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Elementary Extensions of Recursively Saturated Models of Arithmetic

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A countable recursively saturated model of arithmetic has, up to isomorphism, a unique countable recursively saturated elementary end extension. As one might guess from the countability restriction, this isomorphism is not canonical. Indeed, if $\mathfrak{M}$ is a recursively saturated model of arithmetic and $\mathfrak{N}$ is an isomorphic copy thereof into which we want to embed $\mathfrak{M}$ as an elementary initial segment, then there are continuum many positions in $\mathfrak{N}$ in which to place $\mathfrak{M}$. In this paper we note that, in fact, $\mathfrak{M}$ can be so embedded in $\mathfrak{N}$ in continuum many decidedly distinct ways.

We are concerned with structures of the form,

$$(\mathfrak{N};\mathfrak{M}) = (|\mathfrak{N}|;|\mathfrak{M}|;+,\cdot,',0,\ldots),$$

where $\mathfrak{M} \prec_e \mathfrak{N}$ are countable recursively saturated models of PA. Our main theorem asserts great variety: for fixed $\mathfrak{M}$ (or, equivalently, for fixed $\mathfrak{N}$) there are continuum many elementarily inequivalent such structures. If, however, we assume $(\mathfrak{N};\mathfrak{M})$ to be recursively saturated, the situation becomes more tractable: there is only a countable infinity of elementarily inequivalent such structures.

Following the introduction, we get down to business in Section 1 where we first discuss cofinal extensions instead of end extensions. The proofs of the corresponding results are simpler and, besides, somewhat cute and deserving of display. End extensions are then discussed in Section 2.

This paper is not self-contained and the reader is advised to have copies of [1] and [6] at hand. Our notation is that of the latter and is reasonably standard. The merely near standard alphabetical exceptions are: Gothic capitals, $\mathfrak{M}$, $\mathfrak{N}$, denote models of arithmetic, which we take to mean *nonstandard* models of arithmetic. Lower case Latin letters are generally integers of various kinds:

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$a, b, c, \ldots$ denote elements of models; $x, y, z, \ldots$ natural numbers *qua* elements of models; and i, j, k, m, n natural numbers *qua* subscripts, etc. Subscripts e and c indicate that an extension is an end or cofinal extension, respectively. Latin capitals have primarily two uses: F, G, H are reserved for functions on models, and I, J, K for initial segments closed under successor.

Finally, a convention: *we assume all models to be countable.* We shall simply have to do this because we shall be relying on results only known or true in the countable case. Besides, if we are going to give exact cardinal computations, we must use exact cardinal input.

1 Cofinal extensions Cofinal extensions are generally more complicated objects (more fashionably, morphisms) than end extensions. If $\mathfrak{M}$ is recursively saturated, any elementary end extension $\mathfrak{N}$ has the same theory and standard system as $\mathfrak{M}$. Thus, if $\mathfrak{N}$ is also recursively saturated, it is isomorphic to $\mathfrak{M}$. If, however, $\mathfrak{N}$ is an elementary cofinal extension, then $\mathfrak{M}$ and $\mathfrak{N}$ need only share a common theory and not a standard system. There are, in fact, continuum many possibilities for the standard system and so a continuum of nonisomorphic recursively saturated elementary cofinal extensions $\mathfrak{N}$ of $\mathfrak{M}$. If we declare ourselves only to be interested in the case in which the standard system is not violated, we again conclude $\mathfrak{N}$ to be isomorphic to $\mathfrak{M}$ and uniqueness is reinstated. But only temporarily.

1.1 Theorem *Let $\mathfrak{M}$ be a recursively saturated model of arithmetic. There are continuum many distinct theories, $Th(\mathfrak{N};\mathfrak{M})$, of structures $(\mathfrak{N};\mathfrak{M})$, where $\mathfrak{N}$ is a recursively saturated elementary cofinal extension of $\mathfrak{M}$ with the same standard system as $\mathfrak{M}$.*

Proof: We simply cite a few results from the literature. Jensen and Ehrenfeucht, in [1], have proven there to be a continuum of elementarily inequivalent initial segments $I \subset_e \mathfrak{M}$ modeling *PA*, and hence closed under addition and multiplication. But, Paris and Mills [5] have shown that, if $I \subset_e \mathfrak{M}$ is closed under addition and multiplication, then there is an elementary cofinal extension $\mathfrak{N} \succ_c \mathfrak{M}$ in which I is the *greatest common initial segment*:

$$I = \mathrm{GCIS}(\mathfrak{M};\mathfrak{N}) = \{a \in |\mathfrak{M}|: \forall b \in |\mathfrak{N}| - |\mathfrak{M}|(a < b)\}.$$

Now, if the original model $\mathfrak{M}$ is recursively saturated, then, by a result of Smoryński and Stavi [8], $\mathfrak{N}$ is also recursively saturated.

Putting everything together, we get continuum many theories $\mathrm{Th}(\mathfrak{N};\mathfrak{M})$ of structures $(\mathfrak{N};\mathfrak{M})$ given by recursively saturated elementary cofinal extensions $\mathfrak{N}$ of our given recursively saturated model $\mathfrak{M}$ by taking continuum many elementarily inequivalent initial segments $I \subset_e \mathfrak{M}$ modeling *PA* and finding cofinal extensions $\mathfrak{N}_I$ of $\mathfrak{M}$ in which these segments I are the greatest common initial segments. The evident uniform first-order definability of $I = \mathrm{GCIS}(\mathfrak{M};\mathfrak{N})$ in $(\mathfrak{N}_I;\mathfrak{M})$ yields the elementary inequivalence of these pairs.

The only thing left is the small comment that at most one of the extensions $\mathfrak{N}_I$ has a standard system distinct from that of $\mathfrak{M}$. For, unless $I = \omega$, the inclusions $I \subset_e \mathfrak{M}$, $I \subset_e \mathfrak{N}_I$ entail the identities of the standard systems of I and $\mathfrak{M}$ and of those of I and $\mathfrak{N}_I$, whence that of the standard systems of $\mathfrak{M}$ and $\mathfrak{N}_I$.

QED

We cannot resist noting the following cute (albeit pointless) byproduct of our proof:

1.2 Fact Let $\mathfrak{M}$ be recursively saturated, $\mathfrak{M} \prec_c \mathfrak{N}$, and $I = \mathrm{GCIS}(\mathfrak{M};\mathfrak{N})$. If $I \neq \omega$, then $\mathfrak{N}$ is isomorphic to $\mathfrak{M}$.

Proof: As already noted, $\mathfrak{N}$ is recursively saturated and has the same theory and standard system as $\mathfrak{M}$. QED

Returning to our original problem, we note that our proof of Theorem 1.1 merely shows how to find a great many theories. It does not come anywhere near cataloguing them. Even if we fix $\mathfrak{M}$, $\mathfrak{N}$, and $I \subset_e \mathfrak{M}$ to be $\mathrm{GCIS}(\mathfrak{M};\mathfrak{N})$, the desired embedding can be accomplished in more than one way and occasionally (usually?) the results are elementarily inequivalent.

If we assume the entire structure $(\mathfrak{N};\mathfrak{M})$ to be recursively saturated, we get a degree of tractability:

1.3 Theorem *Let $\mathfrak{M}$ be a recursively saturated model of arithmetic. There is a countable infinity of distinct theories, $Th(\mathfrak{N};\mathfrak{M})$, of recursively saturated structures $(\mathfrak{N};\mathfrak{M})$, where $\mathfrak{N}$ is an elementary cofinal extension of $\mathfrak{M}$.*

As was the situation with Theorem 1.1, we again get no catalogue, but merely identical upper and lower bounds on the number of such theories. The key to obtaining these bounds is the following:

1.4 Lemma *Let $\mathfrak{M}$ be recursively saturated and let $T = Th(\mathfrak{N}_0;\mathfrak{M}_0)$ for some $\mathfrak{M}_0 \prec_c \mathfrak{N}_0$ with $Th(\mathfrak{M}_0) = Th(\mathfrak{M})$. In order that T be the theory of a recursively saturated structure $(\mathfrak{N};\mathfrak{M})$, it is necessary and sufficient that T be coded in the standard system, $SSy(\mathfrak{M})$, of $\mathfrak{M}$.*

This is really little more than a lifting to the noninductive theory of pairs of nonstandard models of the existence and uniqueness theorem for recursively saturated models of arithmetic (in terms of their theories and standard systems) and is, thus, obvious. For the reader who does not feel like noting how the old proofs (cf., e.g., [6]) generalize, we offer the following shorter, but *ad hoc*, argument.

Proof: Assume first that $(\mathfrak{N};\mathfrak{M})$ is recursively saturated and let $T = \mathrm{Th}(\mathfrak{N};\mathfrak{M})$. Let Pv be the unary predicate defining $|\mathfrak{M}|$ in $(\mathfrak{N};\mathfrak{M})$ and define

$$\tau v\colon \{\phi \longleftrightarrow \ulcorner\phi\urcorner \in D_v\colon \phi \text{ is a sentence}\} \cup \{Pv\},$$

where D_v denotes the finite set with canonical index v. Evidently τv is a recursive type over $(\mathfrak{N};\mathfrak{M})$. If $a \in |\mathfrak{N}|$ realizes τv, then $a \in |\mathfrak{M}|$ and

$$D_a \cap \omega = \{\ulcorner\phi\urcorner\colon (\mathfrak{N};\mathfrak{M}) \models \phi\} = \mathrm{Th}(\mathfrak{N};\mathfrak{M}) = T,$$

whence $T \in \mathrm{SSy}(\mathfrak{M})$.

Conversely, assume $T = \mathrm{Th}(\mathfrak{N}_0;\mathfrak{M}_0) \in \mathrm{SSy}(\mathfrak{M})$. If $\mathfrak{N}$ is isomorphic to $\mathfrak{M}$, then $\mathrm{Th}(\mathfrak{N}) = \mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{M}_0) = \mathrm{Th}(\mathfrak{N}_0)$ and T is consistent with the elementary diagram of $\mathfrak{N}$, i.e., $\mathfrak{N}$ is elementarily embeddable in a model, say $\mathfrak{N}_1$, expandable to $(\mathfrak{N}_1;\mathfrak{M}_1) \models T$. By the resplendence of $\mathfrak{N}$, since $T \in \mathrm{SSy}(\mathfrak{N})$ ($= \mathrm{SSy}(\mathfrak{M})$), $\mathfrak{N}$ has a recursively saturated expansion $(\mathfrak{N};\mathfrak{M}_2) \models T$, with $\mathfrak{M}_2 \prec_c \mathfrak{N}$. But pseudo-uniqueness (or Fact 1.2 in conjunction with Theorem 1.5.i, below) yields $\mathfrak{M}_2 \cong \mathfrak{N}$ and we already know $\mathfrak{N} \cong \mathfrak{M}$, whence $\mathfrak{M}_2 \cong \mathfrak{M}$. QED

Proof of Theorem 1.3: First note that Lemma 1.4 trivializes the upper bound: every theory, $\mathrm{Th}(\mathfrak{N};\mathfrak{M})$, of a recursively saturated pair $(\mathfrak{N};\mathfrak{M})$ is coded in the standard system of $\mathfrak{M}$ and $\mathrm{SSy}(\mathfrak{M})$ is countable.

For the lower bound we need only concoct a sequence $\{T_n\}_n$ of distinct theories of such pairs coded in $\mathrm{SSy}(\mathfrak{M})$. For then Lemma 1.4 constructs the models.

First, augment $\mathrm{Th}(\mathfrak{M})$ by a unary predicate symbol Pv and axioms asserting Pv to define an elementary cofinal submodel of the universe. This basic theory, say T_{-1}, is recursive in $\mathrm{Th}(\mathfrak{M})$, whence coded in $\mathrm{SSy}(\mathfrak{M})$. Now let $\{T_n^0\}_n$ be a sequence of distinct theories of initial segments of $\mathfrak{M}$ individually coded in $\mathrm{SSy}(\mathfrak{M})$ (cf. [1] for an existence proof) and let, for each n, T_n^1 be the extension of T_{-1} asserting the greatest common initial segment of the universe and the submodel defined by Pv to model T_n^0. Finally, simply choose $T_n \in \mathrm{SSy}(\mathfrak{M})$ to complete T_n^1. QED

Before proceeding to the study of end extensions, we note a few structural consequences of the recursive saturation of $(\mathfrak{N};\mathfrak{M})$.

1.5 Theorem *Let $(\mathfrak{N};\mathfrak{M})$ be recursively saturated with $\mathfrak{M} \prec_c \mathfrak{N}$, and let $I = GCIS(\mathfrak{M};\mathfrak{N})$ be the greatest common initial segment of the extension. Then:*

i. $\omega \subsetneq I$
ii. *I is closed under successor, addition, and multiplication*
iii. *For some nonstandard exponent $e \in I$, I is closed under the function: $a \to a^e$.*

A quick note before proving the theorem. The closures of I under successor, addition, and multiplication do not depend on the recursive saturation of the pair $(\mathfrak{N};\mathfrak{M})$ and, indeed, are the closure properties characteristic of the greatest common initial segments of elementary cofinal extensions. It follows that parts i and iii of the theorem, by going beyond closure under multiplication, are special consequences of recursive saturation.

Proof of Theorem 1.5: The simple proof of ii appears in [7] and we omit it. (The not-so-simple proof of its equally omitted converse is in [5].) To prove i and iii, we note first that I is first-order definable in $(\mathfrak{N};\mathfrak{M})$.

i. Obviously $\omega \subset I$. To see that the extension is proper, note simply that the set

$$\tau v\colon \{v \in I\} \cup \{\neg v = \bar{x} \colon x \in \omega\}$$

is a recursive type over $(\mathfrak{N};\mathfrak{M})$.

iii. Since I is closed under multiplication,

$$(\mathfrak{N};\mathfrak{M}) \models \forall v \in I(v^{\bar{x}} \in I)$$

for each $x \in \omega$. Thus

$$\tau v\colon \{\neg v = \bar{x} \colon x \in \omega\} \cup \{\forall v_0 \in I(v_0^v \in I)\}$$

is a recursive type over $(\mathfrak{N};\mathfrak{M})$. QED

We note that iii cannot be improved to full closure under exponentiation. For the existence of initial segments of $\mathfrak{M}$ closed under polynomial functions but not exponentiation, together with the method of proof of Theorem 1.3, yields the existence of recursively saturated pairs $(\mathfrak{N};\mathfrak{M})$ whose greatest common initial segments are not closed under exponentiation.

*2 **End extensions*** Elementary end extensions of nonstandard models of arithmetic exhibit a great variety of behavior. For each model $\mathfrak{M}$, there are continuum many nonisomorphic models $\mathfrak{N}$ into which $\mathfrak{M}$ can be elementarily initially embedded. Moreover, there are a variety of ways in which $\mathfrak{M}$ can be embedded in such models. The extension can be minimal or nonminimal, conservative or nonconservative; and $\mathfrak{M}$ can be, e.g., regular or nonregular, strong or weak, in $\mathfrak{N}$. When we insist that $\mathfrak{M}$ and $\mathfrak{N}$ both be recursively saturated, we lose many of these options: $\mathfrak{N}$ must be isomorphic to $\mathfrak{M}$ and is neither a minimal nor, as Kotlarski noted, a conservative extension of $\mathfrak{M}$. It is, however, still possible for $\mathfrak{M}$ to be either regular or nonregular, strong or weak, in $\mathfrak{N}$. Since strength and regularity are first-order properties of pairs $(\mathfrak{N};\mathfrak{M})$ with $\mathfrak{M} \subset_e \mathfrak{N}$, this last remark tells us that even for fixed recursively saturated $\mathfrak{M}$, there are a number of theories, $\mathrm{Th}(\mathfrak{N};\mathfrak{M})$, of recursively saturated elementary end extensions. As remarked already in the introduction, this number is the cardinality of the continuum, unless we further insist the structure $(\mathfrak{N};\mathfrak{M})$ to be recursively saturated, in which case there is only a countable infinity of such theories.

Having used initial segments to provide a continuum of theories of cofinal extensions, it is only fair to use the notion of cofinality to obtain our continuum of theories of end extensions. Briefly, we do this by defining the *cofinality* of $\mathfrak{M}$ in $\mathfrak{N}$ in the structure $(\mathfrak{N};\mathfrak{M})$ and noting that there is a continuum of theories of these cofinalities. Before doing this, however, we accomplish the simpler task of finding a continuum of nonisomorphic models $(\mathfrak{N};\mathfrak{M})$ with both $\mathfrak{M}$ and $\mathfrak{N}$ recursively saturated.

First a definition:

2.1 Definition Let $\mathfrak{M}$ be a nonstandard model and $a, b \in |\mathfrak{M}|$. We say a and b are in the same *sky*, say $\mathrm{Sk}(a)$, if there are parameter-free definable functions F, G such that $b < Fa$ and $a < Gb$ in $\mathfrak{M}$. Skies inherit the ordering of $\mathfrak{M}$:

$$\mathrm{Sk}(a) < \mathrm{Sk}(b) \text{ iff } \mathrm{Sk}(a) \neq \mathrm{Sk}(b) \text{ and } a < b.$$

We write $a << b$ to indicate $\mathrm{Sk}(a) < \mathrm{Sk}(b)$. (A quick caution. We have defined skies only for models of arithmetic. When we refer to skies in a model $(\mathfrak{N};\mathfrak{M})$ with $\mathfrak{M} \prec \mathfrak{N}$, we mean the skies of $\mathfrak{N}$ or of $\mathfrak{M}$—only the parameter-free definable functions of $\mathfrak{N}$ and $\mathfrak{M}$, not those of $(\mathfrak{N};\mathfrak{M})$, are used.)

Following [8], we can code huge ascending sequences of skies in a recursively saturated model and use them to generate recursively saturated elementary initial segments.

2.2 Lemma *Let $\mathfrak{N}$ be recursively saturated and $a \in |\mathfrak{N}|$. There is a $b \in |\mathfrak{N}|$ coding a sequence of length a such that for all $c < a$, $(b)_c << (b)_{c+1}$.*

Proof: There is a recursive type,

$$\tau v\bar{a}\colon \forall v_0 < \bar{a}[(v)_{v_0} << (v)_{v_0+\bar{1}}]. \qquad \text{QED}$$

Let $\mathfrak{N}$ be a given recursively saturated model, $a \in |\mathfrak{N}|$ nonstandard, $b \in |\mathfrak{N}|$ as in the lemma, and $I \subset_e \mathfrak{N}$ an initial segment closed under successor, but bounded by a: $I < a$. With all this, we define

$$\mathfrak{M}(I, b) = \bigcup_{e \in I} [0, (b)_e],$$

where $[0, c]$ is the obvious notation for the initial segment $\{d \in |\mathfrak{N}| : d \leqslant c\}$. One sees easily that $\mathfrak{M}(I, b)$ is a recursively saturated elementary initial segment of $\mathfrak{N}$.

2.3 Theorem *Let* $\mathfrak{M}$ *be a given recursively saturated model of arithmetic. There are continuum many nonisomorphic structures* $(\mathfrak{N}; \mathfrak{M})$ *with* $\mathfrak{N}$ *a recursively saturated elementary end extension of* $\mathfrak{M}$.

Proof: Let $\mathfrak{N} \cong \mathfrak{M}$. For a, b, I as above, we obviously have $\mathfrak{M}(I, b) \cong \mathfrak{M}$ and it suffices to find a continuum of nonisomorphic structures $(\mathfrak{N}; \mathfrak{M}(I, b))$.

We know already that there are a continuum of elementarily inequivalent—hence nonisomorphic—possibilities for I. We further note that I is definable via the parameter b over $(\mathfrak{N}; \mathfrak{M}(I, b))$: If Pv defines the submodel $\mathfrak{M}(I, b)$, then $P((\bar{b})_v)$ defines I.

We conclude immediately the existence of a continuum of elementarily inequivalent models of the form $(\mathfrak{N}; \mathfrak{M}(I, b); b)$, where $b \in |\mathfrak{N}| - |\mathfrak{M}|$ is a new designated element. While this does not allow us to conclude the existence of a continuum of elementarily inequivalent models $(\mathfrak{N}; \mathfrak{M})$, it does give us the continuum of nonisomorphic models, for there are only countably many parameters $b \in |\mathfrak{N}|$. QED

Our task now is to eliminate the parameter b from the above. Since all the elements of $|\mathfrak{N}|$ first-order definable in $\mathfrak{N}$ without parameters are in the lowest sky, Sk(0), and b is somewhat larger, there is no hope of defining b in $\mathfrak{N}$. We also see no way of defining b in $(\mathfrak{N}; \mathfrak{M})$. However, if I has sufficiently strong closure properties in $\mathfrak{N}$, it can be recovered from $|\mathfrak{M}|$ in $(\mathfrak{N}; \mathfrak{M})$.

The following analogue to the set-theoretic notion of regularity was introduced by Kirby and Paris (in [2] and [3]).

2.4 Definition Let $\mathfrak{N}$ be nonstandard, $I \subset_e \mathfrak{N}$ an initial segment closed under successor. I is *semi-regular* in $\mathfrak{N}$ if for every $a \in I$ and every parametrically definable function $F\colon |\mathfrak{N}| \to |\mathfrak{N}|$, the intersection $F''[0, a] \cap I$ is bounded in I, i.e., for some b, $F''[0, a] \cap I < b \in I$.

We shall apply the following property of semi-regularity borrowed from Kirby's thesis [2].

2.5 Lemma *Let I, J, K be initial segments of a nonstandard model* $\mathfrak{N}$, *with I semi-regular in* $\mathfrak{N}$. *If there are parametrically definable monotone functions F, G which map I, J, respectively, cofinally into K, then* $I \subset J$.

Since the simple proof of this result has not yet been published, we repeat it here:

Proof: Since I, J are initial segments, either $I \subset J$ or $J < a \in I$ for some a. Assume the latter. Since $F{:}I \to K$ and $G{:}J \to K$ are cofinal, the function

$$Hb = \begin{cases} \mu c[Fc \geqslant Gb], & \text{if such exists} \\ 0, & \text{otherwise,} \end{cases}$$

is well-defined as well as parametrically defined. By semiregularity, $I \cap H''[0, a]$ is bounded in I. But, by the cofinality of the ranges of F, G in K,

$$I \cap H''[0, a] \supset I \cap H''J$$

is cofinal in I, a contradiction. QED

We can now prove the following.

2.6 Theorem *Let $\mathfrak{M}$ be a recursively saturated model of arithmetic. There are continuum many distinct theories, $Th(\mathfrak{N}; \mathfrak{M})$, of structures $(\mathfrak{N}; \mathfrak{M})$, where $\mathfrak{N}$ is a recursively saturated elementary end extension of $\mathfrak{M}$.*

Proof: We repeat the construction of the proof of Theorem 2.3. Let $\mathfrak{M}, \mathfrak{N}, I, b$ be as before, only now we also assume I to be semiregular in $\mathfrak{N}$. (There is still a continuum of elementarily inequivalent such segments—as noted explicitly in [2] and implicitly in [3], all initial segments modeling PA can be re-embedded in $\mathfrak{N}$ as semiregular initial segments.) We conclude that the resulting continuum of models $(\mathfrak{N}; \mathfrak{M}(I, b))$ is the desired family of models by uniformly recovering I from $(\mathfrak{N}; \mathfrak{M}(I, b))$ in a first-order way.

By the lemma, I is the least initial segment that can be cofinally mapped into $|\mathfrak{M}(I, b)|$. Moreover, the function accomplishing this is of a particularly simple form: $F_b d = (b)_d$. We define I by saying in a first-order way: $a \in I$ iff a is in the domain of every increasing function of the form $F_c d = (c)_d$ whose range intersects cofinally with $|\mathfrak{M}(I, b)|$. QED

As before, when we assume the recursive saturation of the pair $(\mathfrak{N}; \mathfrak{M})$, the situation becomes more tractable:

2.7 Theorem *Let $\mathfrak{M}$ be a recursively saturated model of arithmetic. There is a countable infinity of distinct theories, $Th(\mathfrak{N}; \mathfrak{M})$, of recursively saturated structures $(\mathfrak{N}; \mathfrak{M})$, with $\mathfrak{M} \prec_e \mathfrak{N}$.*

Also as before, these theories are precisely those coded in SSy($\mathfrak{M}$). We omit the proof.

Once again history repeats itself: the recursive saturation of $(\mathfrak{N}; \mathfrak{M})$ has certain structural consequences. The following is moderately interesting.

2.8 Theorem *Let $(\mathfrak{N}; \mathfrak{M})$ be recursively saturated, with $\mathfrak{M} \prec_e \mathfrak{N}$. Then $\mathfrak{M} = \mathfrak{M}(I, b)$ for some $I \subset_e \mathfrak{N}$, $b \in |\mathfrak{N}|$.*

Proof: Let τv be the set of all formulae of the forms:

$$\forall v_0[(v)_{v_0} << (v)_{v_0+\bar{1}}] \tag{1}$$
$$\forall v_0 \in |\mathfrak{M}| \, \exists v_1[v_0 < (v)_{v_1} \in |\mathfrak{M}|]. \tag{2}$$

We claim that τv is a type. For a finite collection $F_0, \ldots, F_{n-1}$ of monotone functions definable in $\mathfrak{N}$ without parameters, let $Fv = \Sigma F_i v + 1$. Pick $a > |\mathfrak{M}|$ and let

$$c = (0, F0, \ldots, F^{a-1}0),$$

i.e., $lh(c) = a$ and $(c)_e = F^e 0$ is the e-fold application of F to 0. The instances of (1) corresponding to $F_0, \ldots, F_{n-1}$ (i.e., $F_j((c)_e) < (c)_{e+1}$) are clearly satisfied; moreover, (2) is satisfied since, if $d \in |\mathfrak{M}|$,

$$d < (c)_{d+1} = F^{d+1}0 \in |\mathfrak{M}|.$$

Thus, τv is a recursive type over $(\mathfrak{N}; \mathfrak{M})$, and hence realized therein by some $b \in |\mathfrak{N}|$. Simply let I be the inverse image of $|\mathfrak{M}|$:

$$I = \{a \in |\mathfrak{N}|: (b)_a \in |\mathfrak{M}|\}. \qquad \text{QED}$$

We note that the argument shows that we can take $I = |\mathfrak{M}|$. If $\mathfrak{M}$ is semi-regular in $\mathfrak{N}$, then Lemma 2.5 shows we must take $I = |\mathfrak{M}|$. Moreover, the recursive saturation of $(\mathfrak{N}; \mathfrak{M})$ along with the parametric definability of I shows I to be recursively saturated. Hence, we can*not* take $I = \omega$. Whatever I happens to be, Theorem 2.8 tells us that our proof of Theorem 2.3 (and, perhaps, that of 2.6 as well) was not as *ad hoc* as it might have seemed.

A rather more mundane structural consequence of the recursive saturation of the pair is the following.

2.9 Theorem *Let $(\mathfrak{N}; \mathfrak{M})$ be recursively saturated with $\mathfrak{M} \prec_e \mathfrak{N}$. Then*

i. $|\mathfrak{M}|$ *has no highest sky*
ii. $|\mathfrak{N}| - |\mathfrak{M}|$ *has no lowest sky.*

Part i actually requires only the recursive saturation of $\mathfrak{M}$—the skies of any recursively saturated model of arithmetic are densely ordered with a first but no last element. This fact (whence part i) and part ii are trivially proven by exhibiting the proper types, a task we leave to the reader.

The interest in Theorem 2.9 is that, if we only assume $\mathfrak{N}$ to be recursively saturated and $\mathfrak{M} \prec_e \mathfrak{N}$, then we obtain two more tractable cases:

2.10 Theorem *Let $\mathfrak{N}$ be recursively saturated. There is a countable infinity of nonisomorphic structures $(\mathfrak{N}; \mathfrak{M})$ with $\mathfrak{M} \prec_e \mathfrak{N}$ in each of the following cases:*

i. $|\mathfrak{M}|$ *has a highest sky*
ii. $|\mathfrak{N}| - |\mathfrak{M}|$ *has a lowest sky.*

This result is due independently to Kotlarski and the author.

Proof idea: In each case, the isomorphism type of $(\mathfrak{N}; \mathfrak{M})$ is determined uniquely by the extremal sky. Since there are only countably many skies, the upper bound is obvious.

To establish the lower bound, one constructs a sequence of partial types $\tau_0 v, \tau_1 v, \ldots$ with two special properties:

i. For $i \neq j$, $\tau_i v$ and $\tau_j v$ cannot be realized in a common sky
ii. Each $\tau_n v$ is recursive in $\mathrm{Th}(\mathfrak{N})$.

Details of the construction of these types can be found in [7].

Let types $\tau_0 v$, $\tau_1 v$, . . . with these properties be given. By ii, they are realized in $\mathfrak{N}$ by (say) a_0, a_1, ... We use the a_n's to construct models $\mathfrak{M}_n$. Suppose we want, e.g., a highest sky. Then we define

$$|\mathfrak{M}_n| = \bigcup_{a \leqslant a_n} \mathrm{Sk}(a),$$

and note that no two structures $(\mathfrak{N}; \mathfrak{M}_m)$ and $(\mathfrak{N}; \mathfrak{M}_n)$ for $m \neq n$ are isomorphic. QED

Note that we have said nothing about the number of theories of such structures. Obviously, they are of at most a countable infinity. That this bound is achieved requires a little work. The following serves as a lemma in this direction.

2.11 Theorem *Let $\mathfrak{N}$ be recursively saturated and let $\mathfrak{M} \prec_e \mathfrak{N}$ be such that either $|\mathfrak{M}|$ has a highest sky or $|\mathfrak{N}| - |\mathfrak{M}|$ has a lowest sky. Then the following are definable without parameters in $(\mathfrak{N}; \mathfrak{M})$:*

i. ω
ii. *the truth definition for $\mathfrak{M}$*
iii. *the extremal sky.*

Moreover, the definitions are uniform in each of the two given cases.

Proof: We will consider only the easier of the two cases, that in which $|\mathfrak{M}|$ has a highest sky.

i. Following [4], we note that ω codes $\mathfrak{M}$ in $\mathfrak{N}$, i.e., we can do the following: let $a \in |\mathfrak{M}|$ be of the highest sky and let $G_0, G_1, \ldots$ enumerate all parameter-free Skolem functions of $\mathfrak{N}$. Letting further $F_n v = n + \sum_{i \leqslant n} G_n v$, we define

$$\tau v \bar{a}: \{\forall v_0[(v)_{v_0} < (v)_{v_0 + \bar{1}}]\} \cup \{(v)_{\bar{n}} = F_n \bar{a}: n \in \omega\}.$$

But $\tau v \bar{a}$ is, as usual, a recursive type over $\mathfrak{N}$. Let b realize $\tau v \bar{a}$ and note that, for any $c \in |\mathfrak{M}|$,

$$c \in \omega \text{ iff } (b)_c \in |\mathfrak{M}|.$$

By the definability of $|\mathfrak{M}|$ in $(\mathfrak{N}; \mathfrak{M})$, we obtain the parametric definability of ω in $(\mathfrak{N}; \mathfrak{M})$. Since ω is semiregular in every nonstandard model, the trick used in proving Theorem 2.6 can be repeated to eliminate the parameter.

ii. By [8], for each $a \in |\mathfrak{N}|$ there is a $b \in |\mathfrak{N}|$ coding satisfaction for parameters less than a: For all $c < a$ and all formulas ϕv with only v free,

$$\mathfrak{N} \models < \bar{c}, \ulcorner \phi \urcorner > \in D_{\bar{b}} \longleftrightarrow \phi \bar{c}.$$

Letting $a \in |\mathfrak{N}| - |\mathfrak{M}|$, define satisfaction for $\mathfrak{M}$ by

$$\mathrm{Sat}_{\mathfrak{M}}(v_0, v_1): \langle v_0, v_1 \rangle \in D_{\bar{b}} \wedge v_1 \in \omega \wedge Fm_1(v_1),$$

where $Fm_1(v)$ is the usual arithmetization of "v codes a formula with one free variable". To eliminate the parameter, simply quantify it out:

$$\forall v_2[\text{“}D_{v_2} \cap \omega \text{ satisfies the clauses of a truth definition for } \mathfrak{M}\text{”} \rightarrow \ldots].$$

iii. Write

$$v_0 \in \text{Bigsky}: \forall v_1 \in |\mathfrak{M}| [v_1 > v_0 \to \exists F[\text{Sat}_{\mathfrak{M}}((v_0, v_1), \ulcorner F((v)_{\bar{0}}) > (v)_{\bar{1}} \urcorner)]],$$

where F ranges over parameter-free Skolem functions. QED

We are now in position to sketch a proof of the following:

2.12 Theorem *Let $\mathfrak{N}$ be recursively saturated. There is a countable infinity of theories, $Th(\mathfrak{N}; \mathfrak{M})$, of structures $(\mathfrak{N}; \mathfrak{M})$ with $\mathfrak{M} \prec_e \mathfrak{N}$ in each of the following cases:*

i. $|\mathfrak{M}|$ *has a highest sky*
ii. $|\mathfrak{N}| - |\mathfrak{M}|$ *has a lowest sky.*

Proof sketch: We consider the case of a highest sky. Let $\tau_0 v$, $\tau_1 v$, . . . be the sequence of types with the properties,

i. for $i \neq j$, $\tau_i v$ and $\tau_j v$ cannot be realized in a common sky
ii. each $\tau_n v$ is recursive in $\text{Th}(\mathfrak{N})$,

which we used in constructing nonisomorphic models $(\mathfrak{N}; \mathfrak{M}_n)$ by insisting $\tau_n v$ be realized in the highest sky of $\mathfrak{M}_n$.

We note simply that, thanks to Theorem 2.11, we can uniformly assert in a first-order way that a given model $(\mathfrak{N}; \mathfrak{M}_n)$ realizes a given type $\tau_m v$ in its highest sky. For since $\text{Th}(\mathfrak{N}) = \text{Th}(\mathfrak{M}_n)$ and the truth definition for $\mathfrak{M}_n$ and ω are uniformly definable in the models $(\mathfrak{N}; \mathfrak{M}_n)$, $\text{Th}(\mathfrak{N})$ (coded as a subset of ω) is similarly uniformly definable. Again, since ω and $\text{Th}(\mathfrak{N})$ are uniformly definable, any set recursive in $\text{Th}(\mathfrak{N})$ is so definable. For each $m \in \omega$, let $T_m v$ define $\tau_m v$ in all models $(\mathfrak{N}; \mathfrak{M}_n)$:

$$(\mathfrak{N}; \mathfrak{M}_n) \models T_m(\ulcorner \phi v \urcorner) \quad \text{iff} \quad \phi v \in \tau_m v.$$

We say $\tau_m v$ is realized in the highest sky of $|\mathfrak{M}|$ as follows:

$$S_m: \exists v_0[\text{Bigsky}(v_0) \wedge \forall v_1[T_m(v_1) \to \text{Sat}_{\mathfrak{M}}(v_0, v_1)]].$$

But

$$(\mathfrak{N}; \mathfrak{M}_n) \models S_m \text{ iff } m = n,$$

whence these models are elementarily inequivalent. QED

It is an open problem whether or not all nonisomorphic models $(\mathfrak{N}; \mathfrak{M})$ with extremal skies are elementarily inequivalent.

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