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DIFFERENTIAL REPRESENTATIONS OF VECTOR FIELDS

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In the present paper we are concerned with Lie algebra representations of $\mathcal{A}(M)$, where $\mathcal{A}(M)$ denotes the Lie algebra of vector fields on a smooth manifold M. It is well-known that the Lie derivatives acting on various tensor spaces give rise to representations of $\mathcal{A}(M)$. Another type of representations of $\mathcal{A}(M)$ arises from the considerations of the adjoint representation of $\mathcal{A}(M)$ on the space of differential operators with a finite order. These examples lead us to introduce a notion of differential representations of $\mathcal{A}(M)$. Specifically, φ is called a differential representation of $\mathcal{A}(M)$ on $\Gamma(E)$, E being a certain real vector bundle, if φ is a Lie algebra representation of $\mathcal{A}(M)$ to $\mathrm{Hom}(\Gamma(E), \Gamma(E))$ with $\mathrm{supp}\,\varphi(X)\sigma\subset\mathrm{supp}\,X\cap\mathrm{supp}\,\sigma$. We do not know whether all the differential representations of $\mathcal{A}(M)$ arouse geometric interest or not. However, the differential representations mentioned above have some geometric nature, which is characterized as connection-type. Here we use the terminology "connection" in a wide sense.

The main result of the present paper can be stated as follows. If on $\Gamma(E)$ there exists a differential representation of connection type of $\mathcal{A}(M)$, then we have

$$\mathrm{Pont}\,(E)\subset\mathrm{PONT}\,(M)\,.$$

Here $\mathrm{Pont}\,(E)$ denotes the subalgebra of $H^*(M;\boldsymbol{R})$ which is generated by the Pontrjagin classes $p_i(i\geqq 1)$ of E, while $\mathrm{PONT}\,(M)$ denotes the ideal of $H^*(M;\boldsymbol{R})$ which is generated by the Pontrjagin classes of M. This result, of course, involves that there is a topological obstruction to the existence of differential representation of connection type on $\Gamma(E)$. To find this obstruction, we extend the Chern-Weil theory on characteristic classes so as to be adaptable not only for the de Rham cohomology ring but also for more general cohomology rings. Then the theory especially applies to the Losik cohomology ring which yields a desired topological obstruction. Some analogous results will be also obtained for the Lie algebra consisting of the vector fields of type (1, 0) on a complex manifold.

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