NUMERICAL SEMIGROUPS WITH FIXED MULTIPLICITY AND CONCENTRATION

J. C. ROSALES, M. B. BRANCO, AND M. A. TRAESEL

Abstract. We define the concentration of a numerical semigroup S as $\mathsf{C}(S) = \max\{\mathrm{next}_S(s) - s \mid s \in S\backslash\{0\}\}$ wherein $\mathrm{next}_S(s) = \min\{x \in S \mid s < x\}$. In this paper, we study the class of numerical semigroups with multiplicity m and concentration less than or equal to k, denoted by $\mathsf{C}_k[m]$. We give algorithms to calculate the whole set $\mathsf{C}_k[m]$ with given genus or Frobenius number. In addition, we prove that if $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S verifies the Wilf's conjecture.

1. Introduction

We start by recalling some terminology related to numerical semigroups and setting some notations to be used along the paper. Let $\mathbb{Z}$ be the set of integers and let $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\}$ be the set of nonnegative integers. A submonoid of $(\mathbb{N}, +)$ is a subset of $\mathbb{N}$ closed under addition and containing 0. A numerical semigroup is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N}\backslash S = \{n \in \mathbb{N} \mid n \notin S\}$ is finite.

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S\backslash\{0\})$, $\mathrm{F}(S) = \max(\mathbb{Z}\backslash S)$ and $\mathrm{g}(S)$ the cardinality of $\mathbb{N}\backslash S$ are three important invariants of S known as **multiplicity**, **Frobenius number** and **genus** of S, respectively.

If S is a numerical semigroup and $s \in S\backslash\{0\}$, we denote by $\mathrm{next}_S(s) = \min\{x \in S \mid s < x\}$ and by $\mathrm{prev}_S(s) = \max\{x \in S \mid x < s\}$. We define the concentration of a numerical semigroup S as $\mathsf{C}(S) = \max\{\mathrm{next}_S(s) - s \mid s \in S\backslash\{0\}\}$.

For m a positive integer, the semigroup $\triangle(m) = \{0, m, \rightarrow\}$ is called half-line or ordinary which has concentration 1. Moreover, every numerical semigroup with concentration 1 is of this form. The family of numerical semigroups with concentration 2 has been studied in [16].

Given m and k positive integers, we denote by

$$\mathscr{L}(m) = \{S \mid S \text{ is a numerical semigroup with } \mathrm{m}(S) = m\}$$

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and by, $\mathsf{C}_k[m] = \{S \mid S \in \mathscr{L}(m) \text{ with } \mathsf{C}(S) \leq k\}$.

Observe that if $k \geq m$, then $\mathsf{C}_k[m] = \mathscr{L}(m)$. The purpose of the present paper is to study the set of numerical semigroups $\mathsf{C}_k[m]$ when $m \geq 3$ and $k \in \{2, \dots, m-1\}$.

If $\mathcal{X}$ is a nonempty subset of $\mathbb{N}$, we denote by $\langle \mathcal{X} \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by $\mathcal{X}$, that is,

$$\langle \mathcal{X} \rangle = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N} \setminus \{0\},\ x_1, \dots, x_n \in \mathcal{X}, \text{and } \lambda_1, \dots, \lambda_n \in \mathbb{N} \right\},$$

which is a numerical semigroup if and only if $\gcd(\mathcal{X}) = 1$ (see [19]).

If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle \mathcal{X} \rangle$ then we say that $\mathcal{X}$ is a system of generators of M. Moreover, if $M \neq \langle \mathcal{Y} \rangle$ for all $\mathcal{Y} \subsetneq \mathcal{X}$, then we say that $\mathcal{X}$ is a minimal system of generators of S. In [19, Corollary 2.8] it is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which is finite. We denote by $\operatorname{msg}(M)$ the minimal system of generators of M, its cardinality is called the embedding dimension of M and is denoted by $\mathrm{e}(M)$.

This paper is organized as follows. In Section 2, we will show that if S is a numerical semigroup, then $\mathsf{C}(S) = \max\{\operatorname{next}_S(x) - x \mid x \in \operatorname{msg}(S)\}$. Also, we will see that if $S \in \mathsf{C}_k[m]$ and $S \neq \triangle(m)$ then $S \cup \{\mathrm{F}(S)\} \in \mathsf{C}_k[m]$. This will allow us to order the elements of $\mathsf{C}_k[m]$ making a tree with root $\triangle(m)$. We will characterize the sons of an arbitrary vertex of this tree and this will give us an algorithmic procedure to compute the elements of $\mathsf{C}_k[m]$ with a given genus. Besides, we will prove that $\mathsf{C}_k[m]$ is an infinite set if and only if there exists $d \in \{2, \dots, k\}$ wherein d divides m.

Given $S \in \mathscr{L}(m)$, denote by $\theta(S) = S \cap \{m+1, \dots, 2m-1\}$. An (k, m)-set is a set A fulfilling that $A = \theta(S)$, for some $S \in \mathsf{C}_k[m]$. We will start the Section 3, by proving that the set $\mathsf{C}_k[m]$ is equal to $\Big\{S \in \mathscr{L}(m) \mid A \subseteq S \text{ for some } (k,m)-\text{set } A\Big\}$. From this, we will show that $\mathsf{C}_k[m]$ is the union of finitely many Frobenius pseudo-varieties (see [14]).

Following the notation introduced in [17], we say that S is an elementary numerical semigroup if $\mathrm{F}(S) < 2m$. Denote by $\mathcal{E}(m) = \{S \in \mathscr{L}(m) \mid S \text{ is elementary}\}$ and by $\mathcal{E}\big(\mathsf{C}_k[m]\big) = \{S \in \mathsf{C}_k[m] \mid S \text{ is elementary}\}$. In Section 3, we will prove that the set $\mathcal{E}\big(\mathsf{C}_k[m]\big)$ is equal to $\Big\{\{0\} \cup \big\{m, m+x_1, m+x_1+x_2, \dots, m+x_1+x_2+\cdots+x_p\big\} \cup \{2m, \rightarrow\} \mid \big(x_1, x_2, \dots, x_p, x_{p+1}\big) \in \{1, \dots, k\}^{p+1} \text{ and } x_1 + x_2 + \cdots + x_{p+1} = m\Big\}$. As a consequence of this result, we will be able to give an algorithm to compute the whole set $\mathcal{E}\big(\mathsf{C}_k[m]\big)$ with given genus and Frobenius number.

Let $\mathsf{C}_k[m, F] = \{S \in \mathsf{C}_k[m] \mid \mathrm{F}(S) = F\}$. In Section 4, we will give an algorithm to compute the whole set $\mathsf{C}_k[m, F]$ (note that the case $\mathrm{F}(S) < 2m$ has been studied in the Section 3). Using the terminology introduced in [18] a numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Denote by $\mathscr{I}\big(\mathsf{C}_k[m,F]\big) = \{S \in \mathsf{C}_k[m,F] \mid S \text{ is irreducible}\}$. In this section, we define an equivalence relation $\sim$ over $\mathsf{C}_k[m,F]$ such that $\mathsf{C}_k[m,F]/\sim = \big\{[S] \mid S \in \mathscr{I}\big(\mathsf{C}_k[m,F]\big)\big\}$ where $[S]$ denotes the equivalence class of S with respect to $\sim$.

This way, in order to determine explicitly the elements in the set $\mathsf{C}_k[m,F]$ we need:

(1) an algorithm to compute the set $\mathscr{I}\big(\mathsf{C}_k[m,F]\big)$;
(2) an algorithm to compute the class $[S]$, for each $S \in \mathscr{I}\big(\mathsf{C}_k[m,F]\big)$.

Since (1) is solved in [3], we only need to solve (2).

An element s in S is a small element if $s < \mathrm{F}(S)$. Denote by $N(S)$ the set of all small elements in S and by $n(S)$ its cardinality. In 1978, H. S. Wilf (see [20]) conjectured that $\mathrm{g}(S)$ is bounded above by $(\mathrm{e}(S)-1)n(S)$. This question has been solved in some special cases, but remains open in general, and it is one of the most important issues in Numerical Semigroups Theory. In Section 5, we will show that if $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S satisfies Wilf's conjecture.

2. The tree associated to $\mathsf{C}_k[m]$

Throughout this paper, m and k are positive integers such that $2 \leq k \leq m-1$. From [19, Lemma 2.3] we can deduce the following result.

Lemma 1. *Let M be a submonoid of $(\mathbb{N},+)$ such that $M \neq \{0\}$ and $M^* = M \backslash \{0\}$. Then $\mathrm{msg}(M) = M^* \backslash (M^* + M^*)$.*

The next result gives us characterizations for numerical semigroups with multiplicity m and concentration less than or equal to k.

Proposition 2. *Let S be a numerical semigroup with $\mathrm{m}(S) = m$. The following conditions are equivalent:*

(1) *S belongs to $\mathsf{C}_k[m]$.*
(2) *if $h \in \mathbb{N}\backslash S$ such that $h > \mathrm{m}(S)$, then $\{h+1,\ldots,h+k-1\} \cap S \neq \emptyset$.*
(3) *$\{s+1,\ldots,s+k\} \cap S \neq \emptyset$ for all $s \in S\backslash\{0\}$.*
(4) *$\{x+1,\ldots,x+k\} \cap S \neq \emptyset$ for all $x \in \mathrm{msg}(S)$.*

Proof. (1) *implies* (2). Let $s \in S\backslash\{0\}$ such that $s < h < \mathrm{next}_S(s)$. As $h > \mathrm{m}(S)$, then $s \geq m$ and thus $\mathrm{next}_S(s) - s \leq k$. Hence, $h + (\mathrm{next}_S(s) - h) \in S$ and $\mathrm{next}_S(s) - h \in \{1,\ldots,k-1\}$.

(2) *implies* (3). If $s+1 \notin S$ then by (2), we deduce that $\{s+2,\ldots,s+k\} \cap S \neq \emptyset$.

(3) *implies* (4). Trivial.

(4) *implies* (1). If $s \in S\backslash\{0\}$, then there exists $x \in \mathrm{msg}(S)$ and $t \in S$ such that $s = x+t$. Let $i \in \{1,\ldots,k\}$ such that $x+i \in S$, then $s+i = x+i+t \in S$ and thus $\mathrm{next}_S(s) - s \leq s+i-s \leq k$. $\square$

From the proof of the previous proposition we obtain the following.

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Corollary 3. *If S is a numerical semigroup, then* $\mathsf{C}(S) = \max\{\mathrm{next}_S(x) - x \mid x \in \mathrm{msg}(S)\}$.

Example 4. Using the previous results, we deduce that $S = \langle 5, 7, 9\rangle$ is a numerical semigroup with $\mathsf{C}(S) = 2$, because $\max\big\{\mathrm{next}_S(5) - 5, \mathrm{next}_S(7) - 7, \mathrm{next}_S(9) - 9\big\} = \max\{7-5, 9-7, 10-9\} = 2$.

The following result can be deduced from Proposition 2 item (2).

Corollary 5. *If $\{S, T\} \subseteq \mathscr{L}(m)$ such that $S \subset T$ and $S \in \mathsf{C}_k[m]$, then $T \in \mathsf{C}_k[m]$.*

It is well known that if S is a numerical semigroup such that $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup. As a consequence of Corollary 5, we have:

Corollary 6. *If $S \in \mathsf{C}_k[m]$ such that $S \neq \triangle(m)$, then $S \cup \{\mathrm{F}(S)\} \in \mathsf{C}_k[m]$.*

The previous result enables us, given an element $S \in \mathsf{C}_k[m]$, to define recursively the following sequence of elements in $\mathsf{C}_k[m]$:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \triangle(m) \\ \triangle(m) & \text{otherwise.} \end{cases}$

The next result is easy to prove.

Proposition 7. *If $S \in \mathsf{C}_k[m]$ and $\{S_n \mid n \in \mathbb{N}\}$ is the previous sequence of numerical semigroups, then $S_{\mathrm{g}(S)-m+1} = \triangle(m)$.*

A graph $G = (V, E)$ consists of a set V and a collection E of ordered pairs (v, w) of distinct elements from V. Elements of V are called vertices and elements of E are called edges. A path of length n connecting the vertices u and v of G is a sequence of n distinct edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ with $v_0 = u$ and $v_n = v$.

A graph G is a tree if there exists a vertex r (known as the root of G) such that for every other vertex v of G, there exists a unique path connecting v and r. If (u, v) is an edge of the tree then we say that u is a son of v.

We define the graph $G\big(\mathsf{C}_k[m]\big)$ as the graph whose vertices are elements of $\mathsf{C}_k[m]$ and $(S, T) \in \mathsf{C}_k[m] \times \mathsf{C}_k[m]$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$.

As a consequence of Proposition 7, we have the following result.

Theorem 8. *The graph $G\big(\mathsf{C}_k[m]\big)$ is a tree with root equal to $\triangle(m)$.*

From this, it is possible to construct recursively the elements of the set $G\big(\mathsf{C}_k[m]\big)$, starting in $\triangle(m)$, we connect each vertex with its sons. Hence, we need to characterize the sons of an arbitrary vertex of this tree and so we need the next result.

Lemma 9. [15, Lemma 1.7] *Let S be a numerical semigroup and $x \in S$. Then $S \backslash \{x\}$ is a numerical semigroup if and only if $x \in msg(S)$.*

The following result is easy to prove.

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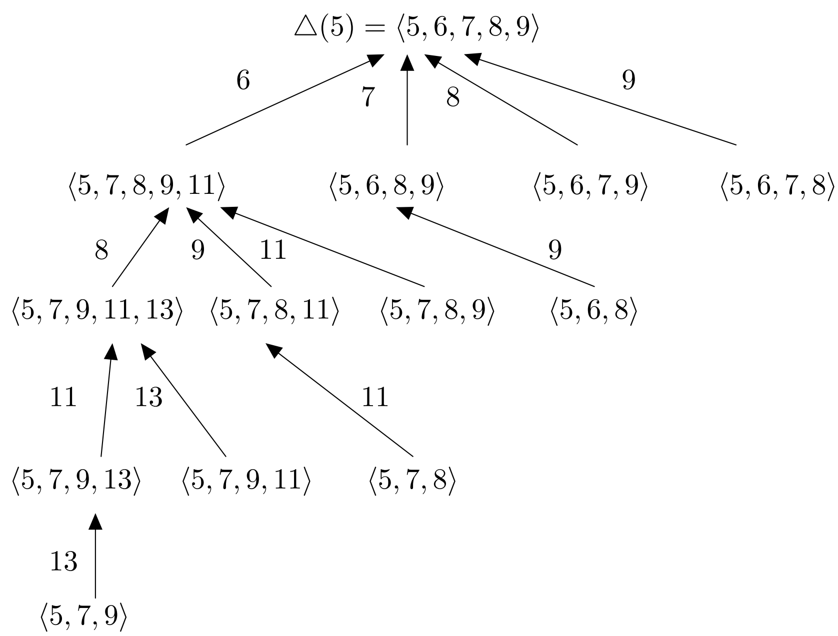
Proposition 10. *If $S \in \mathsf{C}_k[m]$, then the set of sons of S in the tree $G\big(\mathsf{C}_k[m]\big)$ is equal to* $\Big\{S\backslash\{x\} \mid x \in msg(S),\ x \geq \mathrm{F}(S)+2\Big\} \cup \Big\{S\backslash\{\mathrm{F}(S)+1\} \mid \mathrm{F}(S)+1 \in msg(S),\ \mathrm{F}(S)+1 \neq m \text{ and } \mathrm{F}(S)+2-prev_S(\mathrm{F}(S)+1) \leq k\Big\}$.

Example 11. From previous proposition, we construct the tree $G\big(\mathsf{C}_2[5]\big)$.



Note that the number x appearing on either side of an edge (Q,P) means that $Q = P\backslash\{x\}$ and $\mathrm{F}(Q) = x$.

We have that $G\big(\mathsf{C}_2[5]\big)$ is finite, in fact by [16, Proposition 12] we already knew that $G\big(\mathsf{C}_2[5]\big)$ is finite but, for example, $G\big(\mathsf{C}_2[4]\big)$ is infinite. Our next goal is to characterize the pair of positive integers (k,m) such that $\mathsf{C}_k[m]$ is finite.

If S is a numerical semigroup, then $\mathbb{N}\backslash S$ is a finite set. Hence we can announce the next result.

Lemma 12. *If S is a numerical semigroup, then $\{T \mid T$ is a numerical semigroup and $S \subseteq T\}$ is a finite set.*

Proposition 13. *The set $\mathsf{C}_k[m]$ is infinite if and only if there exists a divisor d of m such that $2 \leq d \leq k$.*

Proof. Necessity. If $S \in \mathsf{C}_k[m]$, then $\mathrm{next}_S(m) \in \{m+1,\ldots,m+k\}$ and so $\mathsf{C}_k[m] \subseteq \big\{S \mid S$ is a numerical semigroup and $\langle m, m+i\rangle \subseteq S$, for some $i \in \{1,\ldots,k\}\big\}$. By using Lemma 12, we can deduce that there exists $i \in \{1,\ldots,k\}$ such that $\gcd(m, m+i) = d \neq 1$. Hence, d is a divisor of m such that $2 \leq d \leq k$.

Sufficiency. For each $t \in \mathbb{N}$ define $S(t) = \{0\} \cup \big(\{m\} + \langle d\rangle\big) \cup \{m + t.d, \rightarrow\}$. It is clear that $S(t)$ is a numerical semigroup in $\mathsf{C}_k[m]$ with $\mathrm{F}\big(S(t)\big) = m + t.d - 1$. Therefore, $\mathsf{C}_k[m]$ is an infinite set. □

Example 14. Let $d = 2$, $m = 14$ and $k = 3$. As 2 divides 14 and $2 \le 3$, by Proposition 13, we have that $\mathsf{C}_3[14]$ has infinite cardinality.

On the other hand, none of the elements of $\{2, 3, 4\}$ divides 25, then $\mathsf{C}_4[25]$ has finite cardinality.

Let us finish this section by giving an algorithm that allows us to compute the whole set $\mathsf{C}_k[m]$ with a given genus.

Let G be a tree with root, and v one of its vertices. The depth of the vertex v is the length of the path that connects v to the root of G, denoted by $d_G(v)$. Given $n \in \mathbb{N}$, denote by

$$N(G, n) = \{v \mid d_G(v) = n\}\,.$$

The height of the tree G is defined as $h(G) = \ max\,\{k \in \mathbb{N} \mid N(G, k) \ne \emptyset\}$.

The following result is easy to prove.

Lemma 15. *With the above notation, we have:*

(1) $N\big(G(\mathsf{C}_k[m]), n\big) = \{S \in \mathsf{C}_k[m] \mid \mathrm{g}(S) = m - 1 + n\}$,
(2) $N\big(G(\mathsf{C}_k[m]), n{+}1\big) = \big\{S \mid S \textit{ is a son of an element in } N\big(G(\mathsf{C}_k[m], n)\big)\big\}$.

Algorithm 16.

INPUT: Integers m, g such that $g \ge m - 1$.
OUTPUT: The set $\{S \in \mathsf{C}_k[m] \mid \mathrm{g}(S) = g\}$

(1) $A = \{\langle m, m + 1, \ldots, 2m - 1\rangle\}$, $i = m - 1$.
(2) If $i = g$ then return A.
(3) For each $S \in A$ compute $B_S = \{T \mid T \text{ is a son of } S \in G(\mathsf{C}_k[m])\}$.
(4) If $\bigcup_{S\in A} B_S = \emptyset$, then return $\emptyset$.
(5) $A := \bigcup_{S\in A} B_S$, $i = i + 1$ and go to step 2.

Example 17. Let us compute the set $\{S \in \mathsf{C}_3[4] \mid \mathrm{g}(S) = 6\}$.

(1) Start with $A = \{\langle 4, 5, 6, 7\rangle\}$, $i = 3$.
(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$ and then $A = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$, $i = 4$.
(3) The second loop constructs $B_{\langle 4,6,7,9\rangle} = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$, $B_{\langle 4,5,7\rangle} = \{\langle 4, 5, 11\rangle\}$ and $B_{\langle 4,5,6\rangle} = \emptyset$ and then $A = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle, \langle 4, 5, 11\rangle\}$, $i = 5$.
(4) The third loop constructs $B_{\langle 4,7,9,10\rangle} = \{\langle 4, 7, 10, 13\rangle, \langle 4, 7, 9\rangle\}$, $B_{\langle 4,6,9,11\rangle} = \{\langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle\}$, $B_{\langle 4,6,7\rangle} = \emptyset$ and $B_{\langle 4,5,11\rangle} = \{\langle 4, 5\rangle\}$ and then $A = \{\langle 4, 7, 10, 13\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle, \langle 4, 5\rangle\}$, $i = 6$.
(5) Return $\big\{\langle 4, 7, 10, 13\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle, \langle 4, 6, 9\rangle, \langle 4, 5\rangle\big\}$.

3. (k,m)-SETS

Recall that a (k,m)-set is the set of the form $\theta(S) = S \cap \{m+1,\ldots,2m-1\}$ with $S \in \mathsf{C}_k[m]$. The next result characterizes the set $\mathsf{C}_k[m]$.

Theorem 18. *With the above notation,* $\mathsf{C}_k[m] = \Big\{S \in \mathscr{L}(m) \mid A \subseteq S$ *for some* $(k,m)-$*set* $A\Big\}$.

Proof. If $S \in \mathsf{C}_k[m]$, then S is a numerical semigroup with multiplicity m and $\theta(S) \subseteq S$.

Conversely, if A is a (k,m)-set, $S \in \mathscr{L}(m)$ such that $A \subseteq S$, we distinguish two cases:

(1) If $\gcd(\{m\} \cup A) = 1$, then $T = \langle \{m\} \cup A\rangle \in \mathscr{L}(m)$. From (4) Proposition 2, we deduce that $T \in \mathsf{C}_k[m]$. Since $S \in \mathscr{L}(m)$, $T \subseteq S$ and $T \in \mathsf{C}_k[m]$, then by Corollary 5, we have that $S \in \mathsf{C}_k[m]$.
(2) If $\gcd(\{m\} \cup A) = d \neq 1$, then $T = \langle \{m\} \cup A\rangle \cup \{\mathrm{F}(S)+1,\rightarrow\} \in \mathsf{C}_k[m]$ and $T \subseteq S$. By applying Corollary 5 again, we get $S \in \mathsf{C}_k[m]$. □

Let $\Omega = \{A \mid A \text{ is a } (k,m)-\text{set}\}$. Clearly, the binary relation $\subseteq$ is a partial order on Ω (i.e. reflexive, transitive and antisymmetric). If ω is a subset of Ω, we denote by $\mathrm{Minimals}_{\subseteq}(\omega)$ the set of minimal elements of ω with the order $\subseteq$.

Following the notation introduced in [14], a Frobenius pseudo-variety is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfills the following conditions:

(1) $\mathcal{P}$ has a maximum element (with respect to the order $\subseteq$);
(2) If $\{S,T\} \subseteq \mathcal{P}$, then $S \cap T \in \mathcal{P}$;
(3) If $S \in \mathcal{P}$ and $S \neq \max \mathcal{P}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

The next result is easy to prove.

Lemma 19. *If* $A \subseteq \{m,\rightarrow\}$, *then* $\mathcal{P}(A) = \{S \in \mathscr{L}(m) \mid A \subseteq S\}$ *is a Frobenius pseudo-variety with* $\max(\mathcal{P}(A)) = \triangle(m)$.

Note that, if the set $\mathrm{Minimals}_{\subseteq}\{A \mid A \text{ is a } (k,m)-\text{set}\} = \{A_1, A_2, \ldots, A_p\}$ then, as a consequence of Theorem 18, we have that $\mathsf{C}_k[m] = \bigcup_{i=1}^{p} \mathcal{P}(A_i)$. From this fact we have:

Proposition 20. *With the above notation,* $\mathsf{C}_k[m]$ *is the union of finitely many Frobenius pseudo-varieties.*

Our next goal is to give an algorithm to compute all (k,m)-sets, with given k and m positive integers. To this end, we need to introduce some concepts and results.

Given a (k,m)-set A such that $A \neq \{m+1, m+2, \ldots, 2m-1\}$, denote by

$$\mathcal{B}(A) = \max\big(\{m+1, m+2, \ldots, 2m-1\} \setminus A\big).$$

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The following result is easy to prove.

Lemma 21. *If A is a $\big(k,m\big)$-set and $A \subseteq B \subseteq \{m+1, m+2, \ldots, 2m-1\}$, then B is also a $\big(k,m\big)$-set.*

The previous result enables us, given a $\big(k,m\big)$-set A , to define recursively the following sequence of $\big(k,m\big)$-sets:

- $A_0 = A$,
- $A_{n+1} = \begin{cases} A_n \cup \{\mathcal{B}(A_n)\} & \text{if } A_n \neq \{m+1, m+2, \ldots, 2m-1\} \\ \{m+1, m+2, \ldots, 2m-1\} & \text{otherwise.} \end{cases}$

Let $\mathscr{C}\big(k,m\big) = \big\{A \mid A \text{ is a } \big(k,m\big) - \text{set}\big\}$. We define the graph $G\Big(\mathscr{C}\big(k,m\big)\Big)$ as the graph whose vertices are elements of $\mathscr{C}\big(k,m\big)$ and $(X,Y) \in \mathscr{C}\big(k,m\big) \times \mathscr{C}\big(k,m\big)$ is an edge if $Y = X \cup \{\mathcal{B}(X)\}$. From previous results it is easy to prove the next one.

Proposition 22. *The graph $G\Big(\mathscr{C}\big(k,m\big)\Big)$ is a tree with root equal to $\{m+1, m+2, \ldots, 2m-1\}$. Moreover, if A is a $\big(k,m\big)$-set, then the set of sons of A in the tree $G\Big(\mathscr{C}\big(k,m\big)\Big)$ is equal to $\Big\{A\backslash\{a\} \mid \{a, a+1, \ldots, 2m-1\} \subseteq A$ and either $a \leq m+k-1$ or $\{a-1, a-2, \ldots, a-(k-1)\} \cap A \neq \emptyset\Big\}$.*

Example 23. By using Proposition 22, we construct the tree $G\Big(\mathscr{C}\big(2,4\big)\Big)$.

$$\begin{tikzcd}
 & \{5,6,7\} & \\
\{6,7\} \arrow[ur, "5"] & \{5,7\} \arrow[u, "6"'] & \{5,6\} \arrow[ul, "7"'] \\
\{6\} \arrow[u, "7"'] & &
\end{tikzcd}$$

Observe that $\text{Minimals}_{\subseteq}\big(\mathscr{C}\big(2,4\big)\big) = \big\{\{6\}, \{5,7\}\big\}$. Hence, by Theorem 18, we obtain that $\mathsf{C}_2[4] = \{S \mid S \in \mathscr{L}(4) \text{ with } \{6\} \subset S \text{ or } \{5,7\} \subset S\}$. Moreover, by Proposition 20, $\mathsf{C}_2[4]$ is the union of the Frobenius pseudo-varieties $\{S \mid S \in \mathscr{L}(4) \text{ with } \{6\} \subset S\}$ and $\{S \mid S \in \mathscr{L}(4) \text{ with } \{5,7\} \subset S\}$.

Now, our goal in this section, is to compute all $\big(k,m\big)$-sets with a given cardinality.

Proposition 24. *Let p be a positive integer. The following conditions are equivalent.*

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(1) *A is an $\big(k,m\big)$-set with cardinality p,*
(2) $A = \Big\{m + x_1, m + x_1 + x_2, \dots, m + x_1 + x_2 + \cdots + x_p \mid (x_1, x_2, \dots, x_p, x_{p+1}) \in \{1, \dots, k\}^{p+1}$ *and* $x_1 + x_2 + \cdots + x_p + x_{p+1} = m\Big\}$.

Proof. 1) *implies* 2). Assume that $A = \{a_1 < a_2 < \cdots < a_p\}$, $x_1 = a_1 - m$, $x_{i+1} = a_{i+1} - a_i$ for all $i \in \{1, \dots, p-1\}$ and $x_{p+1} = 2m - a_p$. Then, we can conclude that $(x_1, x_2, \dots, x_p, x_{p+1}) \in \{1, \dots, k\}^{p+1}$, $x_1 + x_2 + \cdots + x_p + x_{p+1} = m$ and $A = \{m + x_1, m + x_1 + x_2, \dots, m + x_1 + x_2 + \cdots + x_p\}$.

2) *implies* 1). It is clear that, under desired conditions, every $A = \big\{m + x_1, m + x_1 + x_2, \dots, m + x_1 + x_2 + \cdots + x_p\big\}$ is contained in $\big\{m+1, m+2, \dots, 2m-1\big\}$. Then, we have that $S = \{0, m\} \cup A \cup \{2m, \rightarrow\} \in \mathsf{C}_k[m]$ and $A = \theta(S)$. Hence, A is a $\big(k,m\big)$-set with cardinality p. □

Given $q \in \mathbb{Q}$ and $p \in \mathbb{N}\backslash\{0\}$, we denote by $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \leq z\}$ and by $n(k,m,p) = \#\big\{A \mid A \text{ is a } \big(k,m\big) - \text{set with cardinality } p\big\}$ (where $\#A$ stands for cardinality of A).

As a consequence of Proposition 24, we obtain the following result.

Corollary 25. *With the above notation, we have that $n(k,m,p) \neq 0$ if and only if $\lceil \frac{m}{k} \rceil - 1 \leq p \leq m-1$. Furthermore, $n(k,m,p) = \#\big\{(x_1, x_2, \dots, x_{p+1}) \in \{1, \dots, k\}^{p+1} \mid x_1 + x_2 + \cdots + x_p + x_{p+1} = m\big\}$.*

Example 26. By using Proposition 24 let us calculate:
(1) the set $\big\{A \mid A \text{ is a } \big(2,4\big) - \text{set with cardinality } 2\big\}$, which is equal to $\big\{\{4 + x_1, 4 + x_1 + x_2\} \mid (x_1, x_2, x_3) \in \{1,2\}^3 \text{ and } x_1 + x_2 + x_3 = 4\big\}$. Since $\big\{(x_1, x_2, x_3) \in \{1,2\}^3 \mid x_1 + x_2 + x_3 = 4\big\} = \big\{(1,1,2), (1,2,1), (2,1,1)\big\}$, then the solution is $\big\{\{5,6\}, \{5,7\}, \{6,7\}\big\}$;
(2) the set $\big\{A \mid A \text{ is a } \big(2,4\big) - \text{set with cardinality } 1\big\}$, which is equal to $\big\{\{4 + x_1\} \mid (x_1, x_2) \in \{1,2\}^2 \text{ and } x_1 + x_2 = 4\big\}$. As $\big\{(x_1, x_2) \in \{1,2\}^2 \mid x_1 + x_2 = 4\big\} = \big\{(2,2)\big\}$, then the solution is equal to $\big\{\{6\}\big\}$.

Recall that S is an elementary numerical semigroup if $\mathrm{F}(S) < 2m(S)$. We denote by $\mathcal{E}(m)$ the set of elementary numerical semigroups with multiplicity m.

Proposition 27. [17, Lemma 1] *Let A be a subset of $\{m+1, \dots, 2m-1\}$. Then $\{0, m\} \cup A \cup \{2m, \rightarrow\}$ is an elementary numerical semigroup with multiplicity m. Moreover, every elementary numerical semigroup with multiplicity m is of this form.*

Denote by

$$\mathcal{E}\big(\mathsf{C}_k[m]\big) = \{S \in \mathsf{C}_k[m] \mid S \text{ is elementary}\}.$$

Thus, from Theorem 18 and Propositions 24 and 27, we deduce the next result.

Proposition 28. *With the above notation,* $\mathcal{E}\big(\mathsf{C}_k[m]\big) = \Big\{\{0\} \cup \{m, m + x_1, m+x_1+x_2, \ldots, m+x_1+x_2+\cdots+x_p\} \cup \{2m, \rightarrow\} \mid \big(x_1, x_2, \ldots, x_p, x_{p+1}\big) \in \{1, \ldots, k\}^{p+1}$ *and* $x_1 + x_2 + \cdots + x_{p+1} = m\Big\}$.

As a consequence of the previous proposition we have:

Corollary 29. *Let* g *be a positive integer. Then, the set of* $\big\{S \in \mathcal{E}\big(\mathsf{C}_k[m]\big)$ *with* $\mathrm{g}(S) = g\big\}$ *is equal to* $\Big\{\{0\} \cup \{m, m + x_1, m + x_1 + x_2, \ldots, m+x_1+x_2+\cdots+x_{2m-g-2}\} \cup \{2m, \rightarrow\} \mid \big(x_1, x_2, \ldots, x_p, x_{2m-g-1}\big) \in \{1, \ldots, k\}^{2m-g-1}$ *and* $x_1 + x_2 + \cdots + x_{2m-g-1} = m\Big\}$.

Example 30. Let us calculate all numerical semigroups S in $\mathcal{E}\big(\mathsf{C}_3[5]\big)$ with $\mathrm{g}(S) = 6$. By Corollary 29, we have that $\big\{S \in \mathcal{E}\big(\mathsf{C}_3[5]\big)$ with $\mathrm{g}(S) = 6\big\} = \Big\{\{0\} \cup \{5, 5+x_1, 5+x_1+x_2\} \cup \{10, \rightarrow\} \mid \big(x_1, x_2, x_3\big) \in \{1, 2, 3\}^3$ and $x_1 + x_2 + x_3 = 5\Big\}$. As $\big\{(x_1, x_2, x_3) \in \{1, 2, 3\}^3$ and $x_1 + x_2 + x_3 = 5\big\} = \big\{(1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1)\big\}$, then $\big\{S \in \mathcal{E}\big(\mathsf{C}_3[5]\big) \mid \mathrm{g}(S) = 6\big\} = \{0\} \cup A \cup \{10, \rightarrow\}$ such that A belongs to $\big\{\{5, 6, 7\}, \{5, 6, 8\}, \{5, 6, 9\}, \{5, 7, 8\}, \{5, 7, 9\}, \{5, 8, 9\}\big\}$.

Given a positive integer F, denote by

$$\mathsf{C}_k[m, F] = \{S \in \mathsf{C}_k[m] \mid \mathrm{F}(S) = F\}.$$

We will finish this section studying the elementary elements in $\mathsf{C}_k[m, F]$. The next result is easy to prove.

Proposition 31. *With the above notation, we have the following:*

(1) *If* $F = m - 1$ *then* $\mathsf{C}_k[m, F] = \{\triangle(m)\}$*;*
(2) *If* $F = m + r$ *with* $1 \leq r \leq k - 1$*, then* $\mathsf{C}_k[m, F] = \Big\{\{0, m\} \cup A \cup \{m + r + 1, \rightarrow\} \mid A \subseteq \{m + 1, \ldots, m + r - 1\}\Big\}$*;*
(3) *If* $F = m + r$ *with* $k \leq r < m$*, then* $\mathsf{C}_k[m, F] = \Big\{\{0, m\} \cup A \cup \{m + r + 1, \rightarrow\} \mid A = \{m + x_1, m + x_1 + x_2, \ldots, m + x_1 + x_2 + \cdots + x_p\}$*,* $(x_1, x_2, \ldots, x_p, x_{p+1}) \in \{1, \ldots, k\}^{p+1}$ *with* $x_1 + x_2 + \cdots + x_p + x_{p+1} = r + 1$ *and* $x_{p+1} \geq 2\Big\}$.

Example 32. Let us calculate the set of numerical semigroups $\mathsf{C}_2[5, 8]$. By Proposition 31, we get that $\mathsf{C}_2[5, 8] = \Big\{\{0, 5\} \cup A \cup \{9, \rightarrow\} \mid A = \{5 + x_1\},\ (x_1, x_2) \in \{1, 2\}^2$ with $x_1 + x_2 = 4$ and $x_2 \geq 2\Big\} \cup \Big\{\{5 + x_1, 5 + x_1 + x_2\},\ (x_1, x_2, x_3) \in \{1, 2\}^3$ with $x_1 + x_2 + x_3 = 4$ and $x_3 \geq 2\Big\}$. Then, $\mathsf{C}_2[5, 8] = \Big\{\{0, 5\} \cup A \cup \{9, \rightarrow\} \mid A \in \big\{\{7\}, \{6, 7\}\big\}\Big\}$.

4. Non elementary elements of $\mathsf{C}_k[m,F]$

The aim of this section is to give an algorithm to compute all elements in the set $\mathsf{C}_k[m,F]$ with $F > 2m$.

Following the terminology introduced in [18] a numerical semigroup is irreducible if it cannot be expressed as the intersection of two numerical semigroups properly containing it. This class of numerical semigroups has been widely studied in the literature and depending on the parity of the Frobenius number, which is odd or even, are called symmetric or pseudo-symmetric, respectively (see[11] and [1]). Of the many characterizations of irreducible numerical semigroups existing in the literature, we have the following results.

Proposition 33. [19, Corollary 4.5] *Let S be a numerical semigroup.*

(1) *S is symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$.*
(2) *S is pseudo-symmetric if and only if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+2}{2}$.*

Note that if S is a numerical semigroup, then $\mathrm{g}(S) \geq \frac{\mathrm{F}(S)+1}{2}$ (see, [19, Lemma 2.14]. As a consequence of Proposition 33, we obtain that the irreducible numerical semigroups are those with the least possible genus in terms of their Frobenius number.

Proposition 34. [2, Lemma 4] *Let S be a numerical semigroup. The following conditions hold:*

(1) *S is irreducible if and only if S is maximal in the set of all numerical semigroups with Frobenius number $\mathrm{F}(S)$.*
(2) *if $h = \max\left\{x \in \mathbb{N}\backslash S \mid \mathrm{F}(S) - x \notin S \text{ and } x \neq \frac{\mathrm{F}(S)}{2}\right\}$, then $S \cup \{h\}$ is also a numerical semigroup with Frobenius number $\mathrm{F}(S)$.*
(3) *S is irreducible if and only if $\left\{x \in \mathbb{N}\backslash S \mid \mathrm{F}(S) - x \notin S \text{ and } x \neq \frac{\mathrm{F}(S)}{2}\right\} = \emptyset$.*

Let S be a non-irreducible numerical semigroup. Denote by $\alpha(S) = \max\left\{x \in \mathbb{N}\backslash S \mid \mathrm{F}(S) - x \notin S \text{ and } x \neq \frac{\mathrm{F}(S)}{2}\right\}$. If S is an irreducible numerical semigroup, then by definition $\alpha(S) = 0$. Observe that, if $\alpha(S) \neq 0$ then $\frac{\mathrm{F}(S)}{2} < \alpha(S) < \mathrm{F}(S)$.

If $S \in \mathsf{C}_k[m,F]$, as a consequence of Corollary 5 and Proposition 34, then we can define the following sequence of elements in $\mathsf{C}_k[m,F]$:

- $S_0 = S$,
- $S_{n+1} = S_n \cup \{\alpha(S_n)\}$

It is easy to prove the next result.

Proposition 35. *Let $S \in \mathsf{C}_k[m,F]$ and let $(S_n)_{n\in\mathbb{N}}$ be the previous sequence. Then there exists a non negative integer p such that S_p is an irreducible numerical semigroup in $\mathsf{C}_k[m,F]$.*

We denote by $\mathbf{I}(S)$ the irreducible S_p obtained from a numerical semigroup S.

We define the following equivalence relation over $\mathsf{C}_k[m,F]$:

$$S \sim T \text{ if and only if } \mathbf{I}(S) = \mathbf{I}(T).$$

Denote the equivalence class modulo $\sim$ by $[S] = \{T \in \mathsf{C}_k[m,F] \mid S \sim T\}$ and by $\mathscr{I}\big(\mathsf{C}_k[m,F]\big) = \{S \in \mathsf{C}_k[m,F] \mid S \text{ is irreducible}\}$. As a consequence of Proposition 35, we deduce the next result.

Theorem 36. *The quotient set $\mathsf{C}_k[m,F]/\sim$ is equal to $\big\{[S] \mid S \in \mathscr{I}\big(\mathsf{C}_k[m,F]\big)\big\}$. Moreover, if $\{S,T\} \subseteq \mathscr{I}\big(\mathsf{C}_k[m,F]\big)$ and $S \neq T$ then $[S] \cap [T] = \emptyset$.*

In view of Theorem 36, in order to determine explicitly the elements in the set $\mathsf{C}_k[m,F]$ we need:

(1) an algorithm to compute the set $\mathscr{I}\big(\mathsf{C}_k[m,F]\big)$.
(2) an algorithm to compute the class $[S]$, for each $S \in \mathscr{I}\big(\mathsf{C}_k[m,F]\big)$.

In [3] an efficient algorithm was given to compute all irreducible numerical semigroups with fixed multiplicity m and Frobenius number F. By using Proposition 2, we choose those with concentration less than or equal to k and thus we have solved (1). Our goal now is to provide an algorithm to solve (2).

Let ∇ be an element in $\mathscr{I}\big(\mathsf{C}_k[m,F]\big)$. Let $G([\nabla])$ be the graph with vertex set $[\nabla]$ and $(S,T) \in [\nabla]\times[\nabla]$ is an edge if and only if $T = S \cup \{\alpha(S)\}$ and $\alpha(S) \neq 0$.

Proposition 37. *If $\nabla \in \mathscr{I}\big(\mathsf{C}_k[m,F]\big)$, then the graph $G([\nabla])$ is a tree with root equal to ∇. Moreover, the sons of a vertex T is $\big\{T\backslash\{x\} \mid x \in \mathrm{msg}(T),\ \frac{F}{2} < x < F,\ \alpha(T) < x,\ next_T(x) - prev_T(x) \leq k \text{ and } x > m\big\}$.*

Proof. If S is a son of T, then $T = S \cup \{\alpha(S)\}$ with $\alpha(S) \neq 0$ and thus $T\backslash\{\alpha(S)\} = S$. By Lemma 9, we have that $\alpha(S) \in \mathrm{msg}(T)$. Clearly, $\frac{F}{2} < \alpha(S) < F$, $\alpha(T) < \alpha(S)$ and $\mathrm{next}_T(\alpha(S)) - \mathrm{prev}_T(\alpha(S)) \leq k$.

Conversely, if $x \in \mathrm{msg}(T)$, $\frac{F}{2} < x < F$, $\mathrm{next}_T(x) - \mathrm{prev}_T(x) \leq k$, then we get that $T\backslash\{x\} \in \mathsf{C}_k[m,F]$. If $\alpha(T) < x$, then we have $\alpha(T\backslash\{x\}) = x$. Therefore, $T = (T\backslash\{x\}) \cup \{\alpha(T\backslash\{x\}\}$ and thus $T\backslash\{x\}$ is a son of T. □

Algorithm 38.

INPUT: $\nabla \in \mathscr{I}\big(\mathsf{C}_k[m,F]\big)$.
OUTPUT: The set $[\nabla]$.

1. $A = \{\nabla\}$ and $C = \{\nabla\}$.
2. For each $S \in C$ compute the set $B_S = \big\{T \mid T \text{ is a son of } S \text{ in the tree } G([\nabla])\big\}$.
3. $C = \bigcup_{S\in C} B_S$.
4. If $C = \emptyset$ then return A.
5. $A = A \cup C$ go to step 2.

Example 39. By Propositions 2 and 33 we have $\langle 5,7,9,11\rangle \in \mathscr{I}\big(\mathsf{C}_3[5,13]\big)$. Let us compute $[\langle 5,7,9,11\rangle]$.

- Start with $A = \big\{\langle 5,7,9,11\rangle\big\}$ and $C = \big\{\langle 5,7,9,11\rangle\big\}$.
- The first loop constructs $B_{\langle 5,7,9,11\rangle} = \Big\{\langle 5,7,11\rangle, \langle 5,7,9\rangle\Big\}$, then $C = \Big\{\langle 5,7,11\rangle, \langle 5,7,9\rangle\Big\}$ and thus $A = \Big\{\langle 5,7,9,11\rangle, \langle 5,7,11\rangle, \langle 5,7,9\rangle\Big\}$
- The second loop constructs $B_{\langle 5,7,11\rangle} = \big\{\langle 5,7,16,18\rangle,\big\}$ and $B_{\langle 5,7,9\rangle} = \emptyset$, then $C = \big\{\langle 5,7,16,18\rangle,\big\}$ and thus $A = \Big\{\langle 5,7,9,11\rangle, \langle 5,7,11\rangle, \langle 5,7,9\rangle, \langle 5,7,16,18\rangle\Big\}$.
- The third loop constructs $B_{\langle 5,7,16,18\rangle} = \emptyset$, then $C = \emptyset$.
- Hence, $[\langle 5,7,9,11\rangle] = \Big\{\langle 5,7,9,11\rangle, \langle 5,7,11\rangle, \langle 5,7,9\rangle, \langle 5,7,16,18\rangle\Big\}$.

5. Wilf's conjecture

We say that s is a small element in S if $s < \mathrm{F}(S)$. Denote by $N(S)$ the set of all small elements in S and by $n(S)$ its cardinality.

In 1978, H. S. Wilf (see [20]) conjectured an upper bound for $\mathrm{g}(S)$, namely $\mathrm{g}(S) \le (\mathrm{e}(S)-1)n(S)$. Nowadays, Wilf's conjecture remains unanswered in general, but for some specific families of numerical semigroups this conjecture is known to be true (see for example [6], [7], [9], [8], [10], [12], [13], [4] and [5]).

The next result appears in [8, Corollary 6.5].

Lemma 40. *If S is a numerical semigroup with $\mathrm{F}(S)+1 \le 3\mathrm{m}(S)$, then S verifies Wilf's conjecture.*

It is clear that $\{0,1,\ldots,\mathrm{F}(S)\} = N(S)\cup(\mathbb{N}\backslash S)$ and so $\mathrm{F}(S)+1 = \mathrm{g}(S)+n(S)$. Hence, we have that $\mathrm{F}(S)+1 \le \mathrm{e}(S)n(S)$ is another way to present Wilf's conjecture.

Theorem 41. *If $S \in \mathscr{L}(m)$, $p = \#\theta(S)$ and $2m \le (p+1)^2$, then S satisfies Wilf's conjecture.*

Proof. Let $q \in \mathbb{N}$ and $r \in \{1,\ldots,m-1\}$ such that $\mathrm{F}(S) = q.m + r$. If $q \in \{0,1,2\}$, then by Lemma 40, S satisfies Wilf's conjecture.

Now, we suppose that $q \ge 3$. By Lemma 1, we know that $\theta(S)\cup\{m\} \subseteq \mathrm{msg}(S)$ and thus $p+1 \le \mathrm{e}(S)$. Clearly, we have that $\{0\}$, $\theta(S)\cup\{m\}$, $\{m\}+\big(\theta(S)\cup\{m\}\big)$, $\{2m\}+\big(\theta(S)\cup\{m\}\big)$, …, $\{(q-2)m\}+\big(\theta(S)\cup\{m\}\big)$ are disjoint subsets of the set $N(S)$ and so we can conclude that $(q-1)(p+1)+1 \le n(S)$.

As by hypothesis $2m \le (p+1)^2$, then $2(q-1)m \le (q-1)(p+1)^2$. If $q \ge 3$, then $q+1 \le 2(q-1)$ and so we deduce that $(q+1)m \le (q-1)(p+1)^2$. Since $r \le m-1$, then we get $\mathrm{F}(S)+1 = (q+1)m \le (q-1)(p+1)^2$. As $p+1 \le e(S)$ thus $F(S)+1 \le (q-1)(p+1)e(S)$. Applying the inequality $(q-1)(p+1)+1 \le n(S)$, we obtain $\mathrm{F}(S)+1 \le \mathrm{e}(S)(n(S)-1)$. Hence, we have $\mathrm{F}(S)+1 \le \mathrm{e}(S)n(S)$ and thus S verifies Wilf's conjecture. ☐

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Corollary 42. *If $S \in \mathsf{C}_k[m]$ with $k \leq \sqrt{\frac{m}{2}}$, then S satisfies the Wilf's conjecture.*

Proof. If $S \in \mathsf{C}_k[m]$, by Proposition 24, $\theta(S) = \Big\{m+x_1, m+x_1+x_2, \ldots, m+x_1+x_2+\cdots+x_p \mid (x_1, x_2, \ldots, x_p, x_{p+1}) \in \{1, \ldots, k\}^{p+1}$ and $x_1+x_2+\cdots+x_p+x_{p+1} = m\Big\}$. From Corollary 25, we have that $\#\theta(S) \geq \lceil \frac{m}{k} \rceil - 1$. If $k \leq \sqrt{\frac{m}{2}}$, then $2m \leq (\frac{m}{k})^2$. This implies that $2m \leq (p+1)^2$. Finally, by Theorem 41, we conclude that S satisfies Wilf's conjecture. □

Example 43. If $S \in \mathsf{C}_5[100]$, by Corollary 42, S satisfies Wilf's conjecture.

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Departamento de Álgebra, Universidad de Granada, E-18071 Granada, Spain
Email address: jrosales@ugr.es

Departamento de Matemática, Universidade de Évora, 7000 Évora, Portugal
Email address: mbb@uevora.pt

Departamento de Matemática, Instituto Federal de São Paulo, Caraguatatuba, SP, Brazil
Email address: marciotraesel@ifsp.edu.br

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