\mathrm{G.gldim}\Lambda$, is defined as the supremum of the Gorenstein projective dimension of all Λ-modules. Analogously, the *PGF-dimension* $\mathrm{PGF\text{-}dim}_\Lambda M$ of any Λ-module M, and the *PGF-global dimension* $\mathrm{PGF\text{-}gldim}\Lambda$ are defined. The Gorenstein projective dimension is a refinement of the PGF-dimension in the sense that $\mathrm{Gpd}_\Lambda M \le \mathrm{PGF\text{-}dim}_\Lambda M$ for any module M, and $\mathrm{Gpd}_\Lambda M = \mathrm{PGF\text{-}dim}_\Lambda M$ if $\mathrm{PGF\text{-}dim}_\Lambda M < \infty$; see [11, Corollary 3.7].

The above result shows that the comparison between $\mathrm{sfli} RG$ and $\mathrm{silf} RG$ is essentially about a problem on the relation between the projective and flat dimensions of modules. Moreover, we have the following.

Corollary 5.3. *Let R be a commutative Gorenstein ring, G be any group. If $\mathrm{silf} RG$ is finite, then the following hold:*

(1) $\mathrm{Ghd}_R G < \infty$, *and every Gorenstein projective RG-module is Gorenstein flat.*
(2) *For any Gorenstein flat RG-module M, $\mathrm{PGF\text{-}dim}_{RG} M = \mathrm{Gpd}_{RG} M < \infty$.*

Proof. By Proposition 5.2, the finiteness of $\mathrm{silf}\,RG$ implies both $\mathrm{sfli}\,RG < \infty$ and $\mathrm{splf}\,RG < \infty$. Then the assertion (1) follows by Theorem 3.7 and Proposition 5.1, and the assertion (2) is immediate from [11, Proposition 3.9] together with [11, Corollary 3.7 (2)]. $\square$

Corollary 5.4. *Let R be a commutative Gorenstein ring. For any group G, the following are equivalent:*

(1) $\mathrm{silf}\,RG < \infty$.
(2) $\mathrm{Ghd}_R G < \infty$, *every Gorenstein flat RG-module has finite Gorenstein projective dimension and every flat Gorenstein projective RG-module is projective.*

Proof. Note that for any commutative Gorenstein ring R, $\mathrm{spli}R = \mathrm{silp}R < \infty$. Then, $\mathrm{sfli}R \leq \mathrm{spli}R < \infty$, and by Theorem 3.7 we have $\mathrm{Ghd}_R G < \infty$ if and only if $\mathrm{sfli}\,RG < \infty$. It follows from [10, Proposition 3.2] that $\mathrm{splf}\,RG < \infty$ if and only if every Gorenstein flat RG-module has finite Gorenstein projective dimension and every flat Gorenstein projective RG-module is projective. Hence, the assertion follows immediately from Proposition 5.2. $\square$

It follows from [18, Proposition 10.2.3] that any Gorenstein projective module of finite projective dimension is necessarily projective. For modules over group rings, the hypothesis on the finiteness of the projective dimension may be relaxed as shown below.

Corollary 5.5. *Let R be a commutative Gorenstein ring, and G be a group. If $\mathrm{silf}\,RG < \infty$, then any Gorenstein projective RG-module of finite flat dimension is necessarily projective.*

It is trivially true that for any finite group G, the integral group ring $\mathbb{Z}G$ is noetherian, while this question is quite subtle for infinite G. It follows from [22, Theorem 1.1] that for any virtually polycyclic group G, the group ring $\mathbb{Z}G$ is noetherian. Recall that a group G is polycyclic if there exists a subnormal sequence

$$G = G_1 \rhd G_2 \rhd \cdots \rhd G_n = 1$$

such that G_k/G_{k+1} is cyclic for all $1 \leq k < n$. The minimal length of such a subnormal sequence is the Hirsch length of G. A group is virtually polycyclic, also named a polycyclic-by-finite group, if it has a polycyclic subgroup of finite index.

Proposition 5.6. *Let R be a commutative Gorenstein ring. Let G be a group such that RG is a left noetherian ring. Then $\mathrm{sfli}\,RG < \infty$ if and only if $\mathrm{silf}\,RG < \infty$. In this case, one has $\mathrm{sfli}\,RG = \mathrm{silf}\,RG$.*

Proof. The "if" part follows from Proposition 5.2, that is, $\mathrm{silf}\,RG < \infty$ implies $\mathrm{sfli}\,RG < \infty$. By [12, Proposition 4.2] we have an inequality $\mathrm{sfli}\,RG \leq \mathrm{silp}\,RG$. Moreover, it follows from [14, Proposition 2.1] that $\mathrm{silf}\,RG = \mathrm{silp}\,RG$; see also [1, Theorem 3.3]. Hence, we have $\mathrm{sfli}\,RG \leq \mathrm{silf}\,RG$. Here, it is not necessary to assume that the group ring RG is noetherian.

For the "only if" part, we assume RG is a left noetherian ring, and $\mathrm{sfli}\,RG = n < \infty$. Let F be any flat left RG-module. Consider the exact sequence of left RG-modules

$$0 \longrightarrow F \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow L \longrightarrow 0,$$

for which I_i $(0 \leq i \leq n-1)$ are injective modules. Denote by $(-)^+$ the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. Then we get an exact sequence of right RG-modules

$$0 \longrightarrow L^+ \longrightarrow I_{n-1}^+ \longrightarrow \cdots \longrightarrow I_1^+ \longrightarrow I_0^+ \longrightarrow F^+ \longrightarrow 0.$$

It is clear that F^+ is an injective RG-module. Since RG is left noetherian, it follows that I_i^+ are flat right RG-modules for $0 \leq i \leq n-1$; see for example [18,

Corollary 3.2.17]. Then, the assumption $\text{sfli}RG = n$ implies that $\text{sfli}(RG)^{op} = n$, and furthermore L^+ is a flat right RG-module. Hence, L is an injective left RG-module by [18, Corollary 3.2.17]. Consequently, we get the desired inequality $\text{silf}RG \leq n$. This completes the proof. □

It is well known that $\text{G.gldim}\Lambda < \infty$ if and only if $\text{spli}\Lambda = \text{silp}\Lambda < \infty$; in this case, one has $\text{G.gldim}\Lambda = \text{spli}\Lambda = \text{silp}\Lambda$; see for example [13, Theorem 4.1]. For PGF-global dimension, it follows from [11, Theorem 5.1] that $\text{PGF-gldim}\Lambda < \infty$ if and only if $\text{spli}\Lambda = \text{silp}\Lambda < \infty$ and $\text{sfli}\Lambda = \text{sfli}\Lambda^{op} < \infty$; in this case one has

$$\text{PGF-gldim}\Lambda = \text{spli}\Lambda = \text{silp}\Lambda = \text{G.gldim}\Lambda.$$

Moreover, we have the following.

Proposition 5.7. *Let R be a commutative ring, and G be a group. Then $\text{spli}RG < \infty$ if and only if $\text{PGF-gldim}RG = \text{G.gldim}RG < \infty$; in this case, $\text{PGF-gldim}RG = \text{spli}RG$.*

Proof. It suffices to prove the "only if" part. It follows from [11, Corollary 5.4] and [20, 1.6] that $\text{silp}RG \leq \text{spli}RG$ with the equality if $\text{spli}RG < \infty$. Invoking the finiteness of $\text{spli}RG$, we infer that $\text{sfli}(RG)^{op} = \text{sfli}RG < \infty$. Then the assertion and the equality $\text{PGF-gldim}RG = \text{spli}RG$ follow immediately from [11, Theorem 5.1]. □

It is well known that the flat dimension of any module is less than or equal to its projective dimension, while it is not clear if the inequality $\text{Gfd}M \leq \text{Gpd}M$ holds "locally" for any module M. However, for group rings the inequality follows "globally" as shown below.

Proposition 5.8. *Let R be a commutative ring and G a group. Then there is an inequality*

$$\text{G.wgldim}RG \leq \text{PGF-gldim}RG = \text{G.gldim}RG \leq \text{G.wgldim}RG + \text{splf}RG.$$

Proof. The inequality $\text{G.wgldim}RG \leq \text{G.gldim}RG$ follows from [5, Corollary 1.2 (1)]. For completeness, we briefly include an argument in [13, Remark 5.4 (ii)], which is different from that of [5]. The inequality is obvious if $\text{G.gldim}RG = \infty$ and hence we may assume that $\text{G.gldim}RG = n < \infty$. Then it follows from [13, Theorem 4.1] that $\text{silp}RG = \text{spli}RG = n$. By the definition, we have $\text{sfli}RG \leq \text{spli}RG$. Hence, it follows from [13, Theorem 5.3] that $\text{G.wgldim}RG = \text{sfli}RG \leq n$.

The inequality $\text{G.gldim}RG \leq \text{G.wgldim}RG + \text{splf}RG$ is immediate from [10, Theorem 3.3]. It remains to prove the equality $\text{PGF-gldim}RG = \text{G.gldim}RG$. It follows from [27, Theorem 4.4] that every PGF-module is Gorenstein projective, and then the inequality $\text{G.gldim}RG \leq \text{PGF-gldim}RG$ is clear. Conversely, in proving the inequality $\text{G.gldim}RG \geq \text{PGF-gldim}RG$, it suffices to assume that $\text{G.gldim}RG = n$ is finite. Then $\text{sfli}RG \leq \text{spli}RG = \text{silp}RG = n$. Hence, we infer from [11, Theorem 5.1] that $\text{PGF-gldim}RG < \infty$, and moreover $\text{G.gldim}RG = \text{PGF-gldim}RG$. This completes the proof. □

References

[1] J. Asadollahi, A. Bahlekeh, A. Hajizamani and S. Salarian, On certain homological invariants of groups, *J. Algebra* **335** (2011) 18-35.

[2] M. Auslander and M. Bridger, *Stable module category*, Mem. Amer. Math. Soc. 94., 1969.

[3] A. Bahlekeh, F. Dembegioti and O. Talelli, Gorenstein dimension and proper actions, *Bull. Lond. Math. Soc.* **41** (2009) 859-871.

[4] D. Bennis, Rings over which the class of Gorenstein flat modules is closed under extensions, *Comm. Algebra* **37**(3) (2009) 855-868.

[5] D. Bennis and N. Mahdou, Global Gorenstein dimensions, *Proc. Amer. Math. Soc.* **138** (2010) 461-465.

[6] D.J. Benson and J.P.C. Greenlees, Commutative algebra for cohomology rings of virtual duality groups, *J. Algebra* **192** (1997) 678-700.

[7] S. Bouchiba, On Gorenstein flat dimension, *J. Algebra Appl.* **14** (2015) 1550096.

[8] K.S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer, Berlin-Heidelberg-New York, 1982.

[9] X.W. Chen and W. Ren, Frobenius functors and Gorenstein homological properties, *J. Algebra* **610** (2022) 18-37.

[10] L.W. Christensen, S. Estrada and P. Thompson, Gorenstein weak global dimension is symmetric, *Math. Nachr.* **294** (2021) 2121-2128.

[11] G. Dalezios and I. Emmanouil, Homological dimension based on a class of Gorenstein flat modules, *C. R. Math. Acad. Sci. Paris* to appear; arXiv:2208.05692.

[12] I. Emmanouil, On certain cohomological invariants of groups, *Adv. Math.* **225** (2010) 3446-3462.

[13] I. Emmanouil, On the finiteness of Gorenstein homological dimensions, *J. Algebra* **372** (2012) 376-396.

[14] I. Emmanouil and O. Talelli, On the flat length of injective modules, *J. Lond. Math. Soc.* **84**(2) (2011) 408-432.

[15] I. Emmanouil and O. Talelli, Gorenstein dimension and group cohomology with group ring coefficients, *J. Lond. Math. Soc.* **97**(2) (2018) 306-324.

[16] I. Emmanouil and O. Talelli, Characteristic modules and Gorenstein (co-)homological dimension of groups, preprint.

[17] E.E. Enochs and J.A. López-Ramos, Kaplansky classes. *Rend. Semin. Mat. Univ. Padova* **107** (2002) 67-79.

[18] E.E. Enochs and O.M.G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Mathematics no. 30, New York: Walter De Gruyter, 2000.

[19] E.E. Enochs, O.M.G. Jenda and B. Torrecillas, Gorenstein flat modules, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **10** (1993) 1-9.

[20] T.V. Gedrich and K.W. Gruenberg, Complete cohomological functors on groups, *Topology Appl.* **25** (1987) 203-223.

[21] R. Gilmer, *Commutative Semigroup Rings*, The University of Chicago Press, Ltd., London. 1984.

[22] P. Hall, Finiteness conditions for soluble groups, *Proc. Lond. Math. Soc.* **s3-4**(1) (1954) 419-436.

[23] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra* **189** (2004) 167-193.

[24] B.M. Ikenaga, Homological dimension and Farrell cohomology, *J. Algebra* **87** (1984) 422-457.

[25] C.U. Jensen, *Les foncteurs dérivés de $\varprojlim$ et leurs applications en théorie des modules*, Lecture Notes in Math., vol. 254, Springer, Berlin, 1972.

[26] W. Ren, Gorenstein projective and injective dimensions over Frobenius extensions, *Comm. Algebra* **46**(12) (2018) 5348-5354.

[27] J. Šaroch and J. Šťovíček, Singular compactness and definability for Σ-cotorsion and Gorenstein modules, *Selecta Math. (N.S.)* **26**(23) (2020) (40 pages).

[28] G. Yang and X.S. Da, On Ding projective complexes. *Acta Math. Sin. (Engl. Ser.)* **34**(11) (2018) 1718-1730.

[29] G. Yang and Z.K. Liu, Gorenstein flat covers over GF-closed rings, *Comm. Algebra* **40**(5) (2012) 1632-1640.

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