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ON MEROMORPHIC FUNCTIONS OF ORDER ZERO

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1. In this paper we shall investigate a relation between the maximum modulus and the minimum modulus of a meromorphic function of order zero. Throughout the paper we assume familiarity with the standard notions of the Nevanlinna theory (see e.g. [4], [5]). We denote the Valiron deficiency of the value a for $f(z)$ by $\Delta(a,f)$. We define the maximum modulus $M(r,f)$ and the minimum modulus $\mu(r,f)$ of $f(z)$ by

$$M(r,f)=\sup|f(z)| \qquad (|z|=r),$$

$$\mu(r,f)=\inf|f(z)| \qquad (|z|=r)$$

respectively. We shall assume that $f(z)$ is transcendental i.e. that

$$\log r=o(T(r,f)) \qquad (r\to\infty).$$

If E is a measurable set on $(0,\infty)$ we define its density as

$$\lim_{r\to\infty}\frac{m\{E\cap(0,r)\}}{r}$$

if the limit exists, and its upper density by replacing lim by lim sup, where $m\{E\cap(0,r)\}$ denotes the measure of $E\cap(0,r)$.

It is well known that if $f(z)$ is an entire function of order zero then

$$\log\mu(r,f)\sim\log M(r,f)\sim T(r,f)$$

in a set of r of upper density 1 [3]. We shall show an analogous result for meromorphic functions of order zero.

THEOREM 1. *Let $f(z)$ be a meromorphic function of order zero. If $\delta(\infty,f)>0$, then*

$$\log\mu(r,f)\sim\log M(r,f)$$

and

$$\delta(\infty,f)\leqq\liminf_{r\to\infty}\frac{\log\mu(r,f)}{T(r,f)}\leqq\limsup_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}\leqq\Delta(\infty,f)$$

in a set of r of upper density 1. *Hence if $\delta(\infty,f)=\Delta(\infty,f)>0$, then*

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$$\log \mu(r,f) \sim \log M(r,f) \sim \delta(\infty,f) T(r,f)$$

in a set of r of upper density 1.

Ostrovskii [6] showed that $\mu(r,f)$ is sometimes large if $f(z)$ is of lower order ρ, $0 \leqq \rho < 1/2$, namely,

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,f)}{T(r,f)} \geqq \frac{\pi\rho}{\sin \pi\rho} \{\cos \pi\rho - 1 + \delta(\infty,f)\}.$$

In particular if $\rho = 0$,

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,f)}{T(r,f)} \geqq \delta(\infty,f).$$

Theorem 1 indicates that if $f(z)$ is of order zero and $\delta(\infty,f) > 0$ then $\mu(r,f)$ is large for a considerable proportion of the values of r.

If the hypothesis $\delta(\infty,f) > 0$ is omitted, the conclusion of Theorem 1 is no longer true. For instance, consider the function

$$f_0(z) = \Pi\left(1 + \frac{z}{e^n}\right) \Big/ \Pi\left(1 - \frac{z}{e^n}\right).$$

Then $f_0(z)$ is of order zero and obviously $\log \mu(r,f_0) = -\log M(r,f_0)$.

Here we note that if there exists an unbounded sequence $\{r_n\}$ of positive numbers such that

$$\Delta = \liminf_{n\to\infty} \frac{\log \mu(r_n,f)}{T(r_n,f)} > 0$$

then $\Delta(\infty,f) \geqq \Delta$. To prove this we assume that $f(z)$ satisfies $T(r,f) \sim N(r,0)$ and $f(0)=1$; this restriction is not essential. By Jensen's formula we have

$$\log \mu(r,f) \leqq N(r,0) - N(r,\infty),$$

whence we obtain

$$\Delta(\infty,f) = 1 - \liminf_{r\to\infty} \frac{N(r,\infty)}{T(r,f)} \geqq \liminf_{n\to\infty} \frac{N(r_n,0) - N(r_n,\infty)}{T(r_n,f)} \geqq \Delta.$$

On the other hand, let $g(z)$ be an entire function of order zero with $\Delta(0,g) > 0$; the existence of such a function was shown by Anderson-Clunie [1]. Then the function $f(z) = g(z)^{-1}$ is meromorphic, of order zero and satisfies $\Delta(\infty,f) > 0$. However $\mu(r,f)$ is bounded since

$$\log \mu(r,f) = -\log M(r,g).$$

Now it is natural to ask whether there is a meromorphic function of order zero such that $\delta(\infty,f) = 0$, $\Delta(\infty,f) > 0$ and

$$\liminf_{r\to\infty} \frac{\log \mu(r, f)}{T(r, f)} > 0$$

in a set of r of upper density 1. In § 4 we shall show that there exists a meromorphic function of order zero having the following two properties: (1) $\delta(\infty, f)=0$, $\Delta(\infty, f)>0$ and (2)

$$\log \mu(r, f) \sim \log M(r, f) \sim \Delta(\infty, f) T(r, f)$$

in a set of r of upper density 1.

Next we shall consider meromorphic functions of order zero satisfying

$$\lim_{r\to\infty} \frac{T(\sigma r, f)}{T(r, f)} = 1 \tag{1.1}$$

for a number $\sigma>1$. For such a function we shall prove the following

THEOREM 2. *Let $f(z)$ be a meromorphic function of order zero satisfying* (1.1) *for a number $\sigma>1$. If $\delta(\infty, f)>0$, then*

$$\log \mu(r, f) \sim \log M(r, f)$$

and

$$\delta(\infty, f) \leqq \liminf_{r\to\infty} \frac{\log \mu(r, f)}{T(r, f)} \leqq \limsup_{r\to\infty} \frac{\log M(r, f)}{T(r, f)} \leqq \Delta(\infty, f)$$

in a set of r of density 1. *Conversely, if*

$$\liminf_{r\to\infty} \frac{\log \mu(r, f)}{T(r, f)} > 0$$

in a set of r of density 1, *then $\delta(\infty, f)>0$.*

COROLLARY 1. *Let $f(z)$ be a meromorphic function of order zero satisfying* (1.1) *for a number $\sigma>1$. If $f(z)$ possesses a Nevanlinna deficient value, then it possesses no other Valiron deficient values.*

2. In order to prove Theorem 1 we need the following two lemmas. They are essentially the same as lemmas in Boas [2] and Cartwright [3], whence we omit their proofs.

LEMMA 1. *If $f(z)$ is a meromorphic function of order less than one with $f(0)=1$, then for every η $(0<\eta<(8/3)e)$ we have*

$$|\log|f(z)| - \{N(2R, 0) - N(2R, \infty)\}| < \left(1 + \log \frac{4e}{\eta}\right)\{Q(2R, 0) + (2R, \infty)\},$$

$|z|<R$, outside a set of circles the sum of whose radii is at most $2\eta R$, where

$$Q(r, a) = r\int_r^\infty \frac{n(t, a)}{t^2}\, dt.$$

LEMMA 2. *If $f(z)$ is of order zero, then*

$$\liminf_{r\to\infty} \frac{Q(r,0)+Q(r,\infty)}{N(r)}=0,$$

where $N(r)=N(r,0)+N(r,\infty)$.

3. *Proof of Theorem* 1. First we assume that $f(z)$ satisfies

$$T(r,f)\sim N(r,0), \qquad f(0)=1. \tag{3.1}$$

Suppose $\delta(\infty,f)>0$, so that for some positive ρ, $0<\rho<\delta(\infty,f)$, and R_0,

$$N(2R,0)-N(2R,\infty)>\frac{\rho}{2-\rho}N(2R) \qquad (R\geqq R_0).$$

Applying Lemma 1 we have

$$\left|\frac{\log|f(z)|}{N(2R,0)-N(2R,\infty)}-1\right|\leqq\left(1+\log\frac{4e}{\eta}\right)\frac{2-\rho}{\rho}\frac{Q(2R,0)+Q(2R,\infty)}{N(2R)},$$

$|z|<R$, outside a set of circles the sum of whose radii is at most $2\eta R$ provided $R\geqq R_0$ and $0<\eta<(8/3)e$. Let $\varepsilon(>0)$ be given. By Lemma 2 it is possible to choose an arbitrarily large positive number R_ε such that

$$\frac{1-\varepsilon}{1+\varepsilon}\leqq\frac{\log\mu(r,f)}{\log M(r,f)}\leqq 1$$

in a set $E(\eta,\varepsilon)$ of $r(<R_\varepsilon)$ of measure at least $(1-2\eta)R_\varepsilon$. Hence by the first fundamental theorem we have

$$\frac{1-\varepsilon}{1+\varepsilon}\left(1-\frac{N(r,\infty)}{T(r,f)}\right)\leqq\frac{\log\mu(r,f)}{T(r,f)}\leqq\frac{\log M(r,f)}{T(r,f)}\leqq\frac{1+\varepsilon}{1-\varepsilon}\left(1-\frac{N(r,\infty)}{T(r,f)}\right)$$

in $E(\eta,\varepsilon)$. Since η and ε are arbitrary we conclude that

$$\log\mu(r,f)\sim\log M(r,f)$$

and

$$\delta(\infty,f)\leqq\liminf_{r\to\infty}\frac{\log\mu(r,f)}{T(r,f)}\leqq\limsup_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}\leqq\Delta(\infty,f)$$

in a set of r of upper density 1.

If $f(z)$ does not satisfy the asymptotic relation of (3.1), we choose $\gamma(\neq 0)$ such that

$$N\left(r,\frac{1}{f-\gamma}\right)\sim T(r,f) \qquad \text{and} \qquad f(0)\neq\gamma.$$

Put $F(z)=c\{f(z)-\gamma\}$, where $F(0)=1$. Then we have

$$T(r,f)\sim T(r,F), \quad T(r,F)\sim N\left(r,\frac{1}{F}\right) \quad \text{and} \quad N(r,f)=N(r,F).$$

Thus the hypotheses in the theorem and the additional property hold with $f(z)$ replaced by $F(z)$. Hence the conclusion of the theorem holds with $F(z)$. Since

$$\log|F(z)|\to\infty \qquad (|z|\to\infty)$$

in the admitted set, this proves the general validity of the theorem.

4. Now we show that there exists a meromorphic function $f(z)$ of order zero having the following two properties: (1) $\delta(\infty,f)=0$, $\Delta(\infty,f)>0$ and (2)

$$\log\mu(r,f)\sim\log M(r,f)\sim\Delta(\infty,f)T(r,f)$$

in a set of r of upper density 1.

First we prove the following

LEMMA 3. *Let $f(z)$ be a meromorphic function of order zero satisfying*

$$\lim_{r\to\infty}\frac{Q(r,0)+Q(r,\infty)}{T(r,f)}=0 \quad \textit{and} \quad T(r,f)\sim N(r,0). \tag{4.1}$$

If $\Delta(\infty,f)>0$, then

$$\log\mu(r,f)\sim\log M(r,f)\sim\Delta(\infty,f)T(r,f)$$

in a set of r of upper density 1.

Proof. We may assume that $f(z)$ satisfies $f(0)=1$. Suppose $\Delta(\infty,f)>0$. Let $\{R_n\}$ be an unbounded increasing sequence of positive numbers such that

$$\lim_{n\to\infty}\frac{N(2R_n,\infty)}{T(2R_n,f)}=1-\Delta(\infty,f).$$

By Lemma 1 and (4.1) we may assume that

$$\left|\frac{\log|f(z)|}{N(2R_n,0)-N(2R_n,\infty)}-1\right|\leqq\frac{1}{n} \tag{4.2}$$

and

$$\log|f(z)|\geqq\left(1-\frac{1}{n}\right)\left\{\Delta(\infty,f)-\frac{1}{n}\right\}T(2R_n,f) \tag{4.3}$$

$|z|<R_n$, outside a set of circles the sum of whose radii is at most $(1/n)R_n$. Hence we have using (4.2)

$$\log\mu(r,f)\sim\log M(r,f),$$

$$\liminf_{r\to\infty}\frac{\log\mu(r,f)}{T(r,f)}\leqq\limsup_{r\to\infty}\frac{\log M(r,f)}{T(r,f)}\leqq\Delta(\infty,f)$$

and using (4.3)

$$\liminf_{r\to\infty}\frac{\log\mu(r,f)}{T(r,f)}\geqq\varDelta(\infty,f)$$

in a set of r of upper density 1. Thus we proved the lemma.

Next we construct a meromorphic function of order zero satisfying $\delta(\infty,f)=0$, $\varDelta(\infty,f)>0$, $T(r,f)\sim N(r,0)$ and

$$\lim_{r\to\infty}\frac{Q(r,0)+Q(r,\infty)}{T(r,f)}=0.$$

Put $g_1(z)=\Pi_{n=1}^{\infty}(1+z/e^{n/2})$. Then we have $N(r,0;g_1)\sim T(r,g_1)\sim(\log r)^2$.

Let $\{r_m\}$ and $\{R_m\}$ be unbounded increasing sequences of positive numbers such that $r_m<R_m<r_{m+1}$, and let $g_2(z)$ be an entire function of order zero, whose zeros in $R_{m-1}<|z|\leqq r_m$ are e^{ν} ($\nu=[\log R_{m-1}]+1,\cdots,[\log r_m]$) and whose zeros in $r_m<|z|\leqq R_m$ are $e^{\mu/3}$ ($\mu=[3\log r_m]+1,\cdots,[3\log R_m]$). Then we have

$$\begin{aligned}N(r_m,0;g_2)&\leqq\int_{R_{m-1}}^{r_m}\frac{\log t+2\log R_{m-1}+1}{t}dt+N(R_{m-1},0;g_2)\\&=\frac{1}{2}(\log r_m)^2-\frac{1}{2}(\log R_{m-1})^2+2\log R_{m-1}\log r_m-2(\log R_{m-1})^2\\&\quad+\log r_m-\log R_{m-1}+N(R_{m-1},0;g_2)\end{aligned}$$

and

$$\begin{aligned}N(R_m,0;g_2)&\geqq\int_{r_m}^{R_m}\frac{3\log t-3\log r_m-1}{t}dt\\&=\frac{3}{2}(\log R_m)^2-\frac{3}{2}(\log r_m)^2-3\log r_m\log R_m+3(\log r_m)^2-\log R_m+\log r_m.\end{aligned}$$

Hence we can define sequences $\{r_m\}$ and $\{R_m\}$ inductively such that

$$\frac{N(r_m,0;g_2)}{N(r_m,0;g_1)}\leqq\frac{3}{4}\quad\text{and}\quad\frac{N(R_m,0;g_2)}{N(R_m,0;g_1)}\geqq\frac{5}{4}.$$

We consider the function $F(z)=g_1(z)/g_2(z)$. Then $F(z)$ is meromorphic, of order zero and satisfies $\varDelta(\infty,F)\geqq1/4$, $\varDelta(0,F)\geqq1/5$. Further we can varify easily that $T(r,F)=O((\log r)^2)$. Valiron [7] proved that if $T(r,F)=O((\log r)^2)$ then for any two complex numbers a, b,

$$\max\{N(r,a),\ N(r,b)\}\sim T(r,F).$$

By this result we conclude that $F(z)$ satisfies $\delta(\infty,F)=0$. Let γ be a complex number such that $N(r,1/(F-\gamma))\sim T(r,F)$, and put $f(z)=F(z)-\gamma$. Then $f(z)$ satisfies

$\delta(\infty, f)=0$, $\Delta(\infty, f)>0$ and $N(r, 0)\sim T(r, f)$. Moreover

$$\lim_{r\to\infty}\frac{Q(r, 0)+Q(r, \infty)}{T(r, f)}=0,$$

since $n(r, 0)\log r\leqq N(r^2, 0)\leqq T(r^2, 0)+O(\log r)=O((\log r)^2)$.

Thus, combining these results, we established that $f(z)$ has the desired properties.

5. The proof of Theorem 2 depends on the following lemma. First we note that if the condition (1.1) holds with $f(z)$ for a number $\sigma>1$ then it holds for arbitrary $\tau>1$. In fact, $\sigma^n>\tau$ for an integer n, so that

$$1\leqq\frac{T(\tau r, f)}{T(r, f)}\leqq\frac{T(\sigma^n r, f)}{T(r, f)}=\frac{T(\sigma r, f)}{T(r, f)}\cdot\frac{T(\sigma^2 r, f)}{T(\sigma r, f)}\cdot\cdots\cdot\frac{T(\sigma^n r, f)}{T(\sigma^{n-1}r, f)}\to 1 \quad (r\to\infty).$$

LEMMA 4. *If $f(z)$ is a meromorphic function of order zero satisfying* (1.1) *for a number $\sigma>1$ and $N(r, 0)\sim T(r, f)$, $f(0)=1$, then*

$$\lim_{r\to\infty}\frac{Q(r, 0)+Q(r, \infty)}{T(r, f)}=0.$$

Proof. For arbitrary $\tau>1$ we have

$$n(r, 0)\log\tau\leqq N(\tau r, 0)\leqq 2N(r, 0) \tag{5.1}$$

and

$$n(r, \infty)\log\tau\leqq N(\tau r, \infty)\leqq\frac{3}{2}T(\tau r, f)\leqq 2N(r, 0)$$

provided $r>r_\tau$. Hence we have

$$Q(r, 0)+Q(r, \infty)=r\int_r^\infty\frac{n(t, 0)+n(t, \infty)}{t^2}dt\leqq\frac{4}{\log\tau}r\int_r^\infty\frac{N(t, 0)}{t^2}dt \quad (r>r_\tau).$$

Using (5.1) we obtain

$$\lim_{r\to\infty}\frac{n(r, 0)}{N(r, 0)}=0,$$

whence it follows that $r^{-1/2}N(r, 0)$ is decreasing for $r>r'_\tau$. Therefore we have

$$Q(r, 0)+Q(r, \infty)\leqq\frac{8}{\log\tau}N(r, 0)\leqq\frac{8}{\log\tau}T(r, f) \quad (r>r_\tau, r'_\tau).$$

Since τ is arbitrary, we have the desired result.

6. *Proof of Theorem* 2. We may assume that $f(z)$ satisfies $N(r, 0)\sim T(r, f)$ and $f(0)=1$. If $\delta(\infty, f)>0$, applying Lemma 1 and Lemma 4 we obtain the desired result. Suppose that

$$\liminf_{r\to\infty}\frac{\log\mu(r,f)}{T(r,f)}=d>0$$

in a set E of r of density 1. Let $\{\rho_n\}$ be an unbounded increasing sequence of positive numbers such that

$$\lim_{n\to\infty}\frac{N(\rho_n,\infty)}{T(\rho_n,f)}=\limsup_{r\to\infty}\frac{N(r,\infty)}{T(r,f)}.$$

Put $R_n=\sigma\rho_n$. We may assume that $m\{E\cap(0,R_n)\}>(1/\sigma)R_n=\rho_n$. There exists a sequence $\{r_n\}$ of positive numbers satisfying $\rho_n<r_n<R_n$ and $r_n\in E$. By Jensen's formula we have

$$d\leqq\liminf_{n\to\infty}\frac{\log\mu(r_n,f)}{T(r_n,f)}\leqq\liminf_{n\to\infty}\frac{N(r_n,0)-N(r_n,\infty)}{T(r_n,f)}=1-\limsup_{n\to\infty}\frac{N(r_n,\infty)}{T(r_n,f)}.$$

On the other hand, using (1.1) we have

$$\frac{N(\rho_n,\infty)}{T(\rho_n,f)}\leqq\left(1+\frac{d}{2}\right)\frac{N(\rho_n,\infty)}{T(R_n,f)}\leqq\left(1+\frac{d}{2}\right)\frac{N(r_n,\infty)}{T(r_n,f)}\qquad(n\geqq n_0).$$

Hence we obtain

$$\delta(\infty,f)=1-\lim_{n\to\infty}\frac{N(\rho_n,\infty)}{T(\rho_n,f)}\geqq1-\left(1+\frac{d}{2}\right)\limsup_{n\to\infty}\frac{N(r_n,\infty)}{T(r_n,f)}\geqq\frac{d}{2}+\frac{d^2}{2}>0.$$

7. *Proof of Corollary* 1. Assume that $\delta(a,f)>0$ and $\Delta(b,f)>0$ $(a\neq b)$. Using Theorem 2 we conclude that there exists a set E_a of r of density 1 such that $\lim_{E_a\ni r\to\infty}f(re^{i\theta})=a$ uniformly in θ. On the other hand, using Lemma 3 and Lemma 4 we conclude that there exists a set E_b of r of upper density 1 such that $\lim_{E_b\ni r\to\infty}f(re^{i\theta})=b$ uniformly in θ. This is a contradiction.

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