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GLOBAL EXISTENCE OF WAVE MAPS IN $1+2$ DIMENSIONS WITH FINITE ENERGY DATA

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Main result

Let N be a smooth, compact, m-dimensional Riemannian manifold, isometrically embedded in $\mathbb{R}^d$. A smooth map $u : \mathbb{R} \times \mathbb{R}^2 \to N$ is called a *wave map* if

$$u_{tt} - \Delta u \perp T_u N. \tag{1}$$

Let

$$e(t,x) := \tfrac{1}{2}(|u_t|^2 + |\nabla u|^2) \tag{2}$$

denote the energy density. Smooth wave maps satisfy the energy identity

$$E(t) := \int_{\{t\}\times\mathbb{R}^2} e(t,x)\,dx = \text{const.} \tag{3}$$

In this note we show that the Cauchy problem for (1) admits a weak solution if the initial data have finite energy. For notational convenience we suppose $0 \in N$. We write H^1 for the Sobolev spaces $H^{1,2}(\mathbb{R}^2)$ or $H^{1,2}(\mathbb{R}^2;\mathbb{R}^d)$.

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Theorem 1.1. *Suppose that* $(u_0, u_1) \in H^1 \times L^2$ *and* $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)}N$ *a.e. Then there exists a global (forward) weak solution* $u : \mathbb{R}_0^+ \times \mathbb{R}^2 \to N$ *of the Cauchy problem*

$$u_{tt} - \Delta u \perp T_u N \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^2,$$
$$u(0, \cdot\,) = u_0, \quad u_t(0, \cdot\,) = u_1.$$

Moreover, u satisfies

$$E(t) \le E(0) \quad \forall t \ge 0$$

and

$$(\nabla u(t, \cdot\,), u_t(t, \cdot\,)) \to (\nabla u_0, u_1) \quad \text{in } L^2 \times L^2, \text{ as } t \to 0.$$

Here u is called a *weak solution* of (1) if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$, $u(t,x) \in N$ $\mathcal{L}^3$-a.e. in $\mathbb{R}^+ \times \mathbb{R}^2$, if

$$\int_{\mathbb{R}^+ \times \mathbb{R}^2} \langle u_t, \varphi_t \rangle - \langle \nabla u, \nabla \varphi \rangle \, dx \, dt = 0$$

for all $\varphi \in H^1(\mathbb{R}^+ \times \mathbb{R}^2)$ with $\varphi(z) \in T_{u(z)}N$ a.e. and compact support, and if

$$u(t, \cdot\,) \rightharpoonup u_0, \quad u_t(t, \cdot\,) \rightharpoonup u_1$$

in the sense of distributions (see Appendix A of [6] for the equivalence of various notions of a weak solution).

Existence of weak solutions was first established by Shatah [14] if $N = S^k$. His result was recently generalized by Freire [5] and Yi Zhou [15] to homogeneous spaces as targets. Short time existence and uniqueness for smooth data can be proved classically by energy methods. For a slightly modified problem that captures the essential difficulties of problem (1) Klainerman and Machedon [9], [10] established short time existence, uniqueness and continuous dependence for initial data in $H^{1+\delta} \times H^\delta$, $\delta > 0$, through new Strichartz type estimates. This exploits the fact that (1) may be written explicitly as a system of hyperbolic differential equations with a particular null-form structure.

The key ingredient in our proof is a compactness result for wave maps under weak convergence ([7], [6]). Given this result a serious technical problem is to find suitable approximate problems for which existence is easy to prove. In this note we follow Yi Zhou [15] and use the viscous approximation

$$u_{tt} - \Delta u - \varepsilon \Delta u_t \perp T_u N. \tag{4}$$

Alternatively, one can use finite-difference approximations of (1) (see [12]).

To explain the compactness theorem it is useful to rewrite (1) as a first order system for du and the connection form of TN. We assume for the remainder of this section that N is parallelizable. Let $(\bar{e}_1, \dots, \bar{e}_m)$ be a smooth orthonormal frame of TN. Then, for a map $u : \mathbb{R}^+ \times \mathbb{R}^2 \to N$, the choice $e_i = \bar{e}_i \circ u$ yields an orthonormal frame $(e_1, \dots, e_m)$ of the pullback bundle $u^{-1}TN$. Let $\theta_i := \langle du, e_i \rangle$ and let ω_{ij} denote the connection form given by $\omega_{ij} := \langle de_i, e_j \rangle$, where $\langle \, , \rangle$ is the scalar product in $\mathbb{R}^d$. The Lorentzian codifferential δ and the Lorentzian contraction act on 1-forms $\varphi = \varphi_0 dt + \varphi_1 dx^1 + \varphi_2 dx^2 = \varphi_\alpha dx^\alpha$, $\psi = \psi_\alpha dx^\alpha$ by $\delta\varphi = \partial_t \varphi_0 - \partial_1 \varphi_1 - \partial_2 \varphi_2$, $\varphi \cdot \psi = -\varphi_0 \psi_0 + \varphi_1 \psi_1 + \varphi_2 \psi_2$. With this notation equation (1) is equivalent to the system

$$\delta\theta_i + \omega_{ij} \cdot \theta_j = 0, \quad 1 \le i \le m. \tag{5}$$

This is a straightforward calculation for smooth map; for the equivalence of weak solutions see Appendix A of [6].

Other frames $\tilde{e}_i$ of $u^{-1}TN$ can be obtained by the gauge transformation

$$\widetilde{e}_i(x) = R_{ij}(x) e_j(x), \quad R(x) = (R_{ij}(x)) \in \mathrm{SO}(d).$$

This frame invariance can be exploited to (locally) choose a frame for which $\delta_{\mathrm{eucl}} \omega_{ij} = \partial_\alpha \omega_{ij,\alpha} = 0$. Using the identity $d\omega_{ij} = de_i \wedge de_j$, $\mathcal{H}^1$ estimates for Jacobians ([3]), $\mathcal{H}^1$-BMO duality ([4]), and concentration compactness ([11]) one then obtains the following compactness result ([6], Theorem 3.7).

For convenience we state this result in the periodic setting. The Campanato space $L^{2,1}(T^3)$ with norm

$$\|f\|_{L^{2,1}}^2 := \sup_{z \in T^3} \sup_{0<R<1} \frac{1}{R} \int_{B(z,R)} |f|^2 \, d\zeta$$

consists of all $f \in L^2(T^3)$ that satisfy $\|f\|_{2,1} < \infty$. The semiarrow $\rightharpoonup$ denotes weak convergence.

THEOREM 1.2. *Let N be a parallelizable compact m-dimensional Riemannian manifold and suppose that the maps $v^n : T^3 \to N$ satisfy*

$$v^n \rightharpoonup v \quad \text{in } H^1(T^3; \mathbb{R}^d), \quad \|Dv^n\|_{L^{2,1}} \le C.$$

Then there exist orthonormal frames $(\widetilde{e}_1^n, \dots, \widetilde{e}_m^n)$ of $(v^n)^{-1}TN$ such that $\widetilde{e}_i^n \rightharpoonup \widetilde{e}_i$ in $H^1(T^3, \mathbb{R}^d)$, $(\widetilde{e}_1, \dots, \widetilde{e}_m)$ is an orthonormal frame of $v^{-1}TN$ and

$$\omega_{ij}^n \cdot \theta_j^n \rightharpoonup \omega_{ij} \cdot \theta_j + \nu_i \tag{6}$$

in the sense of distributions, where ν_i is a Radon measure that satisfies

$$\operatorname{supp} \nu_i \subset \{z \in T^3 : \limsup_{R \to 0} \limsup_{n \to \infty} \|\chi_{B(z,R)} \theta^n\|_{L^{2,1}} > 0\}. \tag{7}$$

We will see that in this form, the convergence result can be directly applied to show weak convergence of solutions (u_ε) of the Cauchy problem for (4) to (forward) weak solutions of (1), (2).

2. Regularized wave maps

In this section we establish the global existence of solutions $u : \mathbb{R}^+_0 \times \mathbb{R}^2 \to N$ to the Cauchy problem for the regularized equation

$$u_{tt} - \Delta u + \varepsilon \Delta u_t \perp T_u N.$$

We do not assume that N is parallelizable. Throughout this section we suppose

$$0 < \varepsilon \leq 1.$$

We first derive an expression for the normal component of $u_{tt} - \Delta u + \varepsilon \Delta u_t$ for an arbitrary (sufficiently smooth) map $u : \mathbb{R}^+ \times \mathbb{R}^2 \to N$. Let π denote the nearest neighbour projection of a neighbourhood of N to N and let $P = \nabla \pi$. For $u \in N$, the linear map $P(u)$ is the orthogonal projection $\mathbb{R}^d \to T_u N$. Let $Q(u) = \mathrm{Id} - P(u)$ denote the projection on the normal space. We have (with summation over $i \in \{1, 2\}$)

$$\begin{aligned}
P(u)u_{tt} &= (P(u)u_t)_t - P(u)_t u_t \\
&= u_{tt} - \nabla^2 \pi(u)(u_t, u_t), \\
P(u)\Delta u &= \Delta u - \nabla^2 \pi(u)(\partial_i u, \partial_i u), \\
P(u)\Delta u_t &= (P(u)\Delta u)_t - (P(u))_t \Delta u \\
&= \Delta u_t - (\nabla^2 \pi(u)(\partial_i u, \partial_i u))_t - \nabla^2 \pi(u_t, \Delta u),
\end{aligned}$$

and thus the normal component is given by

$$\begin{aligned}
Q(u)(u_{tt} - \Delta u - \varepsilon \Delta u_t) &= \nabla^2 \pi(u)(\partial_t u, \partial_t u) - \nabla^2 \pi(u)(\partial_i u, \partial_i u) \\
&\quad - \varepsilon(\nabla^2 \pi(u)(\partial_i u, \partial_i u))_t - \varepsilon \nabla^2 \pi(u)(u_t, \Delta u) =: T(u).
\end{aligned} \tag{8}$$

In particular, for maps u with values in N equation (4) is equivalent to

$$u_{tt} - \Delta u - \varepsilon \Delta u_t = T(u). \tag{9}$$

We consider Cauchy initial data

$$u(0, \cdot\,) = u_0, \quad u_t(0, \cdot\,) = u_1. \tag{10}$$

LEMMA 2.1. *Suppose that* $(u_0, u_1) \in H^2 \times H^1$, $u_0(x) \in N$, $u_1(x) \in T_{u_0(x)} N$ *a.e. Then the initial value problem* (9), (10) *has a unique global solution* $u : \mathbb{R}^+_0 \times \mathbb{R}^2 \to N$ *in the class*

$$X = H^1_{\mathrm{loc}}([0, \infty); H^2) \cap H^{1,\infty}_{\mathrm{loc}}([0, \infty); H^1) \cap H^2_{\mathrm{loc}}([0, \infty); L^2).$$

Moreover, the energy identity

$$E(t) + \varepsilon \int_0^t \int_{\mathbb{R}^2} |\nabla u_t|^2 \, dx \, dt = E(0)$$

holds.

Proof. This result appears already in Yi Zhou [15]. Since the proof of global existence may not be obvious to non-experts we sketch a proof of Lemma 2.1.

Local existence is established by the usual fixed point argument. For convenience we scale

$$v(t,x) = u(\varepsilon t, \varepsilon x)$$

to achieve $\varepsilon = 1$. To simplify the notation in the following we again write u for v. Consider the spaces

$$X_{t,u_0,u_1} := \{u \in H^1(0,t;H^2) \cap H^{1,\infty}(0,t;H^1) \cap H^2(0,t;L^2) : \\ u(0,\,\cdot\,) = u_0,\ u_t(0,\,\cdot\,) = u_1\}, \\ Y_t := L^2(0,t;L^2)$$

with norms

$$\begin{aligned} \|u\|_{X_t}^2 &:= \int_0^t [\|u\|_{H^2}^2(s) + \|u_t\|_{H^2}^2(s) + \|u_{tt}\|_{L^2}^2(s)]\,ds \\ &\quad + \operatorname*{ess\,sup}_{s\in(0,t)}[\|u\|_{H^2}^2(s) + \|u_t\|_{H^1}^2(s)], \\ \|f\|_{Y_t}^2 &:= \int_0^t \|f\|_{L^2}^2(s)\,ds. \end{aligned}$$

Here $\|u\|_{H^2}(s), \dots$ denote the spatial norms at fixed times and we will abbreviate $\|u\|_2(s) := \|u\|_{L^2}(s)$, etc. and will suppress s when no confusion can occur.

For $f \in Y_t$ the linear equation

$$u_{tt} - \Delta u - \Delta u_t = f \tag{11}$$

has a unique solution in X_t. Testing with Δu_t we obtain the estimate

$$\frac{d}{dt}(\|\nabla u_t\|_2^2 + \|\Delta u\|_2^2)(s) + \|\Delta u_t\|_2^2(s) \le \|f\|_2^2(s) \quad \text{for a.e. } s \in (0,t). \tag{12}$$

It follows that

$$\begin{aligned} &\operatorname*{ess\,sup}_{s\in(0,t)}(\|\nabla u_t\|_2^2 + \|\Delta u\|_2^2)(s) \le \|f\|_{Y_t}^2 + \|u_1\|_{H^1}^2 + \|u_0\|_{H^2}^2, \\ &\|\Delta u_t\|_{Y_t}^2 \le \|f\|_{Y_t}^2 + \|u_1\|_{H^1}^2 + \|u_0\|_{H^2}^2. \end{aligned} \tag{13}$$

In view of the identity $\|u_{tt}\|_Y = \|\Delta u + \varepsilon \Delta u_t + f\|_Y$, the Sobolev estimates (for $s \in (0,t)$),

$$\|u_t(s,\,\cdot\,) - u_1\|_2 \le Ct^{1/2}\|u_{tt}\|_{Y_t}, \tag{14}$$

$$\|\nabla u(s,\,\cdot\,) - \nabla u_0\|_2 \le Ct \operatorname*{ess\,sup}_{s\in(0,t)} \|\nabla u_t\|_2, \tag{15}$$

$$\|u(s,\,\cdot\,) - u_0\|_2 \le Ct \operatorname*{ess\,sup}_{s\in(0,t)} \|u_t\|_2, \tag{16}$$

and the identity $\|\nabla^2 u\|_2 = \|\Delta u\|_2$ we deduce that, for $t \in (0,1)$,

$$\|u\|_{X_t} \le C(1+t^{3/2})(\|f\|_{Y_t} + \|u_0\|_{H^2} + \|u_1\|_{H^1}). \tag{17}$$

To proceed, we globally extend the map π—and hence the operator T in (9) to an arbitrary, sufficiently smooth map $u : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^d$—as follows.

Let $2\delta > 0$ be the radius of a tubular neighbourhood $U_{2\delta}(N)$ of N such that the above projection π is smooth and uniquely defined as a map $\pi : U_{2\delta}(N) \to N$.

Let $\chi \in C^\infty(\mathbb{R})$ denote a function such that $\chi(s) = s$ for $s \le \delta^2$, $\chi(s) = \frac{3}{2}\delta^2$ for $s \ge 2\delta^2$, and $\chi'(s) \ge 0$ for all s. The map ϱ given by

$$\varrho(u) = \chi\biggl(\frac{\operatorname{dist}^2(u,N)}{2}\biggr) \quad \text{for } u \in U_{2\delta}(N)$$

then extends to a smooth map on $\mathbb{R}^d$ with gradient

$$\nabla\varrho(u) = u - \pi(u) \quad \text{for } u \in U_\delta(N).$$

Defining

$$\overline{\pi}(u) = u - \nabla\varrho(u), \quad u \in \mathbb{R}^d,$$

we thus obtain the desired smooth extension of the nearest neighbour projection π to a map $\overline{\pi} : \mathbb{R}^d \to \mathbb{R}^d$. In the following, we again write π for $\overline{\pi}$. Observe that

$$\pi(u) = u \quad \text{for } u \notin U_{2\delta}(N);$$

hence

$$|\nabla^k \pi(u)| \le C_k = C_k(N) \quad \text{for all } u \in \mathbb{R}^d,\ k \ge 1.$$

To establish short time existence of solutions of (9), (10) it suffices to show that the map $\mathcal{T} : u \to T(u)$ has the following properties:

(18) $\quad \mathcal{T}$ maps bounded subsets of X_{t,u_0,u_1} to bounded subsets of Y_t;

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{Y_t} \le C(R)t^{1/4}\|u-v\|_{X_t} \quad \text{for } t \le 1, \tag{19}$$

where $R = \max(\|u\|_{X_t}, \|v\|_{X_t})$.

To show boundedness of $\mathcal{T}$ note that

$$|T(u)| \le C(|u_t|^2 + |\nabla u|^2 + |u_t|\cdot|\nabla u|^2 + |\nabla u|\cdot|\nabla u_t| + |u_t|\cdot|\Delta u|). \tag{20}$$

By Ladyzhenskaya's inequality and the identity $\|\nabla^2 u\|_2 = \|\Delta u\|_2$ we have, at fixed time,

$$\|u_t^2\|_2^2 \le \|u_t\|_4^4 \le C\|u_t\|_2^2\|\nabla u_t\|_2^2, \tag{21}$$

$$\||\nabla u|^2\|_2^2 \le \|\nabla u\|_4^4 \le C\|\nabla u\|_2^2\|\Delta u\|_2^2, \tag{22}$$

$$\||u_t|\cdot|\nabla u|^2 + |u_t\Delta u|\|_2^2 \le C\|u_t\|_\infty^2(1+\|\nabla u\|_2^2)\|\Delta u\|_2^2, \tag{23}$$

$$\||\nabla u|\cdot|\nabla u_t|\|_2^2 \le \|\nabla u\|_4^2\|\nabla u_t\|_4^2 \le C\|\nabla u\|_2\|\Delta u\|_2\|\nabla u_t\|_2\|\Delta u_t\|_2, \tag{24}$$

and hence

$$\| \, |\nabla u| \cdot |\nabla u_t| \, \|_2^2 \le \frac{C}{\delta} \|\nabla u\|_2^2 \|\nabla u_t\|_2^2 \|\Delta u\|_2^2 + \delta \|\Delta u_t\|_2^2. \tag{25}$$

In view of the estimate (see (31) below for a refinement)

$$\|u_t\|_\infty^2 \le C\|u_t\|_{H^1}\|u_t\|_{H^2} \le \frac{C}{\delta}\|u_t\|_{H^1}^2 + \delta\|\Delta u_t\|^2$$

it follows (with the choice $\delta = 1$ above and in (25)) that

$$\|\mathcal{T}(u)\|_{Y_t}^2 \le C(1 + \|u\|_{X_t}^6) \quad \text{for } t \le 1. \tag{26}$$

For future references we also note the finer estimate

$$\begin{aligned} \|\mathcal{T}(u)\|_{Y_t}^2 \le C(\delta)t(1 + [\sup_s(\|u\|_{H^2} + \|u_t\|_{H^1})(s)]^6) \\ + \delta\|\Delta u_t\|_{Y_t}^2(1 + \sup_s \|u\|_{H^2}^4) \quad \text{for all } t \in \mathbb{R}_0^+. \end{aligned} \tag{27}$$

The proof of the Lipschitz estimate (19) is similar with the following modifications. First, instead of the quadratic and cubic expressions in (21)–(24) one has to estimate similar expressions in u and $w := v - u$. Application of the Sobolev estimates in time (14)–(16) to w yields the additional small factor of $t^{1/4}$ since $w_0 = w_1 = 0$. Second, an additional term that can be estimated by

$$C|w|(|u_t|^2 + |\nabla u|^2 + |u_t| \cdot |\nabla u|^2 + |\nabla u| \cdot |\nabla u_t| + |u_t| \cdot |\Delta u|),$$

plus a similar term with u replaced by v, arises.

In view of (20)–(24) and the estimate

$$\|w\|_\infty(s) \le C\|\Delta w\|_2(s) \le t^{1/2}\|w\|_{X_t}^{1/2} \tag{28}$$

this term poses no additional difficulty.

Hence (18) and (19) hold, and (9), (10) has a solution $u : [0, \widetilde{t}\,) \times \mathbb{R}^2 \to \mathbb{R}^d$ up to a time $\widetilde{t} = \widetilde{t}(u_0, u_1) > 0$. To see that u takes values in N observe that for short times u is uniformly close to N in view of the embedding $H^1(0,t;H^2) \hookrightarrow C^0(0,t;C^0)$. Hence the projection $v := \pi \circ u$ takes values in N. Moreover,

$$\begin{aligned} v_t &= \nabla\pi(u)u_t, \\ v_{tt} &= \nabla\pi(u)u_{tt} + \nabla^2\pi(u)(u_t, u_t), \\ \Delta v &= \nabla\pi(u)\Delta u + \nabla^2\pi(u)(\partial_i u, \partial_i u), \\ \Delta v_t &= \nabla\pi(u)\Delta u_t + \nabla^2\pi(u)(u_t, \Delta u) + (\nabla^2\pi(u)(\partial_i u, \partial_i u))_t. \end{aligned}$$

Using (9) we deduce that $w = v - u$ satisfies

$$\begin{aligned} w_{tt} - \Delta w - \varepsilon\Delta w_t &= (\nabla\pi \circ u)T(u) \\ &= [(\nabla\pi \circ u) - (\nabla\pi \circ v)]T(u) \\ &\quad + (\nabla\pi \circ v)(T(u) - T(v)) + (\nabla\pi \circ v)T(v). \end{aligned}$$

Since v takes values in N it follows from (8) that $(\nabla\pi\circ v)T(v) = 0$. Now $w_0 = w_1 = 0$ and thus by (17), (19) and (26),

$$\begin{aligned}\|w\|_{X_t} &\le C[(\sup_{(0,t)}\|w\|_\infty)\|\mathcal{T}(u)\|_{Y_t} + \|\mathcal{T}(u)-\mathcal{T}(v)\|_{Y_t}]\\ &\le C(R)(t^{1/2}+t^{1/4})\|w\|_{X_t} \quad \text{for } t\le 1,\end{aligned}$$

where $R = \max(\|u\|_{X_t}, \|v\|_{X_t}) \le C\|u\|_{X_t}$. Hence $w\equiv 0$ on $(0,\widehat{t}\,)$ for sufficiently small $\widehat{t}$. Thus for each u_0, u_1 as in Lemma 2.1 there exists a solution $u:(0,\widehat{t}\,)\times\mathbb{R}^2 \to N$ for some $\widehat{t}(u_0,u_1)>0$. By the usual continuation argument this solution can be extended to a maximal time interval $(0,t^*)$ and we will see that $t^*=\infty$ unless

$$\limsup_{t\nearrow t^*} \|u\|_{H^2}(t) + \|u_t\|_{H^1}(t) = \infty. \tag{29}$$

Indeed, if $\|u\|_{H^2}(t)+\|u_t\|_{H^1}(t)$ remains bounded by C_0 as $t\nearrow t^*$ then (13), (27) (with $\delta = 1/(2(1+C_0^4))$) and (17) imply that $\|\Delta u_t\|_{Y_t}$ and $\|u\|_{X_t}$ also remain bounded as $t\nearrow t^*$. Hence $u(t,\cdot)\to\overline{u}_0$ in H^2 and $u_t(t,\,\cdot\,)\to\overline{u}_1$ in L^2 (and weakly in H^1) as $t\nearrow t^*$ and thus $\overline{u}_0$ takes values in N and $\overline{u}_1\in T_{\overline{u}_0}N$. Therefore one can solve locally with initial data $\overline{u}_0,\overline{u}_1$ and thus extend the solution beyond $(0,t^*)$.

To establish global existence we use a Gronwall type estimate to show that (29) cannot hold for $t^*<\infty$.

Testing (9) with $u_t\in T_uN$ we obtain the energy identity

$$\frac{1}{2}(\|u_t\|_2^2 + \|\nabla u\|_2^2)(t) + \varepsilon\int_0^t \|\nabla u_t\|_2^2 = E_0. \tag{30}$$

We now return to the estimates (20)–(23) and use the Brezis–Waigner inequality for u_t:

$$\|u_t\|_\infty \le C\|u_t\|_{H^1}\left[1+\ln^{1/2}\left(1+\frac{\|u_t\|_{H^2}}{\|u_t\|_{H^1}}\right)\right]. \tag{31}$$

In view of the estimate $ab\le e^a + b\ln b$ (for $b>0$) we deduce that (for $0<\delta\le 1$)

$$\begin{aligned}\|u_t\|_\infty^2\|\Delta u\|_2^2 &\le C\|u_t\|_{H^1}^2\left[1+\ln\left(1+\frac{\|u_t\|_{H^2}}{\|u_t\|_{H^1}}\right)\right]\|\Delta u\|_2^2\\ &\le C\|u_t\|_{H^1}^2\left[\left(1+\frac{\|u_t\|_{H^2}}{\|u_t\|_{H^1}}\right)+\|\Delta u\|_2^2\,\ln\|\Delta u\|_2^2\right]\\ &\le C\|u_t\|_{H^1}^2 + C\|u_t\|_{H^1}(\|u_t\|_{H^1}+\|\Delta u_t\|_2)\\ &\quad + C\|u_t\|_{H^1}^2\|\Delta u\|_2^2\ln\|\Delta u\|_2^2.\\ &\le \left(\delta\|\Delta u_t\|_2^2+\frac{C}{\delta}\|u_t\|_{H^1}^2\right)+C\|u_t\|_{H^1}^2\|\Delta u\|_2^2\ln\|\Delta u\|_2^2.\end{aligned} \tag{32}$$

If we let

$$h(t) := \tfrac{1}{2}(\|\nabla u_t\|_2^2+\|\Delta u\|_2^2), \quad g(t) := \|\nabla u_t\|_2^2,$$

take $\delta = 1/4$ in (32) as well as in (25) and denote by C constants that only depend on E_0 we deduce from (20)–(25), (30), (32), (12) and (9) that

$$h' \le C(g + C)(h(\ln^+ h + 1) + C).$$

Since $\int_2^\infty \frac{ds}{1+s+s\ln s} = \infty$ and $\int_0^{t^*} g\,dt \le E_0$ it follows that $h(t)$ remains bounded as $t \nearrow t^*$. This contradicts (29). Thus $t^* = \infty$, proving global existence. $\square$

To characterize the singular set $\operatorname{supp}\nu_i$ in Theorem 1.2 we will use the local energy inequality in Lemma 2.2 below. Due to the regularizing term we cannot expect finite speed of propagation but we show that the influence of points outside the backward light cone becomes exponentially small as $\varepsilon \to 0$.

Let

$$\psi(s) = \begin{cases} \exp(s) & \text{if } s \le 0, \\ 2 - \exp(-s) & \text{if } s > 0, \end{cases}$$

and for $t \le t_0$ let

$$\varphi_\varepsilon(t,x) = \psi([(1+\sqrt{\varepsilon})(t_0 - t) - |x - x_0|]/\sqrt{\varepsilon}).$$

LEMMA 2.2. *Let u be the solution of* (9), (10) *in Lemma* 2.1. *Then, for* $0 < s < t < t_0$,

$$\int_{\{t\}\times\mathbb{R}^2} \varphi_\varepsilon e\,dx \le \int_{\{s\}\times\mathbb{R}^2} \varphi_\varepsilon e\,dx. \tag{33}$$

Moreover, for all balls $B(z_0,r) \subset B(z_0,R) \subset \mathbb{R}^+ \times \mathbb{R}^2$,

$$\frac{1}{r}\int_{B(z_0,r)} e\,dz \le \frac{C}{R}\int_{B(z_0,R)} e\,dz + C\exp\Big(-\frac{R}{2\sqrt{\varepsilon}}\Big)E_0. \tag{34}$$

PROOF. We have, for a.e. t, in the sense of distributions in $\mathbb{R}^2$,

$$\begin{aligned} e_t &= \langle u_{tt} - \Delta u, u_t\rangle + \operatorname{div}\langle \nabla u, u_t\rangle \\ &= \langle \varepsilon\Delta u_t, u_t\rangle + \operatorname{div}\langle \nabla u, u_t\rangle \\ &= -\varepsilon|\nabla u_t|^2 + \operatorname{div}(\varepsilon\langle\nabla u_t, u_t\rangle + \langle\nabla u, u_t\rangle) \end{aligned}$$

and thus, abbreviating $\varphi = \varphi_\varepsilon$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \varphi e &= \int_{\mathbb{R}^2} (\varphi e_t + e\varphi_t)\,dx \\ &\le \int_{\mathbb{R}^2} (-\varepsilon|\nabla u_t|^2\varphi + \varepsilon|\nabla u_t|\cdot|u_t|\cdot|\nabla\varphi| + |\nabla u|\cdot|u_t|\cdot|\nabla\varphi| + e\varphi_t)\,dx \\ &\le \int_{\mathbb{R}^2} \Big(-\varepsilon|\nabla u_t|^2\varphi + \frac{\varepsilon^{3/2}}{2}|\nabla u_t|^2|\nabla\varphi| + \frac{\varepsilon^{1/2}}{2}u_t^2|\nabla\varphi| + e(|\nabla\varphi| + \varphi_t)\Big)\,dx \\ &\le \int_{\mathbb{R}^2} [\varepsilon|\nabla u_t|^2(\sqrt{\varepsilon}|\nabla\varphi| - \varphi) + e((1+\sqrt{\varepsilon})|\nabla\varphi| + \varphi_t)]\,dx \\ &\le 0 \quad \text{for a.e. } t. \end{aligned}$$

Since $t \to \int_{\mathbb{R}^2} \varphi e$ is absolutely continuous (in fact in $H^1_{\mathrm{loc}}(t, t_0)$) this proves the first inequality. To establish the second estimate, note that we may assume $r \le R/16$ and that at the expense of increasing the constants we may replace balls by cylinders $C(z_0, r) = [t_0 - r, t_0 + r] \times B(x_0, r)$. It follows from (33) that, for $s \in [t + r - R/4, t]$,

$$\begin{aligned}\int_{\{t\}\times B(x_0,r)} e\,dx &\le \int_{\{s\}\times\mathbb{R}^2} e\psi([(1+\sqrt{\varepsilon})(t-s) + r - |x - x_0|]/\sqrt{\varepsilon})\,dx \\ &\le \int_{\{s\}\times B(x_0,R)} 2e\,dx + \int_{\{s\}\times(\mathbb{R}^2\setminus B(x_0,R))} \exp\bigg(-\frac{R}{2\sqrt{\varepsilon}}\bigg) e\,dx \\ &\le \int_{\{s\}\times B(x_0,R)} 2e\,dx + \exp\bigg(-\frac{R}{2\sqrt{\varepsilon}}\bigg) E_0.\end{aligned}$$

Integration over $t \in [t_0 - r, t_0 + r]$ and over $s \in \big[t_0 - \frac{1}{8}R, t_0 - \frac{1}{16}R_0\big]$ yields the desired estimate for cylinders. □

3. Existence of wave maps

Proof of Theorem 1.1. By a construction of Schoen and Uhlenbeck ([13], Section 4) there exist $u_{0\varepsilon} \in C_0^\infty(\mathbb{R}^2, N)$ and $\widetilde{u}_{1\varepsilon} \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^d)$ such that

$$u_{0\varepsilon} \to u \quad \text{in } H^1, \quad \widetilde{u}_{1\varepsilon} \to u_1 \quad \text{in } L^2. \tag{35}$$

Then $P(u_{0\varepsilon}) \to P(u)$ boundedly a.e. and thus

$$u_{1\varepsilon} := P(u_{0\varepsilon})\widetilde{u}_{1\varepsilon} \to u_1 \quad \text{in } L^2. \tag{36}$$

By Lemma 2.1 there exists a global solution $u_\varepsilon : \mathbb{R}_0^+ \times \mathbb{R}^2 \to N$ of the Cauchy problem

$$\begin{gathered} u_{\varepsilon tt} - \Delta u_\varepsilon - \varepsilon\Delta u_{\varepsilon t} \perp T_{u_\varepsilon} N, \\ u_\varepsilon(0, \cdot\,) = u_{0\varepsilon}, \quad u_{\varepsilon t}(0, \cdot\,) = u_{1\varepsilon}. \end{gathered} \tag{37}$$

In view of the energy identity (30) there exists a sequence $\varepsilon_n \to 0$ such that $u^n := u_{\varepsilon_n}$ satisfies

$$u^n \to u \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}_0^+; \mathbb{R}^2), \tag{38}$$

$$Du^n \overset{*}{\rightharpoonup} Du \quad \text{in } L^\infty(\mathbb{R}_0^+; L^2), \tag{39}$$

$$\varepsilon\nabla u_t^n \to 0 \qquad \text{in } L^2(\mathbb{R}_0^+ \times \mathbb{R}). \tag{40}$$

We claim that u is a weak wave map in $\mathbb{R}^+ \times \mathbb{R}^2$. It suffices to check the assertion for every cube $Q' = Q(z_0, r) = z_0 + (-r, r)^3$ and we may assume $Q(z_0, 2r) \subset \mathbb{R}^+ \times \mathbb{R}^2$. Fix such a cube. We assume for convenience that $r = 1/4$, the general case follows by scaling. By reflection across the planes $z^\alpha - z_0^\alpha = \pm 1/4$ and periodic extension we obtain maps $v^n : T^3 \to N$ that satisfy

$$v^n_{|Q'} = u^n, \quad \|Dv^n\|_{L^2(\{t\}\times T^2)} \le 2\|Du^n\|_{L^2(\{t\}\times Q(x_0,1/4))} \le C.$$

The last estimate implies that Dv^n is bounded in $L^{2,1}(T^3)$.

To proceed, we first make the additional assumption that N is parallelizable. Let $(\widetilde{e}^n_1, \dots, \widetilde{e}^n_m)$ be the frames of $(v^n)^{-1}TN$ whose existence is asserted in Theorem 1.2 and let $\theta^n_i = \langle dv^n, \widetilde{e}^n_i\rangle, \omega^n_{ij} = \langle d\widetilde{e}^n_i, \widetilde{e}^n_j\rangle$. Testing (22) with $\eta\widetilde{e}^n_i$ for $\eta \in C^\infty_0(Q')$ we obtain (cf. Section 1)

$$\delta\theta^n_i + \omega^n_{ij} \cdot \theta^n_j = \operatorname{div}\langle \varepsilon\nabla u_{\varepsilon t}, \widetilde{e}^n_i\rangle - \langle \varepsilon\nabla u_{\varepsilon t}, \nabla\widetilde{e}^n_i\rangle \tag{41}$$

in the sense of distributions in Q'. It follows from (38)–(40) and Theorem 1.2 that

$$\delta\theta_i + \omega_{ij} \cdot \theta_j = \nu_i, \tag{42}$$

in the sense of distributions in Q', where

$$\operatorname{supp}\nu_i \subset S := \{z \in Q' : \limsup_{R\to 0}\limsup_{n\to\infty} \|\chi_{B(z,R)}Du^n\|_{L^{2,1}} > 0\}.$$

To finish the argument we show that S has vanishing $H^{1,2}$ capacity and thus $\nu_i = 0$ since the left hand side of (42) is in $H^{-1} + L^1$ (cf. [6] for further details). Indeed, passing to a further subsequence we may assume that

$$|Du^n|^2 \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(Q'). \tag{43}$$

Since $\varepsilon_n \to 0$ it then follows from the "monotonicity formula" (34) that

$$\limsup_{n\to\infty} \|\chi_{B(z,R)}Du^n\|_{L^{2,1}} \le CR^{-1}\mu(B(z,2R))$$

and hence that

$$S \subset \left\{z \in Q' : \limsup_{R\to 0} \frac{1}{R}\mu(B(z,R)) > 0\right\};$$

see [6] for the details.

Now the set on the right hand side is a countable union of sets of finite one-dimensional Hausdorff measure and hence has vanishing H^1 capacity. Therefore

$$\delta\theta_i + \omega_{ij} \cdot \theta_j = 0 \tag{44}$$

as distributions in Q' and thus u is a weak wave map in $\mathbb{R}^+ \times \mathbb{R}^2$.

If N is not parallelizable we use the fact that by [2] or [8], N is a totally geodesic submanifold of a compact parallelizable Riemannian manifold $\widetilde{N}$, which in turn we may assume to be isometrically embedded in $\mathbb{R}^d$. Let $U \subset \mathbb{R}^+_0 \times \mathbb{R}^2$ be open. Since the second fundamental forms of N and $\widetilde{N}$ agree on $TN \times TN$, we have, for all $v \in H^1(U;N)$ and all $\psi \in H^1(U; v^{-1}T\widetilde{N})$ with $\psi(v(x)) \perp T_{v(x)}N$,

$$\langle \partial_i v, \partial_i\psi\rangle = 0, \quad \langle \partial_t v, \partial_t\psi\rangle = 0. \tag{45}$$

(It suffices to approximate $\partial_i v$ by $(P^N \circ v)(\varrho_\varepsilon * \partial_i v)$, the projection to $T_{v(x)}N$ of the standard mollification.) Let $\Pi(p)$ denote the orthogonal projection $T_p\widetilde{N} \to T_pN$ and extend $\Pi(p)$ as the identity on $(T_p\widetilde{N})^\perp$. Then

$$\|\partial_i[(\Pi \circ v)\varphi]\|_2 \le C(\|\varphi\|_\infty + \|\nabla\varphi\|_2), \quad \forall \varphi \in (H^1 \cap L^\infty)(U;\mathbb{R}^d). \tag{46}$$

Now let $v^n : T^3 \to N \hookrightarrow \widetilde{N}$ as above and apply Theorem 1.2 with N replaced by $\widetilde{N}$. Let $(\widetilde{e}_1^n, \ldots, \widetilde{e}_{m'}^n)$ be the corresponding frame of $\widetilde{N}$ and let $\theta_i^n = \langle dv^n, \widetilde{e}_i^n\rangle$, $\omega_{ij}^n = \langle d\widetilde{e}_i^n, \widetilde{e}_j^n\rangle$. As above we obtain, for $\eta \in C_0^\infty(Q')$,

$$\langle \delta\theta_i^n + \omega_{ij}^n \cdot \theta_j^n, \eta\rangle = \int_{Q'} \langle dv^n, d(\eta\widetilde{e}_i^n)\rangle,$$

and application of (45) and (46) with $\varphi^n = \eta\widetilde{e}_i^n$ and $\Pi^n = \Pi \circ v^n$ yields

$$\begin{aligned}\int_{Q'} \langle dv^n, d\varphi^n\rangle &= \int_{Q'} \langle dv^n, d(\Pi^n\varphi^n)\rangle = \int_{Q'} \langle v_{tt}^n - \Delta v^n, \Pi^n\varphi^n\rangle \\ &= \int_{Q'} \langle \varepsilon_n \Delta v_t^n, \Pi^n\varphi^n\rangle \le \|\varepsilon_n\nabla v_t^n\|_2 \|\nabla(\Pi^n\varphi^n)\|_2 \\ &\le C\varepsilon_n \|\nabla v_t^n\|_2 \to 0\end{aligned}$$

as $n \to \infty$. Hence

$$\delta\theta_i^n + \omega_{ij}^n \cdot \theta_j^n \rightharpoonup 0$$

in the sense of distributions, and as before we conclude that v is a weak wave map (with values in $\widetilde{N}$). Since $v^n \to v$ in L^2_{loc} the limit v is a weak wave map with values in N.

While (44) was derived through the use of special frames the equation is frame-invariant (see [6], Appendix A for the weak setting) and hence holds in particular if θ and ω are defined with respect to the frame given by $e_i = \overline{e}_i \cdot u$, where $(\overline{e}_i)$ is a fixed frame of N. From now on we will work in this frame.

It remains to show that u attains the correct initial values. We know that

$$\begin{aligned}&u^n \text{ is bounded in } L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+; H^1),\\ &u_t^n \text{ is bounded in } L^\infty(\mathbb{R}_0^+; L^2).\end{aligned}$$

Thus

$$\|u^n(t,\,\cdot\,) - u_0^n\|_2 \le Ct \tag{47}$$

and, by (39) and the Aubin–Lions lemma,

$$u^n(t,\,\cdot\,) \rightharpoonup u(t,\,\cdot\,) \quad \text{in } H^1, \text{ for all } t \ge 0.$$

Letting $n \to \infty$ and $t \to 0$ in (47) and taking into account (38), (39) and (35) we deduce

$$u(t,\,\cdot\,) \to u_0 \quad \text{in } L^2, \quad u(t,\,\cdot\,) \rightharpoonup u_0 \quad \text{in } H^1.$$

To establish convergence of u_t recall that the normal component of u^n_{tt} is given by

$$(\mathrm{Id} - P(u^n))u^n_{tt} = (\nabla^2\pi)(u^n)(u^n_t, u^n_t).$$

To estimate the tangential components note that by (45),

$$\langle u^n_{tt}, e^n_i\rangle - \langle u^n_{tt}, \Pi^n e^n_i\rangle = \partial_t\langle u^n_t, e^n_i - \Pi e^n_i\rangle = 0,$$
$$\langle \Delta u^n, e^n_i\rangle - \langle \Delta u^n, \Pi^n e^n_i\rangle = 0,$$

and thus

$$\langle u^n_{tt}, e^n_i\rangle = \mathrm{div}\langle \nabla u^n + \varepsilon\nabla u^n_t, \Pi^n e^n_i\rangle - \langle \nabla u^n + \varepsilon\nabla u^n_t, \nabla(\Pi^n e^n_i)\rangle.$$

In combination with (39) and (40) it follows that

$$u^n_t \text{ is bounded in } L^\infty(\mathbb{R}^+_0; L^2),$$
$$u^n_{tt} \text{ is bounded in } (L^2 + L^\infty)(\mathbb{R}^+_0; (L^\infty \cap H^1)^*).$$

Thus

$$\|u^n_t(t,\,\cdot\,) - u^n_t(s,\,\cdot\,)\|_2 \le C(|s-t|^{1/2} + |s-t|) \tag{48}$$

and by (38) and the Aubin–Lions lemma

$$u^n_t(t,\,\cdot\,) \rightharpoonup u_t(t,\,\cdot\,) \quad \text{in } L^2, \text{ for all } t \ge 0.$$

Letting $n \to \infty$ and $t \to 0$ in (48) and taking into account (36) we obtain $u_t(t,\,\cdot\,) \to u_1$ in $(L^\infty \cap H^1)^*_{\mathrm{loc}}$ and therefore $u_t(t,\,\cdot\,) \rightharpoonup u_1$ in L^2, as $t \to 0$. Thus

$$Du(t,\,\cdot\,) \rightharpoonup (\nabla u_0, u_1) \quad \text{in } L^2, \text{ as } t \to 0, \tag{49}$$

and, for every $t \ge 0$,

$$Du^n(t,\,\cdot\,) \rightharpoonup Du(t,\,\cdot\,) \quad \text{in } L^2, \text{ as } n \to \infty. \tag{50}$$

The energy identity (30) yields

$$\begin{aligned} 2E(t) = \|Du(t,\,\cdot\,)\|_2^2 &\le \liminf_{n\to\infty} \|Du^n(t,\,\cdot\,)\|_2^2 \\ &\le \limsup_{n\to\infty} \|(\nabla u^n_0, u^n_1)\|_2^2 = \|(\nabla u_0, u_1)\|_2^2 = 2E(0). \end{aligned} \tag{51}$$

Thus

$$\limsup_{t\to 0} \|Du(t,\,\cdot\,)\|_2^2 \le \|(\nabla u_0, u_1)\|_2^2$$

and hence strong convergence holds in (49). The proof of Theorem 1.1 is finished.

4. Concluding remarks

A major open problem is the *uniqueness* of wave maps with finite energy in $1+2$ dimensions. Uniqueness in the class of all weak solutions would imply the energy identity $E(t) = E(0)$ for all $t \geq 0$, since otherwise time reversal would yield a weak solution for which the energy increases and which would thus be different from the solution constructed above. From the energy identity and (50), (51) one easily deduces that $Du^n \to Du$ in L^2 for the above approximations (this implies the local energy inequality for u) as well as continuity of the map $\mathbb{R}_0^+ \to L^2$ given by $t \mapsto Du(t, \cdot)$. In particular, concentration of energy would be impossible. It is, however, widely believed that such a phenomenon may occur, as it does, for instance, for the harmonic heat flow $u_t - \Delta u \perp T_u N$ in $1+2$ dimensions (see [1]). Hence, uniqueness is only expected to hold in a more restricted class, defined by conditions such as monotonicity of the energy or local energy inequalitites.

Existence of solutions that enjoy such additional properties remains an open problem. On the other hand, the solutions constructed here still enjoy the *compactness* property originally established only for smooth solutions ([7], [6]). The main point is that in view of (34) the solutions constructed above satisfy, for $B(z_0, r) \subset B(z_0, R) \subset \mathbb{R}^+ \times \mathbb{R}^2$,

$$\frac{1}{r} \int_{B(z_0,r)} |Du|^2 \leq \frac{C}{R} \mu(\overline{B(z_0, R)}),$$

where μ is the Radon measure in (43). If $\{u^l\}$ is a sequence of such solutions with uniformly bounded energy and $u^l \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ then, after passage to a subsequence, we may assume $\mu^l \overset{*}{\rightharpoonup} \widetilde{\mu}$ and

$$\limsup_{l \to \infty} \|\chi_{B(z_0,r)} Du^l\|_{L^{2,1}} \leq \frac{C}{R} \widetilde{\mu}(\overline{B(z_0, 3R)}).$$

It now follows from Theorem 1.2 as above that u is a weak wave map.

References

[1] K. C. Chang, W. Y. Ding and R. G. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom. **36** (1992), 507–515.

[2] D. Christodoulou and A. S. Tahvildar-Zadeh, *On the regularity of spherically symmetric wave maps*, Comm. Pure Appl. Math. **46** (1993), 1041–1091.

[3] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[4] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[5] A. Freire, *Global weak solutions to the wave map system to compact homogeneous spaces*, preprint.

[6] A. Freire, S. Müller and M. Struwe, *Weak compactness of wave maps and harmonic maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire (to appear).

[7] ———, *Weak convergence of wave maps from* $(1+2)$*-dimensional Minkowski space to Riemannian manifolds*, Invent. Math. (to appear).

[8] F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), 591–596.

[9] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), 1221–1268.

[10] ———, *Smoothing estimates for null forms and applications*, Internat. Math. Res. Notes **4** (1994), 383–389.

[11] P.-L. Lions, *The concentration compactness principle in the calculus of variations, the limit case, part II*, Rev. Mat. Iberoamericana **12** (1985), 45–121.

[12] S. Müller, M. Struwe and V. Šverák, in preparation.

[13] R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Differential Geom. **18** (1983), 253–268.

[14] J. Shatah, *Weak solutions and development of singularities in the* $SU(2)\sigma$*-model*, Comm. Pure Appl. Math. **41** (1988), 459–469.

[15] Y. Zhou, *Global weak solutions for* $1+2$ *dimensional wave maps into homogeneous spaces*, preprint, 1995.

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