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INFINITELY MANY SOLUTIONS FOR OPERATOR EQUATIONS INVOLVING DUALITY MAPPINGS ON ORLICZ–SOBOLEV SPACES

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Abstract. Let X be a real reflexive and separable Banach space having the Kadeč–Klee property, compactly imbedded in the real Banach space V and let $G\colon V \to \mathbb{R}$ be a differentiable functional.

By using "fountain theorem" and "dual fountain theorem" (Bartsch [3] and Bartsch–Willem [4], respectively), we will study the multiplicity of solutions for operator equation

$$J_\varphi u = G'(u),$$

where J_φ is the duality mapping on X, corresponding to the gauge function φ.

Equations having the above form with J_φ a duality mapping on Orlicz–Sobolev spaces are considered as applications. As particular cases of the latter results, some multiplicity results concerning duality mappings on Sobolev spaces are derived.

1. Introduction

This paper is concerned with multiplicity results for equations of type

$$J_\varphi u = G'(u), \tag{1.1}$$

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where

(a) X is a real reflexive and separable Banach space having the Kadeč–Klee property, compactly imbedded in the real Banach space V;
(b) $J_\varphi\colon X \to X^*$ is a duality mapping corresponding to the gauge function φ (see Definition 2.2 below);
(c) $G'\colon V \to V^*$ is the differential of the functional $G\colon V \to \mathbb{R}$.

As usual, X^* (resp. V^*) denotes the dual space of X (resp. V) and $\langle\,\cdot\,,\,\cdot\,\rangle_{X,X^*}$ (resp. $\langle\,\cdot\,,\,\cdot\,\rangle_{V,V^*}$) denotes the duality pairing between X^* and X (resp. V^* and V).

Often, we shall omit to indicate the spaces in duality and, simply, we shall write $\langle\,\cdot\,,\,\cdot\,\rangle$.

Our approach is a variational one, the so called "fountain theorem" and "dual fountain theorem" (Bartsch [3] and Bartsch–Willem [4] respectively, see also Willem [19]) being the basic ingredients which are used.

Equations having the form (1.1) with J_φ a duality mapping on Orlicz-Sobolev spaces are considered as applications. As particular cases of these results, some multiplicity results concerning duality mappings on Sobolev spaces are derived.

More particularly, these results apply to many differential operators which are, in fact, duality mappings on some appropriate spaces of functions (for example, if Δ_p, $1 < p < \infty$, is the so called p-Laplacian, then $-\Delta_p$ is the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $t \geq 0$).

2. The main result

Let X be a real reflexive and separable Banach space. It is well known that there are $E = \{e_1, \ldots, e_n, \ldots\} \subset X$ and $F = \{f_1, \ldots, f_n, \ldots\} \subset X^*$ such that $X = \overline{\mathrm{Sp}(E)}$, $X^* = \overline{\mathrm{Sp}(F)}$ and

$$\langle f_i, e_j\rangle = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

For what follows, we shall note

$$X_j = \mathrm{Sp}(\{e_j\}), \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{2.1}$$

Theorem 2.1. *Let X be a real reflexive, smooth and separable Banach space having the Kadeč–Klee property and compactly imbedded in the real Banach space V. Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional having the form*

$$H = \Psi - G, \tag{2.2}$$

where

(a) *at any* $u \in X$, $\Psi(u) = \Phi(\|u\|)$, *with*

$$\Phi(t) = \int_0^t \varphi(\xi)\, d\xi, \quad \textit{for all } t \geq 0, \tag{2.3}$$

$\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ *being a gauge function which satisfies*

$$p^* = \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty;$$

(b) $G: V \to \mathbb{R}$ *satisfies:*

(b)$_1$ $G': V \to V^*$ *is demicontinuous;*

(b)$_2$ *there is a constant* $\theta > p^*$ *such that, at any* $y \in V$,

$$\langle G'(y), y\rangle_{V,V^*} - \theta G(y) \geq C = \text{const.}$$

(c) *for any* $u \in X$, *with* $\|u\|_X > 1$, *one has*

$$H(u) \geq c_1 \|u\|_X^p - c_2 \|i(u)\|_V^q - d, \tag{2.4}$$

where i *stands for the compact injection of* X *in* V *while* $q > p > 0$, $c_1 > 0$, $c_2 > 0$ *and* d *are real constants;*

(d) *for any* $k \in \mathbb{N}^*$ *and* $u \in Y_k$, *with* $\|u\|_X > 1$, *one has*

$$H(u) \leq c_3 \|u\|_X^r - c_4 \|u\|_X^s + c_5,$$

where $s > 0$, $r < s$, $c_4 > 0$, c_3 *and* c_5 *are real constants.*

(e) *there exist* $p_* > 1$ *and the positive constants* c_7, c_8 *such that*

$$|G(y)| \leq c_7 \|y\|_V + c_8 \|y\|_V^{p_*}, \tag{2.5}$$

for any $y \in i(X)$.

Then, the functional H *possesses a sequence of critical positive values which converges to* $+\infty$ *and another one, of critical negative values converging to* 0.

Before proceeding to the proof of Theorem 2.1, we list some results we need.

First, we recall that a real Banach space X is said to be *smooth* if it has the following property: for any $x \in X$, $x \neq 0$, there exists a unique $u^*(x) \in X^*$ such that $\langle u^*(x), x\rangle = \|x\|_X$ and $\|u^*(x)\|_{X^*} = 1$. It is well known (see, for instance, Diestel [8], Zeidler [20]) that the smoothness of X is equivalent with the Gâteaux differentiability of the norm. Consequently, if $(X, \|\cdot\|_X)$ is smooth, then, for any $x \in X$, $x \neq 0$, the only element $u^*(x) \in X^*$ with the properties $\langle u^*(x), x\rangle = \|x\|_X$ and $\|u^*(x)\|_{X^*} = 1$ is $u^*(x) = \|\cdot\|'_X(x)$ (where $\|\cdot\|'_X(x)$ denotes the Gâteaux gradient of the $\|\cdot\|_X$-norm at x).

A function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *gauge* function if φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Definition 2.2. If X is a real smooth Banach space and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a gauge function, the duality mapping on X corresponding to φ is the mapping $J_\varphi\colon X \to X^*$ defined by

$$J_\varphi 0 = 0, \quad J_\varphi x = \varphi(\|x\|_X)\|\cdot\|'_X(x), \quad \text{if } x \neq 0.$$

The following metric properties are consequent:

$$\|J_\varphi x\|_{X^*} = \varphi(\|x\|_X),$$
$$\langle J_\varphi x, x\rangle = \varphi(\|x\|_X)\|x\|_X, \quad \text{for all } x \in X.$$

Definition 2.3. A real Banach space has the Kadeč–Klee property if it is strictly convex and

$$\text{if } x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\| \text{ then } x_n \to x.$$

Remark 2.4. Any locally uniformly convex Banach space (in particular, any uniformly convex Banach space) has the Kadeč–Klee property. For proof, we refer to Diestel [8].

Definition 2.5. Let X be a real Banach space. The operator $T\colon X \to X^*$ is said to satisfy condition $(\mathrm{S})_+$ if and only if, as $n \to \infty$, the following holds:

$$x_n \rightharpoonup x \text{ and } \limsup_{n\to\infty} \langle Tx_n, x_n - x\rangle \le 0 \text{ implies } x_n \to x.$$

Proposition 2.6. *If X is a real smooth Banach space having the Kadeč–Klee property, then, any duality mapping $J_\varphi\colon X \to X^*$ satisfies condition* $(\mathrm{S})_+$ (see [9]).

Proposition 2.7. *Let X be a real reflexive and separable Banach space and let Y_k be the subspaces of X given by* (2.1). *We assume the following:*

$(\mathrm{H})_1$ *The operator $S\colon X \to X^*$ is bounded and satisfies condition* $(\mathrm{S})_+$.

$(\mathrm{H})_2$ *The operator $K\colon X \to X^*$ is compact.*

Then, any bounded sequence $(u_{n_j}) \subset X$ with $u_{n_j} \in Y_{n_j}$ and

$$\|(Su_{n_j} - Ku_{n_j})|_{Y_{n_j}}\|_{Y^*_{n_j}} \to 0 \quad \textit{as } j \to \infty,$$

contains a convergent subsequence.

Proof. There exists a subsequence also denoted $(u_{n_j})_j$ and $u \in X$ such that $u_{n_j} \rightharpoonup u$ as $j \to \infty$. We deduce that $(Su_{n_j})_j$ is bounded and (passing to a subsequence) we can suppose that $Ku_{n_j} \to f^* \in X^*$ as $j \to \infty$.

We will show that

$$\langle Su_{n_j} - Ku_{n_j}, u_{n_j} - u\rangle \to 0 \quad \text{as } j \to \infty. \tag{2.6}$$

One can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$ as $j \to \infty$. But

$$\langle Su_{n_j} - Ku_{n_j}, u_{n_j} - u\rangle = \langle Su_{n_j} - Ku_{n_j}, u_{n_j} - v_{n_j}\rangle + \langle Su_{n_j} - Ku_{n_j}, v_{n_j} - u\rangle$$

Since $u_{n_j} - v_{n_j} \in Y_{n_j}$, we have

$$\begin{aligned}\langle Su_{n_j} - Ku_{n_j}, u_{n_j} - v_{n_j}\rangle &= (Su_{n_j} - Ku_{n_j})\,|_{Y_{n_j}}\,(u_{n_j} - v_{n_j}) \\ &\le \|(Su_{n_j} - Ku_{n_j})\,|_{Y_{n_j}}\,\|_{Y^*_{n_j}}\|u_{n_j} - v_{n_j}\| \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

On the other hand, the sequences $(Su_{n_j})_j$ and $(Ku_{n_j})_j$ are bounded. Taking into account that $v_{n_j} \to u$ as $j \to \infty$, it follows that

$$\langle Su_{n_j} - Ku_{n_j}, v_{n_j} - u\rangle \to 0 \quad \text{as } j \to \infty,$$

therefore (2.6) holds.

Now, since $Ku_{n_j} \to f^*$ as $j \to \infty$ and $u_{n_j} \rightharpoonup u$ as $j \to \infty$, one has

$$\langle Ku_{n_j}, u_{n_j} - u\rangle \to 0 \quad \text{as } j \to \infty,$$

therefore

$$\langle Su_{n_j}, u_{n_j} - u\rangle \to 0 \quad \text{as } j \to \infty.$$

The operator S satisfying condition $(\mathrm{S})_+$, it follows that $u_{n_j} \to u$ as $j \to \infty$ and proposition is proved. □

In order to state the next results, we recall that if X is a real Banach space, $H \in \mathcal{C}^1(X,\mathbb{R})$ and $c \in \mathbb{R}$, we say that H satisfies the $(\mathrm{PS})^*_c$-*condition* (with respect to $(Y_n)_n$), if any sequence $(u_{n_j})_j \subset X$ for which

$$u_{n_j} \in Y_{n_j}, \quad \lim_{j\to\infty} H(u_{n_j}) = c \quad \text{and} \quad \lim_{j\to\infty} \|(H|_{Y_{n_j}})'(u_{n_j})\|_{Y^*_{n_j}} = 0, \tag{2.7}$$

contains a subsequence converging to a critical point of H. Also, we say that H satisfies the *Palai-s-Smale condition at level c on X* ($(\mathrm{PS})_c$-*condition*, for short), if any sequence $(u_n) \subset X$ for which $H(u_n) \to c$ and $H'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence. The $(\mathrm{PS})^*_c$-condition implies the $(\mathrm{PS})_c$-condition (Willem [19, Remark 3.19, a)]).

In what follows, a sequence $(u_{n_j}) \subset X$ satisfying (2.7), will be called a $(\mathrm{PS})^*_c$-*sequence* for H.

One has

Corollary 2.8. *Let X be a real reflexive and separable Banach space ($X = \overline{\mathrm{Sp}(E)}$), compactly imbedded in the real Banach space V and $H \in \mathcal{C}^1(X,\mathbb{R})$ be such that $H'(u) = Su - Nu$, where $S\colon X \to X^*$ is bounded, satisfies condition $(\mathrm{S})_+$ and $N\colon V \to V^*$ is demicontinuous. Let Y_k be the subspaces of X given* *by* (2.1). *If* $c \in \mathbb{R}$, *assume that any* $(\mathrm{PS})^*_c$*-sequence for* H *is bounded. Then,* H *satisfies the* $(\mathrm{PS})^*_c$*-condition for any* $c \in \mathbb{R}$.

Proof. Since H' has the form $H'(u) = Su - Ku$ with $K = i^* \circ N \circ i \colon X \to X^*$ compact, it follows by Proposition 2.7 that, if $(u_{n_j})_j \subset X$ is a bounded $(\mathrm{PS})^*_c$-sequence for H, then $(u_{n_j})_j$ contains a convergent subsequence (also denoted $(u_{n_j})_j$. Therefore $u_{n_j} \to u$ as $j \to \infty$.

We shall show that $H'(u) = 0$. Since $\overline{\mathrm{Sp}(E)} = X$, it is sufficient to show that $\langle H'(u), w\rangle = 0$, for any $w \in \mathrm{Sp}(E)$.

Indeed, if $w \in \mathrm{Sp}(E)$, there exists $p \in \mathbb{N}$ such that $w \in Y_p$, therefore $w \in Y_q$, $q \geq p$. From (2.7), it follows that for any $\varepsilon > 0$, there exists n_ε such that

$$\|(H|_{Y_{n_j}})'(u_{n_j})|_{Y^*_{n_j}} < \varepsilon, \quad \text{for all } j \geq n_\varepsilon.$$

But $w \in Y_{n_j}$, for any $j \geq \max(p, n_\varepsilon)$. Consequently,

$$\lim_{j\to\infty} \langle (H|_{Y_{n_j}})'(u_{n_j}), w\rangle = 0. \tag{2.8}$$

Since

$$\langle H'(u), w\rangle = \langle H'(u) - H'(u_{n_j}), w\rangle + \langle H'(u_{n_j}), w\rangle,$$

taking into account $H \in \mathcal{C}^1(X, \mathbb{R})$ and (2.8), we obtain $\langle H'(u), w\rangle = 0$, for any $w \in \mathrm{Sp}(E)$, therefore $H'(u) = 0$. $\square$

Corollary 2.9. *Let* X *be a real, reflexive and smooth Banach space having the Kadeč–Klee property and compactly imbedded in the real Banach space* V. *Let* $H \in \mathcal{C}^1(X, \mathbb{R})$ *be a functional having the form* $H = \Psi - G$, *where:*

(a) *at any* $u \in X$, $\Psi(u) = \Phi(\|u\|)$ *with*

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \quad \textit{for all } t \geq 0$$

and $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ *being a gauge function which satisfies*

$$\sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} = p^* < \infty;$$

(b) $G \colon V \to \mathbb{R}$ *satisfies:*

(b)$_1$ $G' \colon V \to V^*$ *is demicontinuous;*

(b)$_2$ *there is a constant* $\theta > p^*$ *such that*

$$\langle G'(y), y\rangle_{V,V^*} - \theta G(y) \geq C = \text{const.} \quad \textit{for all } y \in V. \tag{2.9}$$

Then, the functional H *satisfies the* $(\mathrm{PS})^*_c$*-condition, for any* $c \in \mathbb{R}$.

Proof. It suffices to prove that the hypotheses of Corollary 2.8 are fulfilled with $S = J_\varphi$ and $N = G'$. Indeed, according to Asplund's Theorem ([2]) $\Psi' = J_\varphi$, J_φ is bounded and, by Proposition 2.6, J_φ satisfies condition $(\mathrm{S})_+$. The demicontinuity of G' is assumed by $(\mathrm{b})_1$. It remains to be proved that any $(\mathrm{PS})_c^*$-sequence for H is bounded.

Let $(u_{n_j})_j \subset X$ be a $(\mathrm{PS})_c^*$-sequence for H. By putting $\varepsilon_{n_j} = \|H'(u_{n_j})\|_{Y^*_{n_j}}$ and taking into account the boundedness of $H(u_{n_j})$ one has:

$$H(u_{n_j}) - \frac{1}{\theta}\langle H'(u_{n_j}), u_{n_j}\rangle_{X,X^*} \le M + \frac{\varepsilon_{n_j}}{\theta}\|u_{n_j}\|_X, \quad M = \text{const.} \tag{2.10}$$

On the other hand, since, at any $u \in X$, $H(u) = \Psi(u) - G(i(u))$, one has

$$H'(u) = \Psi'(u) - (i^* \circ G' \circ i)(u) = J_\varphi u - (i^* \circ G' \circ i)(u),$$

where, as usual, i stands for the injection of X in V and i^* is its adjoint. Consequently,

$$\begin{aligned}
H(u_{n_j}) &- \frac{1}{\theta}\langle H'(u_{n_j}), u_{n_j}\rangle_{X,X^*} \\
&= \Phi(\|u_{n_j}\|) - G(i(u_{n_j})) - -\frac{1}{\theta}\langle J_\varphi u_{n_j} - (i^* \circ G' \circ i)(u_{n_j}), u_{n_j}\rangle_{X,X^*} \\
&= \left[\Phi(\|u_{n_j}\|) - \frac{1}{\theta}\varphi(\|u_{n_j}\|)\|u_{n_j}\|\right] \\
&\quad + \frac{1}{\theta}[\langle G'(i(u_{n_j})), i(u_{n_j})\rangle_{V,V^*} - \theta G(i(u_{n_j}))].
\end{aligned}$$

From p^* definition, $\varphi(\|u_{n_j}\|)\|u_{n_j}\| \le p^*\Phi(\|u_{n_j}\|)$ such that, taking into account (2.9), one obtains

$$H(u_{n_j}) - \frac{1}{\theta}\langle H'(u_{n_j}), u_{n_j}\rangle_{X,X^*} \ge \left(1 - \frac{p^*}{\theta}\right)\Phi(\|u_{n_j}\|) + \frac{C}{\theta}. \tag{2.11}$$

Comparing (2.10) and (2.11), we infer that

$$\left(1 - \frac{p^*}{\theta}\right)\Phi(\|u_{n_j}\|) \le M_1 + \frac{\varepsilon_{n_j}}{\theta}\|u_{n_j}\|, \quad M_1 = M - \frac{C}{\theta}.$$

Since $\Phi(t)/t \to \infty$ as $t \to \infty$ and $\varepsilon_{n_j} \to 0$ as $n \to \infty$, this inequality implies the boundedness of (u_n). $\square$

Next we state the basic result we need for proving Theorem 2.1.

Theorem 2.10. *Let X be a real reflexive and separable Banach space and let X_k, Y_k, Z_k be the subspaces of X given by (2.1). Let $H \in \mathcal{C}^1(X, \mathbb{R})$ be an even functional satisfying the following hypotheses:*

$(\mathrm{H})_1$ *H satisfies the $(\mathrm{PS})_c^*$-condition, for any $c \in \mathbb{R}$;*

$(\mathrm{H})_2$ *For any $k \in \mathbb{N}^*$ there exists $\rho_k > r_k > 0$ such that*

$$a_k = \max_{\substack{u \in Y_k \\ \|u\|_X = \rho_k}} H(u) \le 0 \tag{2.12}$$

and

$$b_k = \inf_{\substack{u \in Z_k \\ \|u\|_X = r_k}} H(u) \to \infty \quad \textit{as } k \to \infty; \tag{2.13}$$

(H)$_3$ *There exists $k_0 \in \mathbb{N}^*$ such that for any $k \geq k_0$ there exist $\varphi_k > r_k > 0$ such that*

$$\inf_{\substack{u \in Z_k \\ \|u\|_X = \varphi_k}} H(u) \geq 0, \tag{2.13}$$

$$b_k = \max_{u \in Y_k, \|u\|_X = r_k} H(u) < 0 \tag{2.14}$$

and

$$d_k = \inf_{\substack{u \in Z_k \\ \|u\|_X \leq \varphi_k}} H(u) \to 0 \quad \textit{as } k \to \infty. \tag{2.15}$$

Then, H possesses a sequence of critical positive values which converges to $+\infty$ and another one, of critical negative values converging to 0*.*

Theorem 2.10 is obtained as a direct consequence of both "fountain theorem" (Bartsch [3]) and "dual fountain theorem" (Bartsch–Willem [4]) as follows: the hypothesis "H satisfies the $(\mathrm{PS})_c^*$-condition for every $c \in [d_{k_0}, 0)$" in the statement of the "dual fountain theorem" is replaced by "H satisfies the $(\mathrm{PS})_c^*$-condition for every $c \in \mathbb{R}$", the fact that $(\mathrm{PS})_c^*$-condition implies the $(\mathrm{PS})_c$-condition is taken into account and then by union of the such modified hypotheses of the two above quoted theorems.

Proof of Theorem 2.1. We shall prove that the hypotheses of Theorem 2.10 are satisfied and then will follow by this theorem that the functional H possesses a sequence of critical positive values which converges to ∞ and another one, of critical negative values converging to 0.

Indeed, according to Corollary 2.9, H satisfies the $(\mathrm{PS})_c^*$-condition for any $c \in \mathbb{R}$. Thus hypothesis (H)$_1$ of Theorem 2.10 is satisfied.

We split in two steps the proof of the fact that hypothesis (H)$_2$ of Theorem 2.10 is also satisfied.

Step 1. Define

$$\alpha_k = \sup\{\|i(u)\|_V \mid u \in Z_k,\ \|u\|_X = 1\},\ k \in \mathbb{N}^*, \tag{2.16}$$

and show that

(a) $0 < \alpha_{k+1} \leq \alpha_k$, for all $k \in \mathbb{N}^*$, and $\alpha_k \to 0$ as $k \to \infty$;

(b)

$$\|i(u)\|_V \leq \alpha_k \|u\|_X, \quad \text{for all } u \in Z_k, k \in \mathbb{N}^*, \tag{2.17}$$

where i stands for the compact injection of X in V.

Indeed, let $C = \text{const.} > 0$ be such that

$$\|i(u)\|_V \le C\|u\|_X, \quad \text{for all } u \in X.$$

Since for any $u \in Z_k$, with $\|u\|_X = 1$ one has $\|i(u)\|_V \le C$, we derive that $\alpha_k \le C$. Since $Z_{k+1} \subset Z_k$ one derives that $\alpha_{k+1} \le \alpha_k$. Since $i(u) \ne i(0) = 0$ for any $u \in X$, $u \ne 0$, one derives that $\|i(u)\|_V > 0$ for any $u \in Z_k$, with $\|u\|_X = 1$. Consequently, $\alpha_k > 0$.

By α_k definition, there is $u_k \in Z_k$, with $\|u_k\|_X = 1$ such that

$$0 \le \alpha_k - \|i(u_k)\|_V < \frac{1}{k}, \quad k \in \mathbb{N}^*. \tag{2.18}$$

We shall prove that $u_k \rightharpoonup 0$ (in X). Since X is reflexive and (u_k) is bounded, it suffices to show that zero is the unique weakly cluster point of (u_k).

Consider a subsequence of (u_k) (still denoted by (u_k)) and an element $u \in X$ such that $u_k \rightharpoonup u$. We shall prove that $u = 0$. Let $p \in \mathbb{N}^*$ be fixed (but arbitrary chosen). One has $f_p(u_k) \to f_p(u)$ as $k \to \infty$. But, for any $k > p$, $f_p(u_k) = 0$ (that's because $u_k \in Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$, $X_j = \mathrm{Sp}(\{e_j\})$ and $f_p(e_j) = 0$ for any $j \ge k$).

Consequently, $f_p(u) = 0$. Since $X^* = \overline{\mathrm{Sp}(\{f_1, \dots, f_n, \dots\})}$, we derive, by density, that $f(u) = 0$, for all $f \in X^*$, thus $u = 0$. Since $u_k \rightharpoonup 0$ (in X), the compactness of i implies $i(u_k) \to 0$ in Y and then, from (2.18), $\alpha_k \to 0$. Clearly, (b) directly follows by the definition of α_k.

Step 2. Define $r_k = (c_1/2c_2\alpha_k^q)^{1/(q-p)}$ and $\rho_k = \max(r_k + 1, t_0)$, $t_0 > 0$ being such that $h(t) = c_3 t^r - c_4 t^s + c_5 \le 0$ for $t \ge t_0$ (since $h(t) \to -\infty$ as $t \to \infty$, such a t_0 exists). Clearly, one has $\rho_k > r_k > 0$. Moreover, we shall show that (2.12) and (2.13) hold.

Let $u \in Y_k$ with $\|u\|_X = \rho_k$. Since $\rho_k > 1$, it follows from (d) that $H(u) \le c_3\rho_k^r - c_4\rho_k^s + c_5 = h(\rho_k)$ and, since $\rho_k \ge t_0$, it follows that $h(\rho_k) \le 0$, thus (2.12) holds.

Let k_0 be such that $r_k > 1$ for any $k \ge k_0$ (since $r_k \to \infty$ as $k \to \infty$, such a k_0 exists). Since $\|i(u)\|_Y \le \alpha_k\|u\|_X$, for any $u \in Z_k$ (see (2.17)), we derive from (2.4) that, for $k \ge k_0$ and $u \in Z_k$ satisfying $\|u\|_X = r_k$,

$$H(u) \ge c_1\|u\|_X^p - c_2\alpha_k^q\|u\|_X^q - d = c_1 r_k^p - c_2\alpha_k^q r_k^q - d = \frac{c_1}{2} r_k^p - d.$$

Consequently, for $k \ge k_0$

$$\inf_{\substack{u \in Z_k \\ \|u\|_X = r_k}} H(u) \ge \frac{c_1}{2} r_k^p - d \to \infty \quad \text{as } k \to \infty.$$

Consequently, (2.13) holds as well, therefore $(\mathrm{H})_2$ is satisfied.

Now, we shall prove that the hypothesis $(\mathrm{H})_3$ of Theorem 2.10 is also satisfied. Let us consider $t_0 > 1$ such that $h(t) = c_3 t^r - c_4 t^s + c_5 < -1$ for $t \ge t_0$ (Since

$h(t) \to -\infty$ as $t \to \infty$, such a t_0 exists). Define $r_k = t_0$, for all $k \in \mathbb{N}^*$. Let $u \in Y_k$ with $\|u\|_X = r_k$. Since $r_k > 1$, it follows from (d) that $H(u) \le c_3 r_k^r - c_4 r_k^s + c_5 = h(r_k)$ and, since $r_k = t_0$, it follows that $h(r_k) < -1$, thus (2.14) holds. Now, we will show that there exists $k_0 \in \mathbb{N}^*$ such that for any $k \ge k_0$ there exists $\varphi_k > r_k > 0$ such that (2.13) holds.

Define $\gamma_k = (c_1/(2c_2\alpha_k^q))^{1/(q-p)}$, whith $(\alpha_k)_k$ given by (2.16).

Since $\lim_{k\to\infty} \alpha_k = 0$, it follows that $\lim_{k\to\infty} \gamma_k = \infty$, therefore there exists $k_1 \in \mathbb{N}^*$ such that, for any $k \ge k_1$, one has $\gamma_k > t_0$.

We derive from (2.4) that, for $k \ge k_1$ and $u \in Z_k$ satisfying $\|u\|_X = \gamma_k$,

$$H(u) \ge c_1\|u\|_X^p - c_2\alpha_k^q\|u\|_X^q - d = c_1\gamma_k^p - c_2\alpha_k^q\gamma_k^q - d = \frac{c_1}{2}\gamma_k^p - d.$$

Since $\lim_{k\to\infty}((c_1/2)\gamma_k^p - d) = \infty$, there exists $k_0 \in \mathbb{N}^*$, $k_0 \ge k_1$ such that, for any $k \ge k_0$,

$$\frac{c_1}{2}\gamma_k^p - d > 0.$$

Define $\varphi_k = \gamma_{k_0} > t_0 = r_k$. Consequently, for $k \ge k_0$ and $u \in Z_k$, $\|u\|_X = \varphi_k$, $H(u) > 0$, therefore (2.13) holds as well.

Now, since $\Psi(u) \ge 0$, for all $u \in X$, we derive from (2.2) and (2.5) that

$$H(u) \ge -G(u) \ge -c_7\|i(u)\|_V - c_8\|i(u)\|_V^{p_*}, \quad \text{for all } u \in X.$$

Consequently, for $k \ge k_0$ and $u \in Z_k$ satisfying $\|u\|_X \le \varphi_k$, one has

$$H(u) \ge -c_7\alpha_k\varphi_k - c_8\alpha_k^{p_*}\varphi_k^{p_*},$$

therefore $d_k \ge -c_7\alpha_k\gamma_{k_0} - c_8\alpha_k^{p_*}\gamma_{k_0}^{p_*}$, for all $k \ge k_0$. Then $\lim_{k\to\infty} d_k \ge 0$.

On the other hand, since $Z_k \cap Y_k \ne \emptyset$ and $r_k < \varphi_k$, it follows that

$$d_k \le b_k < 0, \quad \text{for all } k \ge k_0,$$

therefore $-c_7\alpha_k\gamma_{k_0} - c_8\alpha_k^{p_*}\gamma_{k_0}^{p_*} \le d_k \le b_k < 0$, for all $k \ge k_0$.

Since $\alpha_k \to 0$ as $k \to \infty$, it follows that (2.15) is satisfied. Thus hypothesis $(\mathrm{H})_3$ of Theorem 2.10 is satisfied. The proof is complete. $\square$

3. Applications to Orlicz–Sobolev spaces

Throughout this section Ω denotes a bounded open subset of $\mathbb{R}^N$, $N \ge 2$. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing odd continuous function with $\lim_{t\to\infty} a(t) = \infty$. For $m \in \mathbb{N}^*$, let us denote by $W_0^m E_A(\Omega)$ the Orlicz–Sobolev space generated by the N-function A, given by

$$A(t) = \int_0^t a(s)\,ds. \tag{3.1}$$

We shall always suppose that

$$\lim_{t\to 0}\int_t^1 \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau < \infty, \tag{3.2}$$

replacing, if necessary, A by another N-function equivalent to A near infinity (which determines the same Orlicz space).

Suppose also that

$$\lim_{t\to\infty}\int_1^t \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau=\infty. \tag{3.3}$$

With (3.3) satisfied, we define the *Sobolev conjugate* A_* of A by setting

$$A_*^{-1}(t)=\int_0^t \frac{A^{-1}(\tau)}{\tau^{(N+1)/N}}\,d\tau,\quad t\geq 0.$$

The existence and multiplicity of weak solutions for the boundary value problem

$$J_a u=\sum_{|\alpha|<m}(-1)^{|\alpha|}D^\alpha g_\alpha(x,D^\alpha u)\quad \text{in } \Omega, \tag{3.4}$$

$$D^\alpha u=0\quad \text{on } \partial\Omega,\quad |\alpha|\leq m-1, \tag{3.5}$$

is studied, in this section, in the following functional framework:

- $T[u,v]$ is a nonnegative symmetric bilinear form on the Orlicz–Sobolev space $W_0^m E_A(\Omega)$, involving the only generalized derivatives of order m of the functions $u,v\in W_0^m E_A(\Omega)$, satisfying

$$c_1\sum_{|\alpha|=m}(D^\alpha u)^2\leq T[u,u]\leq c_2\sum_{|\alpha|=m}(D^\alpha u)^2,\quad \text{for all } u\in W_0^m L_A(\Omega),$$

with c_1, c_2 be positive constants;
- $\|u\|_{m,A}=\|\sqrt{T[u,u]}\|_{(A)}$ is a norm on $W_0^m E_A(\Omega)$, $\|\cdot\|_{(A)}$ designating the Luxemburg norm on the Orlicz space $L_A(\Omega)$;
- $J_a\colon (W_0^m E_A(\Omega),\|\cdot\|_{m,A})\to (W_0^m E_A(\Omega),\|\cdot\|_{m,A})^*$ is the duality mapping on $(W_0^m E_A(\Omega),\|\cdot\|_{m,A})$ subordinated to the gauge function a;
- $g_\alpha\colon\Omega\times\mathbb{R}\to\mathbb{R}$, $|\alpha|<m$, are Carathéodory functions satisfying hypotheses:

(H)$_1$ there exist the N-functions M_α, $|\alpha|<m$, which increase essentially more slowly than A_* near infinity and satisfy the Δ_2-condition, such that

$$|g_\alpha(x,s)|\leq c_\alpha(x)+d_\alpha\overline{M}_\alpha^{-1}(M_\alpha(s)),\quad x\in\Omega,\ s\in\mathbb{R},\ |\alpha|<m, \tag{3.6}$$

where $\overline{M}_\alpha$ are the complementary N-functions to M_α, $c_\alpha\in K_{\overline{M}_\alpha}$ (the Orlicz class generated by the N-function $\overline{M}_\alpha$) and d_α are positive constants;

(H)$_2$ for any α with $|\alpha|<m$, there exist $s_\alpha>0$ and $\theta_\alpha>p^*=\sup_{t>0}ta(t)/A(t)$ such that

$$0<\theta_\alpha G_\alpha(x,s)\leq s g_\alpha(x,s),$$

for almost every $x \in \Omega$ and all s with $|s| \geq s_\alpha$, where

$$G_\alpha(x,s) = \int_0^s g_\alpha(x,\tau)\,d\tau. \tag{3.7}$$

Assume also that

(H)$_3$ the function $a(t)/t$ is nondecreasing on $(0,\infty)$, (3.2) and (3.3) being fulfilled as well (see the beginning of this section).

By (*weak*) *solution* of the problem (3.4)–(3.5), we understand a solution of the equation

$$J_a u = G'(u), \tag{3.8}$$

in the following functional framework:

(i) $X = W_0^m E_A(\Omega)$ endowed with the $\|\cdot\|_{m,A}$-norm;
$V = \bigcap_{|\beta|<m} W^{m-1} L_{M_\beta}(\Omega)$ endowed with the norm

$$\|u\|_V = \sum_{|\beta|<m} \|u\|_{W^{m-1}L_{M_\beta}(\Omega)};$$

(ii) $J_a =$ the duality mapping on $(W_0^m E_A(\Omega), \|\cdot\|_{m,A})$ corresponding to the gauge function a;

(iii) $G': V \to V^*$ is the differential of the functional $G: V \to \mathbb{R}$,

$$G(u) = \sum_{|\alpha|<m} \int_\Omega G_\alpha(x, D^\alpha u(x))\,dx.$$

According to [10, Proposition 6.2], X is compactly imbedded in V.

Proposition 3.1. *Let $A: \mathbb{R} \to \mathbb{R}_+$ be the N-function given by* (3.1)*. Furthermore, we assume that A satisfies* (3.2) *and* (3.3)*, the Δ_2-condition being also satisfied by A and $\overline{A}$. Let $g_\alpha: \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$ be Carathéodory functions satisfying condition* (H)$_1$*. Then, the functional $H: W_0^m E_A(\Omega) \to \mathbb{R}$ defined by*

$$H(u) = \Psi(u) - G(u), \tag{3.9}$$

with

$$\Psi(u) = A(\|u\|_{m,A}), \qquad G(u) = \sum_{|\alpha|<m} \int_\Omega G_\alpha(x, D^\alpha u(x))\,dx,$$

for all $u \in W_0^m E_A(\Omega)$, is well-defined and $\mathcal{C}^1$ on $W_0^m E_A(\Omega)$, with

$$H'(u) = J_a u - \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u).$$

Proof. Clearly, the well-definedness of H on $W_0^m E_A(\Omega)$ reduces to that of G. At its turn, the well-definedness of G on $W_0^m E_A(\Omega)$ is proved in [10, Proposition 7.5].

We shall prove more: G is well-defined on V. Fix α with $|\alpha| \leq m-1$. If $u \in Y$, then $u \in W^{m-1}L_{M_\beta}(\Omega)$, for all β with $|\beta| \leq m-1$. In particular, $u \in W^{m-1}L_{M_\alpha}(\Omega)$, therefore $D^\alpha u \in L_{M_\alpha}(\Omega) = E_{M_\alpha}(\Omega)$.

Taking into account [10, Proposition 7.5 and (7.15)], one has

$$|G_\alpha(x,s)| \leq c_\alpha |s| + 2d_\alpha M_\alpha(|s|).$$

Therefore

$$\int_\Omega G_\alpha(x, D^\alpha u(x))\,dx \leq c_\alpha \int_\Omega |D^\alpha u(x)|\,dx + 2d_\alpha \int_\Omega M_\alpha(|D^\alpha u(x)|)\,dx.$$

Since, $D^\alpha u \in E_{M_\alpha}(\Omega) \hookrightarrow L^1(\Omega)$, it follows that $\int_\Omega |D^\alpha u(x)|\,dx$ makes sense. Also, $\int_\Omega M_\alpha(|D^\alpha u(x)|)\,dx$ makes sense. Consequently,

$$\int_\Omega G_\alpha(x, D^\alpha u(x))\,dx < \infty.$$

In order to prove that $H \in \mathcal{C}^1$, it is sufficient to prove that $\Psi \in \mathcal{C}^1$ and $G \in \mathcal{C}^1$. Indeed, one has ([10, Proposition 7.5]):

$$\Psi'(u) = J_a u, \quad \text{for all } u \in W_0^m E_A(\Omega),$$

where

$$J_a u = \begin{cases} 0 & \text{if } u = 0, \\ a(\|u\|_{m,A}) \|\cdot\|'_{m,A}(u) & \text{if } u \neq 0, \end{cases}$$

and

$$\langle \|\cdot\|'_{m,A}(u), h\rangle = \frac{\displaystyle\int_\Omega a\bigg(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\bigg)\frac{T[u,h](x)}{\sqrt{T[u,u](x)}}\,dx}{\displaystyle\int_\Omega a\bigg(\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\bigg)\frac{\sqrt{T[u,u](x)}}{\|u\|_{m,A}}\,dx},$$

for all $u \in W_0^m E_A(\Omega)$, $u \neq 0$, for all $h \in W_0^m E_A(\Omega)$.

The continuity of the map $u \mapsto \|\cdot\|'_{m,A}(u)$ at any $u \neq 0$ is proved in [10, Theorem 3.6] and for the continuity of Ψ' at $u = 0$, see the proof of Proposition 7.5 in [10]. Thus $\Psi \in \mathcal{C}^1$.

As far as the $\mathcal{C}^1$-regularity of G is concerned, for a later use, we shall prove more: G is $\mathcal{C}^1$ on V and

$$\langle G'(u), h\rangle = \sum_{|\alpha|<m} \int_\Omega g_\alpha(x, D^\alpha u(x)) D^\alpha h(x)\,dx, \quad u, h \in V. \tag{3.10}$$

Indeed, let $u, h \in V$. One has

$$\begin{aligned} |G(u+h) &- G(u) - \langle G'(u), h\rangle| \\ &= \bigg| \sum_{|\alpha|<m} \int_\Omega [G_\alpha(x, D^\alpha u(x) + D^\alpha h(x)) \\ &\qquad - G_\alpha(x, D^\alpha u(x)) - g_\alpha(x, D^\alpha u(x)) D^\alpha h(x)]\,dx \bigg| \end{aligned}$$

$$
\begin{aligned}
&= \bigg| \sum_{|\alpha|<m} \int_\Omega [g_\alpha(x, D^\alpha u(x) + \theta_{D^\alpha h}(x) \cdot D^\alpha h(x)) D^\alpha h(x) \\
&\qquad - g_\alpha(x, D^\alpha u(x)) D^\alpha h(x)]\, dx \bigg| \\
&\le 2 \sum_{|\alpha|<m} \| g_\alpha(x, D^\alpha u(x) + \theta_{D^\alpha h} \cdot D^\alpha h(x)) \\
&\qquad - g_\alpha(x, D^\alpha u(x)) \|_{(\overline{M}_\alpha)} \| D^\alpha h \|_{(M_\alpha)} \\
&\le 2 \|h\|_V \sum_{|\alpha|<m} \| g_\alpha(x, D^\alpha u(x) + \theta_{D^\alpha h} \cdot D^\alpha h(x)) - g_\alpha(x, D^\alpha u(x)) \|_{(\overline{M}_\alpha)},
\end{aligned}
$$

where $0 \le \theta_{D^\alpha h}(x) \le 1$ ([13, Lemma 18.1]) and Hölder's type inequality was used ([13, p. 80]). Consequently,

$$
\begin{aligned}
&\frac{|G(u+h) - G(u) - \langle G'(u), h \rangle|}{\|h\|_V} \\
&\qquad \le 2 \sum_{|\alpha|<m} \| g_\alpha(x, D^\alpha u(x) + \theta_{D^\alpha h} \cdot D^\alpha h(x)) - g_\alpha(x, D^\alpha u(x)) \|_{(\overline{M}_\alpha)}.
\end{aligned}
$$

Suppose $\|h\|_V \to 0$. It follows that

$$
\|h\|_{W^{m-1} L_{M_\alpha}(\Omega)} \to 0, \quad \text{therefore} \quad \|D^\alpha h\|_{(M_\alpha)} \to 0,
$$

for any α with $|\alpha| < m$. Taking into account the continuity of Nemytskij operators (see [13, Theorem 17.6]), it follows that G is Fréchet differentiable on V and G' is given by (3.10).

Moreover, the operator $G' : V \to V^*$ given by (3.10) is continuous (see [10, Proposition 6.3]).

Now, since X is continuously imbedded in V and G is $\mathcal{C}^1$ on V, it follows that G is $\mathcal{C}^1$ on X. □

The main result is the following.

Theorem 3.2. *Let $A : \mathbb{R} \to \mathbb{R}_+$ be the N-function given by* (3.1)*, fulfilling* (3.2), (3.3) *and hypothesis* $(\mathrm{H})_3$*, and let $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, be Carathéodory functions satisfying* $(\mathrm{H})_1$, $(\mathrm{H})_2$ *and being odd in the second argument: $g_\alpha(x, -s) = -g_\alpha(x, s)$. Suppose that the N-functions A, $\overline{A}$ and $\overline{M}_\alpha$, $|\alpha| < m$, satisfy the Δ_2-condition. With*

$$
p_0 = \inf_{t>0} \frac{t a(t)}{A(t)}, \quad p^* = \sup_{t>0} \frac{t a(t)}{A(t)} < \infty, \tag{3.11}
$$

we further assume:

$(\mathrm{H})_4$ $p_0 < \gamma = \max_{|\alpha|<m} \gamma_\alpha$, $\gamma_\alpha = \sup_{t>0} t M'_\alpha(t) / M_\alpha(t)$.

Then, the functional (3.9) *possesses a sequence of critical positive values which converges to* ∞ *and another one, of critical negative values converging to* 0*.*

Proof. Theorem 2.1 applies. Indeed, since $a(t)/t$ is nondecreasing on $(0, \infty)$, $W_0^m E_A(\Omega)$ is uniformly convex ([10, Theorem 3.14]). Consequently, $W_0^m E_A(\Omega)$ is reflexive and has the Kadeč–Klee property. The same space is smooth ([10, Theorem 3.6]), separable ([1, Theorem 8.28]) and compactly imbedded in $V = \bigcap_{|\beta|<m} W^{m-1} L_{M_\beta}(\Omega)$, endowed with the norm

$$\|u\|_V = \sum_{|\beta|<m} \|u\|_{W^{m-1} L_{M_\alpha}(\Omega)}. \tag{3.12}$$

([10, Proposition 6.2]). The functional $H \in \mathcal{C}^1(X, \mathbb{R})$ (Proposition 3.1), is even (since g_α are odd in the second argument) and satisfies the hypotheses (a)–(d) of Theorem 2.1.

Since (3.11) holds, the hypothesis (a) is obviously satisfied with $\varphi = a$. Since $G' \colon V \to V^*$ is continuous ([10, Proposition 6.3]), $(\mathrm{b})_1$ is obviously satisfied.

Taking into account [10, Lemma 7.7]), we infer that there exists a positive constant C such that

$$\sum_{|\alpha|<m} \int_\Omega \left[\frac{1}{\theta} g_\alpha(x, D^\alpha u_n(x)) D^\alpha u_n(x) - G_\alpha(x, D^\alpha u_n(x))\right] dx \geq -C, \tag{3.13}$$

where $\theta = \min_{|\alpha|<m} \theta_\alpha$. We remark that (3.13) can be rewrited as

$$\frac{1}{\theta} \langle G'(u_n), u_n \rangle - G(u) \geq -C,$$

therefore $(\mathrm{b})_2$ in Theorem 2.1 is fulfilled.

We will prove that hypothesis (c) of Theorem 2.1 is fulfilled. For the first term in (3.9), according to [10, Lemma 6.5 a)], we have

$$A(\|u\|_{m,A}) \geq A(1) \|u\|_{m,A}^{p_0}, \tag{3.14}$$

for all $u \in W_0^m E_A(\Omega)$ with $\|u\|_{m,A} > 1$.

We shall now handle the estimations for the second term in (3.9). As in [10, Proposition 7.5, (7.15)], from $(\mathrm{H})_3$ we deduce that for any α with $|\alpha| < m$ one has

$$|G_\alpha(x, s)| \leq |c_\alpha(x)||s| + 2 d_\alpha M_\alpha(|s|), \quad x \in \Omega,\ s \in \mathbb{R}. \tag{3.15}$$

Consequently,

$$\int_\Omega G_\alpha(x, D^\alpha u(x))\, dx \leq \int_\Omega |c_\alpha(x)||D^\alpha u(x)|\, dx + 2 d_\alpha \int_\Omega M_\alpha(|D^\alpha u(x)|)\, dx, \tag{3.16}$$

for all $u \in W_0^m E_A(\Omega)$. From Hölder's type inequality, we derive

$$\left| \int_\Omega c_\alpha(x)|D^\alpha u(x)|\, dx \right| \le 2\|c_\alpha\|_{(\overline{M}_\alpha)} \|D^\alpha u\|_{(M_\alpha)}, \tag{3.17}$$

therefore

$$\left| \int_\Omega c_\alpha(x)|D^\alpha u(x)|\, dx \right| \le 2\|c_\alpha\|_{(\overline{M}_\alpha)},$$

if $\|D^\alpha u\|_{(M_\alpha)} \le 1$ and

$$\left| \int_\Omega c_\alpha(x)|D^\alpha u(x)|\, dx \right| \le 2\|c_\alpha\|_{(\overline{M}_\alpha)} \|D^\alpha u\|^\gamma_{(M_\alpha)},$$

if $\|D^\alpha u\|_{(M_\alpha)} > 1$. Consequently,

$$\left| \int_\Omega c_\alpha(x)|D^\alpha u(x)|\, dx \right| \le k_\alpha(\|u\|_Y^\gamma + 1), \quad \text{for all } u \in W_0^m E_A(\Omega), \tag{3.18}$$

where $k_\alpha = 2\|c_\alpha\|_{(\overline{M}_\alpha)}$.

On the other hand, if $\|D^\alpha u\|_{(M_\alpha)} \le 1$, then

$$\int_\Omega M_\alpha(D^\alpha u(x))\, dx \le 1.$$

If $\|D^\alpha u\|_{(M_\alpha)} > 1$, then from [10, Lemma 6.5, b)]

$$\int_\Omega M_\alpha(D^\alpha(u(x)))\, dx \le \|D^\alpha(u)\|^{\gamma_\alpha}_{(M_\alpha)} \le \|u\|_Y^\gamma, \tag{3.19}$$

therefore

$$\int_\Omega M_\alpha(|D^\alpha u(x)|)\, dx \le \|u\|_Y^\gamma + 1, \quad \text{for all } u \in W_0^m E_A(\Omega). \tag{3.20}$$

Taking into account (3.16), (3.18) and (3.20), it follows that

$$\int_\Omega G_\alpha(x, D^\alpha u(x))\, dx \le (k_\alpha + 1)\|u\|_Y^\gamma + (k_\alpha + 1),$$

for all $u \in W_0^m E_A(\Omega)$, $|\alpha| < m$. Consequently, summing by α, we have

$$\sum_{|\alpha|<m} \int_\Omega G_\alpha(x, D^\alpha u(x))\, dx < c_2\|u\|_Y^\gamma + c_2, \tag{3.21}$$

where $c_2 = \sum_{|\alpha|<m}(k_\alpha + 1)$. Then, from (3.14) and (3.21), one obtains

$$F(u) \ge A(1)\|u\|^{p_0}_{m,A} - c_2\|u\|_Y^\gamma - c_2,$$

if $u \in W_0^m E_A(\Omega)$, $\|u\|_{m,A} > 1$, therefore, the hypothesis (c) of Theorem 2.1 is fulfilled.

Now, we will prove that the hypothesis (d) of Theorem 2.1 is fulfilled. Let Y_k be a finite dimensional subspace of $W_0^m E_A(\Omega)$. According to [10, Lemma 7.6, (7.46)], it is shown that for any α with $|\alpha| < m$, one has

$$G_\alpha(x, s) \ge \gamma_\alpha(x)|s|^{\theta_\alpha}, \quad \text{for a.e. } x \in \Omega \text{ and } |s| \ge s_\alpha,$$

where $\gamma_\alpha \in L^\infty(\Omega)$.

For α with $|\alpha| < m$ and $v \in W_0^m E_A(\Omega)$, we define

$$\Omega^\alpha_\geq = \{x \in \Omega \mid |D^\alpha v(x)| \geq s_\alpha\}, \quad \Omega^\alpha_< = \Omega \setminus \Omega^\alpha_\geq.$$

Then

$$\int_\Omega G_\alpha(x, D^\alpha v(x))\, dx \geq \int_{\Omega^\alpha_\geq} \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx + \int_{\Omega^\alpha_<} G_\alpha(x, D^\alpha v(x))\, dx.$$

But

$$\int_{\Omega^\alpha_\geq} \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx = \int_\Omega \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx - \int_{\Omega^\alpha_<} \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx.$$

Since

$$\int_{\Omega^\alpha_<} \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx \leq \|\gamma_\alpha\|_\infty s_\alpha^{\theta_\alpha} vol(\Omega),$$

we have

$$\int_\Omega G_\alpha(x, D^\alpha v(x))\, dx \geq \int_\Omega \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx + \int_{\Omega^\alpha_<} G_\alpha(x, D^\alpha v(x))\, dx - k_\alpha,$$

where $k_\alpha = \|\gamma_\alpha\|_\infty s_\alpha^{\theta_\alpha} vol(\Omega)$. On the other hand, it follows from (3.15) that

$$\int_{\Omega^\alpha_<} G_\alpha(x, D^\alpha v(x))\, dx \leq \|c_\alpha\|_{L^1(\Omega)} s_\alpha + 2d_\alpha M_\alpha(s_\alpha) vol(\Omega),$$

therefore

$$\int_\Omega G_\alpha(x, D^\alpha v(x))\, dx \geq \int_\Omega \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx - K_\alpha,$$

where $K_\alpha = k_\alpha + \|c_\alpha\|_{L^1(\Omega)} s_\alpha + 2d_\alpha M_\alpha(s_\alpha) vol(\Omega)$. Consequently,

$$F(v) \leq A(\|v\|_{m,A}) - \sum_{|\alpha|<m} \int_\Omega \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx + K,$$

where K is a positive constant and θ_α are given by $(\mathrm{H})_2$. Taking into account the definition of p^*, for $\|v\|_{m,A} > 1$, one obtains

$$F(v) \leq A(1)\|v\|_{m,A}^{p^*} - \sum_{|\alpha|<m} \int_\Omega \gamma_\alpha(x)|D^\alpha v(x)|^{\theta_\alpha}\, dx + K.$$

Now, the functional $\|\cdot\|_\gamma\colon W_0^m E_A(\Omega) \to \mathbb{R}$ defined by

$$\|u\|_\gamma = \sum_{|\alpha|<m} \left(\int_\Omega \gamma_\alpha(x)|D^\alpha u(x)|^{\theta_\alpha}\, dx \right)^{1/\theta_\alpha}$$

is a norm on $W_0^m E_A(\Omega)$. Denoting by

$$\|D^\alpha u\|_{\theta_\alpha} = \left(\int_\Omega \gamma_\alpha(x)|D^\alpha u(x)|^{\theta_\alpha}\, dx \right)^{1/\theta_\alpha},$$

one has

$$\|u\|_\gamma = \sum_{|\alpha|<m} \|D^\alpha u\|_{\theta_\alpha}.$$

Let $\underline{\alpha}$ be a multiindex satisfying

$$\|D^{\underline{\alpha}} u\|_{\theta_{\underline{\alpha}}} = \max_{|\alpha|<m} \|D^\alpha u\|_{\theta_\alpha}.$$

Then $\|u\|_\gamma \le N_0 \|D^{\underline{\alpha}} u\|_{\theta_{\underline{\alpha}}}$, where $N_0 = \sum_{|\alpha|<m} 1$. Therefore

$$\sum_{|\alpha|<m} \int_\Omega \gamma_\alpha(x)|D^\alpha u(x)|^{\theta_\alpha}\,dx \ge \int_\Omega \gamma_{\underline{\alpha}}(x)|D^{\underline{\alpha}} u(x)|^{\theta_{\underline{\alpha}}}\,dx = \|D^{\underline{\alpha}} u\|^{\theta_{\underline{\alpha}}}_{\theta_{\underline{\alpha}}} \ge \frac{1}{N_0}\|u\|^{\theta_{\underline{\alpha}}}_\gamma.$$

Since $\|\cdot\|_{m,A}$-norm and $\|\cdot\|_\gamma$-norm are equivalent on the finite dimensional subspace Y_k, there is a constant $\delta = \delta(Y_k) > 0$ such that

$$\|u\|_{m,A} \le \delta \|u\|_\gamma.$$

Therefore

$$F(v) \le A(1)\|v\|^{p^*}_{m,A} - \frac{1}{N_0 \delta^{\theta_{\underline{\alpha}}}}\|v\|^{\theta_{\underline{\alpha}}}_{m,A} + K,$$

if $v \in Y_k$, $\|v\|_{m,A} > 1$.

Finally, we will prove that the hypothesis (e) of Theorem 2.1 is fulfilled. Indeed, taking into account (3.17) and (3.12), we derive that

$$\sum_{|\alpha|<m} \int_\Omega |c_\alpha(x)||D^\alpha u(x)|\,dx \le 2\|u\|_V \sum_{|\alpha|<m} \|c_\alpha\|_{(\overline{M}_\alpha)}.$$

Also, from (3.19) it follows that

$$2\sum_{|\alpha|<m} d_\alpha \int_\Omega M_\alpha(D^\alpha(u(x)))\,dx \le 2(\|u\|_V + \|u\|^\gamma_V) \sum_{|\alpha|<m} d_\alpha,$$

therefore, taking into account (3.16), one has

$$G(u) = \sum_{|\alpha|<m} \int_\Omega G_\alpha(x, D^\alpha u(x))\,dx \le c_7\|u\|_V + c_8\|u\|^\gamma_V,$$

where

$$c_7 = 2\sum_{|\alpha|<m} \|c_\alpha\|_{(\overline{M}_\alpha)} + 2\sum_{|\alpha|<m} d_\alpha, \quad c_8 = 2\sum_{|\alpha|<m} d_\alpha,$$

that is (2.5). Taking into account Theorem 2.1 , it follows that the functional F possesses a sequence of critical positive values which converges to ∞ and another one, of critical negative values converging to 0. By Proposition 3.1, equation (3.8) possesses two sequences of solutions in $W^m_0 E_A(\Omega)$ or, equivalently, the problem (3.4)–(3.5) possesses two sequences of weak solutions in $W^m_0 E_A(\Omega)$. □

4. Examples

Example 4.1. Consider the problem (3.4)–(3.5), under the following hypotheses:

(a) the function $a:\mathbb{R}\to\mathbb{R}$ is defined by

$$a(t)=\sum_{i=1}^{n}a_i|t|^{p_i-2}t,$$

where $a_i>0$, $1\le i\le n$, $p_{i+1}>p_i\ge 2$, $1\le i\le n-1$, $p_n<N$;

(b) the Carathéodory functions $g_\alpha:\Omega\times\mathbb{R}\to\mathbb{R}$, $|\alpha|<m$, are odd in the second argument:

$$g_\alpha(x,-s)=-g_\alpha(x,s);$$

(c) there exist q_α, $p_1<q_\alpha<Np_n/(N-p_n)$, $|\alpha|<m$, such that

$$|g_\alpha(x,s)|\le a_\alpha+b_\alpha|s|^{q_\alpha-1},\quad x\in\Omega,\ s\in\mathbb{R},\ a_\alpha,b_\alpha \text{ positive constants;} \tag{4.1}$$

(d) if G_α, $|\alpha|<m$, are given by (3.7), then, there exist $s_\alpha>0$ and $\theta_\alpha>p_n$ such that

$$0<\theta_\alpha G_\alpha(x,s)\le sg_\alpha(x,s),\quad \text{for a.e. } x\in\Omega \text{ and all } s \text{ with } |s|\ge s_\alpha. \tag{4.2}$$

Under these conditions, the problem (3.4)–(3.5) has two sequences of weak solutions.

Proof. The idea of the proof is as follows: the preceding assumptions entail that the hypotheses of Theorem 3.2 are fulfilled.

First, we prove that hypothesis $(\mathrm{H})_3$ is satisfied. Since

$$\frac{a(t)}{t}=\sum_{i=1}^{n}a_it^{p_i-2}\quad \text{for all } t>0,$$

it follows that $a(t)/t$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, the following result is needed. (see [10, Lemma 8.1(ii)]).

Lemma 4.2. *Let $A:\mathbb{R}\to\mathbb{R}_+$, $A(t)=\int_0^{|t|}a(s)\,ds$, be an N-function. Assume that*

$$p^*=\sup_{t>0}\frac{ta(t)}{A(t)}<N$$

and there are constants $0<\gamma<N$ and $\delta>0$ such that

$$A(t)\ge Ct^\gamma,\quad \text{for all } t\in(0,A^{-1}(\delta)). \tag{4.3}$$

Then, (3.2) *and* (3.3) *are satisfied* (*consequently, the Sobolev conjugate A_* of A, can be defined*).

In our case, $p^* = p_n$ and $p_n < N$ (by (a)). Since

$$A(t) = \sum_{i=1}^{n} \frac{a_i}{p_i} t^{p_i} \geq \frac{a_1}{p_1} t^{p_1}, \quad \text{for all } t > 0,$$

it follows that (4.3) is satisfied with $C = a_1/p_1$, $\gamma = p_1$ and any $\delta > 0$.

Secondly, we prove that hypothesis $(\mathrm{H})_1$ is satisfied. By setting

$$M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}, \quad |\alpha| < m,\ s \in \mathbb{R},$$

(4.1) rewrites as

$$|g_\alpha(x,s)| \leq a_\alpha + b_\alpha (q_\alpha - 1)^{1/q'_\alpha} \overline{M}_\alpha^{-1}(M_\alpha(s)), \quad x \in \Omega,\ s \in \mathbb{R},\ |\alpha| < m,$$

showing that (3.6) is satisfied.

What it remains to be proved is that M_α, $|\alpha| < m$, satisfy the Δ_2-condition and increase essentially more slowly than A_* near infinity. It is easy to check (by definition) that M_α, $|\alpha| < m$, satisfy the Δ_2-condition.

By using l'Hôspital rule, we also have

$$\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = \lim_{t\to\infty} \underline{c}_\alpha \frac{A^{-1}(t)}{t^{1/q_\alpha + 1/N}} = \lim_{s\to\infty} \underline{c}_\alpha \frac{s}{(A(s))^{1/q_\alpha + 1/N}} = 0, \tag{4.4}$$
$$\underline{c}_\alpha = q_\alpha^{(q_\alpha - 1)/q_\alpha},$$

since, from (c), the degree of denominator is $p_n(1/q_\alpha + 1/N) > 1$. Thus, M_α, $|\alpha| < m$, increase essentially more slowly than A_*.

The hypothesis $(\mathrm{H})_2$ is covered by (d) (with g_α odd functions in the second argument, according to (b)).

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, the following result is needed (see [10, Lemma 8.1(i)]):

Lemma 4.3. *Let $A\colon \mathbb{R} \to \mathbb{R}_+$, $A(t) = \int_0^{|t|} a(s)\,ds$, be an N-function and $\overline{A}$ be the complementary N-function to A. Assume that*

$$p^* = \sup_{t>0} \frac{ta(t)}{A(t)} < \infty \quad \textit{and} \quad p_0 = \inf_{t>0} \frac{ta(t)}{A(t)} > 1.$$

Then, both A and $\overline{A}$ satisfy the Δ_2-condition.

In our case, as already one has seen, $p^* = p_n < N$ and $p_0 = p_1 > 1$ (according to (a)). Since

$$\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}, \quad \frac{1}{q_\alpha} + \frac{1}{q'_\alpha} = 1, \quad |\alpha| < m,\ s \in \mathbb{R},$$

it is easy to check (by definition) that $\overline{M}_\alpha$, $|\alpha| < m$, satisfy the Δ_2-condition.

Finally, hypothesis $(\mathrm{H})_4$ is satisfied. Indeed, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \quad |\alpha| < m,$$

it follows that $p_0 = p_1 < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3.2. □

Example 4.4. Consider the problem (3.4)–(3.5), under the following hypotheses:

(a) the function $a\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$a(t) = |t|^{p-2} t\sqrt{t^2+1}, \quad 2 \le p < N-1;$$

(b) the Carathéodory functions $g_\alpha\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument:

$$g_\alpha(x,-s) = -g_\alpha(x,s);$$

(c) there exist q_α, $p < q_\alpha < N(p+1)/(N-p-1)$, $|\alpha| < m$, such that the growth conditions (4.1) hold;

(d) there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that the conditions (4.2) hold.

Under these conditions, the problem (3.4)–(3.5) has two sequences of weak solutions.

Proof. The idea of the proof is the same with that used for Example 4.1, namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 3.2.

First, we prove that hypothesis $(\mathrm{H})_3$ is satisfied. Since

$$\frac{a(t)}{t} = t^{p-2}\sqrt{t^2+1} \quad \text{for all } t > 0,$$

it follows that $a(t)/t$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, we shall use Lemma 4.2. In our case, $p^* = p+1$ ([10, Example 8.6]) and $p+1 < N$ (by (a)). Since $a(t) \ge t^{p-1}$, $t > 0$, one has

$$A(t) \ge \frac{1}{p} t^p, \quad \text{for all } t > 0,$$

therefore (4.3) is satisfied with $C = 1/p$, $\gamma = p < N$ and any $\delta > 0$.

Secondly, the hypothesis $(\mathrm{H})_1$ in Theorem 3.2 is satisfied with $M_\alpha(s) = |s|^{q_\alpha}/q_\alpha$, $|\alpha| < m$, which, obviously, satisfy the Δ_2-condition. Also, M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity. Indeed, as in (4.4)

$$\lim_{t\to\infty} \frac{A_*^{-1}(t)}{M_\alpha^{-1}(t)} = \lim_{s\to\infty} c_\alpha \frac{s}{(A(s))^{1/q_\alpha + 1/N}}.$$

It suffices to show that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{1/q_\alpha + 1/N}} = 0.$$

Since $a(t) \geq t^p$, for all $t \geq 0$, it follows that $A(t) \geq t^{p+1}/(p+1)$, for all $t \geq 0$. Consequently,

$$\lim_{s\to\infty} \frac{s}{(A(s))^{1/q_\alpha+1/N}} \leq \lim_{s\to\infty} \frac{s}{(p+1)^{1/q_\alpha+1/N} \cdot s^{(1/q_\alpha+1/N)(p+1)}} = 0,$$

since, from (c), the degree of denominator is $(p+1)(1/q_\alpha + 1/N) > 1$.

The hypothesis $(\mathrm{H})_2$ is covered by (d) (with g_α odd functions in the second argument, according to (b)).

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we shall use Lemma 4.3.

In our case, as already one has seen, $p^* = p+1 < N$ and $p_0 = p > 1$ (according to (a)). Also, the functions

$$\overline{M}_\alpha(s) = \frac{|s|^{q'_\alpha}}{q'_\alpha}, \quad \frac{1}{q_\alpha} + \frac{1}{q'_\alpha} = 1, \quad |\alpha| < m,\ s \in \mathbb{R},$$

satisfy the Δ_2-condition.

Finally, the hypothesis $(\mathrm{H})_4$ is satisfied. Indeed, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \quad |\alpha| < m,$$

it follows that $p_0 = p < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3.2. □

Example 4.5. Consider the problem (3.4)–(3.5), under the following hypotheses:

(a) the function $a\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$a(t) = |t|^{p-2} t \ln(1+|t|), \quad 2 \leq p < N-1;$$

(b) the Carathéodory functions $g_\alpha\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument:

$$g_\alpha(x,-s) = -g_\alpha(x,s);$$

(c) there exist q_α, $p < q_\alpha < Np/(N-p)$, $|\alpha| < m$, such that the growth conditions (4.1) hold;

(d) there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that the conditions (4.2) hold.

Under these conditions, the problem (3.4)–(3.5) has a sequence of weak solutions.

Proof. The idea of the proof is the same with that used for Example 4.1, namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 3.2.

First, we prove that hypothesis $(\mathrm{H})_3$ is satisfied. Since $a(t)/t = t^{p-2}\ln(1+t)$ for all $t > 0$, it follows that $a(t)/t$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, we shall use Lemma 4.2.

In our case, $p^* = p+1$ ([10, Example 8.8]) and $p+1 < N$ (by (a)). Since (see [10, Example 8.8, (8.17)])

$$A(t) \geq \frac{2}{p+1} t^{p+1}, \quad \text{for all } t \in (0,\ \underline{\delta} = A^{-1}(\delta)),$$

it follows that (4.3) is satisfied with $C = 2/(p+1)$, $\gamma = p+1 < N$ and any $\delta > 0$.

Secondly, hypothesis $(\mathrm{H})_1$ in Theorem 3.2 is satisfied with $M_\alpha(s) = |s|^{q_\alpha}/q_\alpha$, $|\alpha| < m$, which, obviously, satisfy the Δ_2-condition. Also, M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity. As in the preceding two examples, this turns out to show that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{1/q_\alpha + 1/N}} = 0.$$

This last equality is true since $A(t) \geq A(1)t^p$, for all $t > 1$ ([10, Lemma 6.5a)]), therefore

$$\lim_{s\to\infty} \frac{s}{(A(s))^{1/q_\alpha+1/N}} \leq \lim_{s\to\infty} \frac{s}{(A(1))^{1/q_\alpha+1/N} \cdot s^{(1/q_\alpha+1/N)p}} = 0,$$

since, from (c), the degree of denominator is $p(1/q_\alpha + 1/N) > 1$.

The arguments needed for proving that hypothesis $(\mathrm{H})_2$ of Theorem 3.2 is satisfied are those used in the preceding two examples.

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we shall use Lemma 4.3. In our case, as already one has seen, $p^* = p+1 < N$ and $p_0 = p > 1$ (according to (a)). Also, the functions $\overline{M}_\alpha(s) = |s|^{q'_\alpha}/q'_\alpha$, $1/q_\alpha + 1/q'_\alpha = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, satisfy the Δ_2-condition.

Finally, hypothesis $(\mathrm{H})_4$ is satisfied, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \quad |\alpha| < m,$$

and, by (c), $p_0 = p < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3.2. $\square$

EXAMPLE 4.6. Consider the problem (3.4)–(3.5), under the following hypotheses:

(a) the function $a: \mathbb{R} \to \mathbb{R}$ is defined by $a(t) = |t|^{p-2} t \ln(1 + c + |t|)$, $2 \leq p \leq N-1$, $c = \text{const.} > 0$;

(b) the Carathéodory functions $g_\alpha: \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, are odd in the second argument:

$$g_\alpha(x, -s) = -g_\alpha(x, s);$$

(c) there exist q_α, $p < q_\alpha < Np/(N-p)$, $|\alpha| < m$, such that the growth conditions (4.1) hold;

(d) there exist $s_\alpha > 0$ and $\theta_\alpha > p+1$ such that the conditions (4.2) hold.

Under these conditions, the problem (3.4)–(3.5) has a sequence of weak solutions.

Proof. The idea of the proof is that used for Example 4.1, namely, we shall show that the preceding assumptions entail the fulfillment of those of Theorem 3.2.

First, we prove that hypothesis $(\mathrm{H})_3$ is satisfied. Since $a(t)/t = t^{p-2}\ln(1+c+t)$ for all $t > 0$, it follows that $a(t)/t$ is nondecreasing on $(0,\infty)$. In order to prove that (3.2) and (3.3) are satisfied, we shall use Lemma 4.2. In our case, $p^* \le p+C_0 < p+1$, where $C_0 = 1/(1+\ln(1+c+t_0))$ and $t_0-(1+c)\ln(1+c+t_0) = 0$ ([10, Example 8.10]) and $p+1 \le N$ (by (a)). Since (see [10, Example 8.10, (8.23)])

$$A(t) \ge \frac{\ln(1+c)}{p} t^p, \quad \text{for all } t \ge 0, \tag{4.5}$$

it follows that (4.3) is satisfied with $C = \ln(1+c)/p$, $\gamma = p < N$ and any $\delta > 0$.

Secondly, the hypothesis $(H)_1$ in Theorem 3.2 is satisfied with $M_\alpha(s) = |s|^{q_\alpha}/q_\alpha$, $|\alpha| < m$, which, obviously, satisfy the Δ_2-condition. Also, M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity. As in the preceding three examples, this comes to showing that

$$\lim_{s\to\infty} \frac{s}{(A(s))^{1/q_\alpha+1/N}} = 0.$$

This last equality is true since (4.5) holds, therefore

$$\lim_{s\to\infty} \frac{s}{(A(s))^{1/q_\alpha+1/N}} \le \lim_{s\to\infty} \frac{s}{\left(\dfrac{\ln(1+\alpha)}{p}\right)^{1/q_\alpha+1/N} \cdot s^{(1/q_\alpha+1/N)p}} = 0,$$

since, from (c), the degree of denominator is $p(1/q_\alpha + 1/N) > 1$.

The necessary arguments in order to prove that hypothesis $(\mathrm{H})_2$ of Theorem 3.2 is satisfied are that used in the preceding three examples.

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we shall use Lemma 4.3. In our case, as already one has seen, $p^* \le p + C_0 < N$ and $p_0 = p > 1$ (according to (a)). Also, the functions $\overline{M}_\alpha(s) = |s|^{q'_\alpha}/q'_\alpha$, $1/q_\alpha + 1/q'_\alpha = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, satisfy the Δ_2-condition.

Finally, hypothesis $(\mathrm{H})_4$ is satisfied, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \quad |\alpha| < m,$$

and, by (c), $p_0 = p < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3.2. □

Remark 4.7. The function a in Example 4.1 appears, in a different context, in [11] (see [11, Example 3.1]) while that in Examples 4.5 and 4.6 appears in [6] (see [6, Examples 1 and 2], respectively).

5. Particular cases

In this section we shall prove that some already known multiplicity results for the p-Laplacian may be obtained as particular cases of Theorem 3.2.

Theorem 5.1. *Assume the following:*

(a) $a(t) = |t|^{p-2}t$, $t \in \mathbb{R}$, $2 \le p < N$;

(b) *the Carathéodory functions* $g_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}$, $|\alpha| < m$, *are odd in the second argument:*

$$g_\alpha(x, -s) = -g_\alpha(x, s);$$

(c) *there exist* q_α, $p < q_\alpha < Np/(N-p)$, $|\alpha| < m$, *such that*

$$|g_\alpha(x, s)| \le a_\alpha + b_\alpha |s|^{q_\alpha - 1}, \quad x \in \Omega,\ s \in \mathbb{R},\ a_\alpha, b_\alpha \text{ positive constants;}$$

(d) *if* G_α, $|\alpha| < m$, *are given by* (3.7), *then there exist* $s_\alpha > 0$ *and* $\theta_\alpha > p$ *such that*

$$0 < \theta_\alpha G_\alpha(x, s) \le s g_\alpha(x, s), \quad \text{for a.e. } x \in \Omega \text{ and all } s \text{ with } |s| \ge s_\alpha.$$

Under these conditions, the functional $H : (W_0^{m,p}(\Omega), \|\cdot\|_{m,A}) \to \mathbb{R}$,

$$H(u) = \frac{1}{p^2} \int_\Omega (\sqrt{T[u,u]})^p \, dx - \sum_{|\alpha|<m} \int_\Omega G_\alpha(x, D^\alpha u(x)) \, dx,$$

possesses a sequence of critical positive values which converges to ∞ *and another one, of critical negative values converging to* 0.

Proof. The idea of the proof is as follows: the preceding assumptions entail that the hypotheses of Theorem 3.2 are fulfilled.

First, we prove that hypothesis $(\mathrm{H})_3$ is satisfied. Since $a(t)/t = t^{p-2}$ for all $t > 0$, it follows that $a(t)/t$ is nondecreasing on $(0, \infty)$. In order to prove that (3.2) and (3.3) are satisfied, we will use Lemma 4.2.

In our case, $p^* = p$ and $p < N$ (by (a)). Since

$$A(t) = \frac{1}{p} t^p, \quad \text{for all } t > 0,$$

it follows that (4.3) is satisfied for $C = 1/p$, $\gamma = p$ and any $\delta > 0$.

Secondly, we prove that hypothesis $(\mathrm{H})_1$ is satisfied. By setting

$$M_\alpha(s) = \frac{|s|^{q_\alpha}}{q_\alpha}, \quad |\alpha| < m,\ s \in \mathbb{R},$$

(4.1) may be rewritten as

$$|g_\alpha(x, s)| \le a_\alpha + b_\alpha (q_\alpha - 1)^{1/q'_\alpha} \overline{M}_\alpha^{-1}(M_\alpha(s)), \quad x \in \Omega,\ s \in \mathbb{R},\ |\alpha| < m,$$

thus proving that (3.6) is satisfied.

What remains to be proven is that M_α, $|\alpha| < m$, satisfy the Δ_2-condition and increase essentially more slowly than A_* near infinity, where $A_*(t) = Ct^{Np/(N-p)}$ with $C = ((N-p)/Np^{(p+1)/p})^{Np/(N-p)}$. It is easy to check (by definition) that M_α, $|\alpha| < m$, satisfy the Δ_2-condition. Also, from (c), M_α, $|\alpha| < m$, increase essentially more slowly than A_* near infinity.

The hypothesis $(\mathrm{H})_2$ is covered by (d) (with g_α odd functions in the second argument, according to (b)).

In order to prove that A and $\overline{A}$ satisfy the Δ_2-condition, we will use Lemma 4.3. In our case, as one has already seen, $p^* = p < N$ and $p_0 = p > 1$ (according to (a)). Since $\overline{M}_\alpha(s) = |s|^{q'_\alpha}/q'_\alpha$, $1/q_\alpha + 1/q'_\alpha = 1$, $|\alpha| < m$, $s \in \mathbb{R}$, it is easy to check (by definition) that $\overline{M}_\alpha$, $|\alpha| < m$, satisfy the Δ_2-condition.

Finally, hypothesis $(\mathrm{H})_4$ is satisfied. Indeed, since

$$\gamma_\alpha = \sup_{t>0} \frac{tM'_\alpha(t)}{M_\alpha(t)} = q_\alpha, \quad |\alpha| < m,$$

it follows that $p_0 = p < q_\alpha$, $|\alpha| < m$. The result follows by Theorem 3.2. $\square$

Remark 5.2. Since (see Proposition 3.1)

$$H'(u) = J_a u - \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u),$$

we deduce that the problem

$$\begin{gathered} J_a u = \sum_{|\alpha|<m} (-1)^{|\alpha|} D^\alpha g_\alpha(x, D^\alpha u) \quad \text{in } \Omega, \\ D^\alpha u = 0 \quad \text{on } \partial\Omega,\ |\alpha| \le m-1, \end{gathered}$$

has two sequences of weak solutions in $(W_0^{m,p}(\Omega), \|\cdot\|_{m,A})$.

Remark 5.3. If, under the hypotheses of Theorem 5.1, the quadratic form T is given by $T[u,u] = |\nabla u|^2$, the corresponding results given by Theorem 5.1 and Remark 5.2 are:

(a) the functional $H\colon (W_0^{1,p}(\Omega), \|\cdot\|_{1,A}) \to \mathbb{R}$,

$$H(u) = \frac{1}{p^2} \int_\Omega |\nabla u|^p \, dx - \int_\Omega G(x, u(x)) \, dx,$$

possesses a sequence of critical positive values which converges to ∞ and another one, of critical negative values converging to 0;

(b) the problem

$$\begin{aligned} J_a u &= g(x,u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has two sequences of weak solutions in $(W_0^{1,p}(\Omega), \|\cdot\|_{1,A})$.

REMARK 5.4. It is well-known that the duality mapping

$$\mathcal{J}_a\colon (W_0^{1,p}(\Omega), \|\cdot\|_{1,p}) \to (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})^*,$$
$$\|u\|_{1,p} = \||\nabla u|\|_{L^p(\Omega)},$$

is given by

$$\mathcal{J}_a = -\Delta_p, \quad \Delta_p u = \frac{\partial}{\partial x_i}\bigg(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\bigg).$$

Since $\|u\|_{1,A} = p^{-1/p}\|u\|_{1,p}$, for all $u \in W_0^{1,p}(\Omega)$, one has $J_a = (1/p)\mathcal{J}_a = -(1/p)\Delta_p$. Consequently, under the hypotheses (a)–(d), with $m = 1$, $g_\alpha = g$ and $T[u,u] = |\nabla u|^2$, the problem

$$\begin{aligned} -\Delta_p u &= pg && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

has two sequences of weak solutions in $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$.

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