Notre Dame Journal of Formal Logic
Volume 30, Number 1, Winter 1989

Multiset Theory

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A *multiset* is a collection of objects (called *elements*) in which elements may occur more than once. The number of times an element occurs in a multiset is called its *multiplicity*. The *cardinality* of a multiset is the sum of the multiplicities of its elements. Multisets are of interest in certain areas of mathematics, computer science, physics, and philosophy. Section 1 introduces multisets and surveys the relevant literature. Section 2 develops a first-order two-sorted theory MST for multisets that "contains" classical set theory. The intended interpretation of the atomic formula $x \in^n y$ is "x is an element of y with multiplicity n". In MST, one can extend the classical notion of a function. Section 3 constructs a model of MST in ZFC by interpreting $x \in^n y$ as $y(x) = n$ (multisets are modeled by positive integer-valued functions).

Introduction In [16] Kamke spells out the assumptions underlying classical set theory and thereby classical mathematics as a whole:

> By a set we are to understand, according to G. Cantor, "a collection into a whole, of definite, well-distinguished objects (called the 'elements') of our perception or of our thought . . . ". For a set, the order of succession of its elements shall not matter. . . . Furthermore, the same element shall not be allowed to appear more than once. The number complex 1,2,1,2,3, consequently, becomes a set only after deleting the repeated elements.

*This paper is based on part of the author's doctoral thesis prepared under the supervision of Dr. Robin Gandy at the University of Oxford. The research was supported by a Natural Sciences and Engineering Research Council of Canada Scholarship and a United Kingdom Overseas Research Student Award.

I would like to thank my supervisor, Dr. Robin Gandy, and my examiners, Dr. Dan Isaacson and Professor T. J. Smiley. I thank everyone with whom I have discussed multisets and their possible applications, especially T. Hailperin, G. Miller, A. F. Parker-Rhodes, and R. Rado; also M. Hallett, J. Hickman, J. Hoskins, P. Koepke, S. Lee, Z. Manna, B. Müller, M. Osterheld, F. Sheridan, W. van Stigt, and R. Walker. I also thank the referee for valuable comments, especially his suggestion to modify the H operation which simplified the definition of cardinality.

Received February 24, 1987; revised July 23, 1987