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EXTENSIONS OF THEOREMS OF RATTRAY AND MAKEEV

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Abstract. We consider extensions of the Rattray theorem and two Makeev's theorems, showing that they hold for several maps, measures, or functions simultaneously, when we consider orthonormal k-frames in $\mathbb{R}^n$ instead of orthonormal bases (full frames).

We also present new results on simultaneous partition of several measures into parts by k mutually orthogonal hyperplanes.

In the case $k = 2$ we relate the Rattray and Makeev type results with the well known embedding problem for projective spaces.

1. Introduction

In this paper we consider extensions of the following results of Rattray and Makeev:

(a) any odd continuous map $S^{n-1} \to S^{n-1}$ maps some orthonormal basis to an orthonormal basis, the Rattray theorem [20];

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(b) for any absolutely continuous probabilistic measure μ in $\mathbb{R}^n$ there exist n mutually orthogonal hyperplanes $H_1, \dots, H_n$ such that any two of them partition μ into 4 equal parts, the Makeev theorem [17, Theorem 4].

These results share a common family of possible solutions, the manifold of all orthonormal basis $\mathrm{O}(n)$ in $\mathbb{R}^n$. Moreover, they can be seen as a consequence of a single result, Theorem 1.1, proved implicitly already in [20].

A continuous function $f\colon S^{n-1} \times S^{n-1} \to \mathbb{R}$ will be called

(a) *odd*, if for any $x, y \in S^{n-1}$

$$f(-x, y) = -f(x, y), \qquad f(x, -y) = -f(x, y);$$

(b) *symmetric*, if for any $x, y \in S^{n-1}$

$$f(x, y) = f(y, x).$$

Theorem 1.1. *Suppose $f\colon S^{n-1} \times S^{n-1} \to \mathbb{R}$ is an odd and symmetric function. Then there exists an orthonormal basis $(e_1, \dots, e_n) \in \mathrm{O}(n)$ such that for any $i < j$*

$$f(e_i, e_j) = 0.$$

Proof. Consider a particular case when $f(x, y)$ is a generic symmetric bilinear form. It follows from the diagonalization theorem in linear algebra that the required orthonormal basis $e_1, \dots, e_n$ exists and is unique modulo the action of the group $W_n = (\mathbb{Z}_2)^n \rtimes \Sigma_n \subset \mathrm{O}(n)$. Here the group W_n acts on basis $(e_1, \dots, e_n) \in \mathrm{O}(n)$ by

$$\varepsilon_i \cdot (e_1, \dots, e_n) = (e'_1, \dots, e'_n) \quad \text{where } e'_j = \begin{cases} -e_j & \text{for } j = i, \\ e_j & \text{for } j \neq i, \end{cases}$$

for the generators $\varepsilon_1, \dots, \varepsilon_n$ of the component $(\mathbb{Z}_2)^n$ and by

$$\pi \cdot (e_1, \dots, e_n) = (e_{\pi(1)}, \dots, e_{\pi(n)})$$

for the permutation $\pi \in \Sigma_n$ from the symmetric group component of W_n.

Let us show that:

(a) the differential of the corresponding system of equations evaluated at the solution $e_1, \dots, e_n$ is nonzero, and

(b) the solution set represents a nonzero element of the 0-homology $H_0(\mathrm{O}(n)/W_n; \mathbb{F}_2)$.

Suppose the base vector e_i has coordinates b_{ij}, and

$$f(x, y) = \sum_i \lambda_i x_i y_i$$

in the coordinate representation. Since f is a generic symmetric bilinear form we can assume that $\lambda_1, \dots, \lambda_n$ are distinct real numbers. The solution is $b_{ij} = \delta_{ij}$, and its first order deformation is $b_{ij} = \delta_{ij} + s_{ij}$, where s_{ij} is a skew symmetric $n \times n$ matrix. Consider

$$f(e_k, e_l) = \sum_i \lambda_i b_{ik} b_{il}.$$

The linear part, with respect to s_{ij}, is

$$df(e_k, e_l) = \sum_i \lambda_i \delta_{ik} s_{il} + \sum_i \lambda_i s_{ik} \delta_{il} = \lambda_k s_{kl} + \lambda_l s_{lk} = (\lambda_k - \lambda_l) s_{kl}.$$

Since all values $\lambda_k - \lambda_l$ are nonzero, that the differentials $df(e_k, e_l)$ give together a bijective map from the space of skew symmetric matrices to the space of all symmetric expressions of the form t_{kl} for $k \neq l$.

Since any f can be W_n-deformed (by a convex combination) to this particular case, it follows that for a generic f the solution set represents the generator of $H_0(\mathrm{O}(n)/W_n; \mathbb{F}_2)$ (and is nonempty). Therefore, the solution set must be nonempty for all other f by compactness considerations. □

In this paper we consider the following generalized problems of Rattray and Makeev type.

1.1. Generalized Rattray problem. Determine the set

$$\mathcal{R}_{\mathrm{odd}}^{\mathrm{orth}} \subset \mathbb{N}^3 \qquad [\mathcal{R}_{\mathrm{odd,sym}}^{\mathrm{orth}} \subset \mathbb{N}^3]$$

of all triples (n, m, k) with the property that for any collection $f_1, \dots, f_m$ of m odd [and symmetric] functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$ there exists an orthonormal k-frame $(e_1, \dots, e_k) \in V_n^k$ such that for any $1 \leq l \leq m$ and $1 \leq i < j \leq k$

$$f_l(e_i, e_j) = 0.$$

Here V_n^k stands for the Stiefel manifold of all orthonormal k-frames in $\mathbb{R}^n$.

This problem has a natural variation when the requirement for the vectors $e_1, \dots, e_k$ to be orthonormal is dropped. Determine the set $\mathcal{R}_{\mathrm{odd}} \subset \mathbb{N}^3$ [$\mathcal{R}_{\mathrm{odd,sym}} \subset \mathbb{N}^3$] off all triples (n, m, k) with the property that for any collection $f_1, \dots, f_m$ of m odd [and symmetric] functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$ there exist k unit vectors $e_1, \dots, e_k$ such that for any $1 \leq l \leq m$ and $1 \leq i < j \leq k$

$$f_l(e_i, e_j) = 0.$$

An elementary observation is that $\mathcal{R}_{\mathrm{odd}}^{\mathrm{orth}} \subset \mathcal{R}_{\mathrm{odd}}$ [$\mathcal{R}_{\mathrm{odd,sym}}^{\mathrm{orth}} \subset \mathcal{R}_{\mathrm{odd,sym}}$] and

$$(n, m, k) \in \mathcal{R}_{\mathrm{odd}} \implies (n, m-1, k) \in \mathcal{R}_{\mathrm{odd}}^{\mathrm{orth}}$$
$$\Big[(n, m, k) \in \mathcal{R}_{\mathrm{odd,sym}} \implies (n, m-1, k) \in \mathcal{R}_{\mathrm{odd,sym}}^{\mathrm{orth}}\Big]$$

by puting the inner product on $\mathbb{R}^n$ for f_m.

1.2. Generalized Makeev problem. Let $H = \{x \in \mathbb{R}^n \mid \langle x, v\rangle = \alpha\}$ be an affine hyperplane in $\mathbb{R}^n$. Here v is a vector in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$ some constant. The affine hyperplane H determines two open halfspaces

$$H^- = \{x \in \mathbb{R}^n \mid \langle x, v\rangle < \alpha\} \quad \text{and} \quad H^+ = \{x \in \mathbb{R}^n \mid \langle x, v\rangle > \alpha\}.$$

Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be an arrangement of affine hyperplanes in $\mathbb{R}^d$. An *orthant* of the arrangement $\mathcal{H}$ is an intersection of halfspaces $\mathcal{O} = H_1^{\alpha_1} \cap \ldots \cap H_k^{\alpha_k}$, for some $\alpha_j \in \mathbb{Z}_2$. For convenience we assume that $\mathbb{Z}_2 = (\{+1, -1\}, \cdot)$ with obvious abbreviation $H^{+1} \equiv H^+$ and $H^{-1} \equiv H^-$. There are 2^k orthants determined by $\mathcal{H}$. The orthants are not necessary non-empty. They can be indexed by elements of the group $(\mathbb{Z}_2)^k$ in a natural way.

Let μ be an absolutely continuous probabilistic measure on $\mathbb{R}^n$. The arrangement $\mathcal{H}$ *equiparts* the measure μ if for each orthant $\mathcal{O}$ determined by the arrangement $\mu(\mathcal{O}) = (1/2^k)\mu(\mathbb{R}^n)$.

Generalized Makeev problem is to determine the set $\mathcal{M} \subset \mathbb{N}^4$ [$\mathcal{M}^{\mathrm{orth}} \subset \mathbb{N}^4$] of all quadruples (n, m, k, l), where $1 \le l \le k$, with the property that for every collection of m absolutely continuous probabilistic measures $\mu_1, \dots, \mu_m$ on $\mathbb{R}^n$ there exist k [mutually orthogonal] hyperplanes $H_1, \dots, H_k$ such that any l of them equipart all the measures.

It is obvious that $\mathcal{M}^{\mathrm{orth}} \subset \mathcal{M}$. Moreover, by taking μ_m to be the uniform probability measure on the unit ball in $\mathbb{R}^n$ we can derive that

$$(n, m, k, l) \in \mathcal{M} \;\Rightarrow\; (n, m-1, k, l) \in \mathcal{M}^{\mathrm{orth}}.$$

The generalized Makeev problem for $l = k$ is known as the generalized Grünbaum mass partition problem as introduced by Grünbaum in [12, 4, Remarks (v)] and further studied by Ramos in [19] and Mani-Levitska, S. Vrećica, R. Živaljević in [16].

2. Statement of main results

Let $A = \mathbb{F}_2[t_1, \dots, t_k]$ denote the polynomial algebra with variables $t_1, \dots, t_k$ of degree 1. Then

$$w_1 = t_1 + \ldots + t_k, \dots, w_k = t_1 \ldots t_k$$

are elementary symmetric polynomials in A with the respect to permutation of variables. Set for $l \ge 1$,

$$\overline{w}_l = \sum_{\substack{i_1, \dots, i_k \ge 0 \\ i_1 + \ldots + k i_k = l}} \binom{i_1 + \ldots + i_k}{i_1 \;\ldots\ldots\; i_k} w_1^{i_1} \ldots w_k^{i_k},$$

where $\binom{i_1+\ldots+i_k}{i_1 \;\ldots\ldots\; i_k}$ stands for $\frac{(i_1+\ldots+i_k)!}{(i_1)! \;\ldots\; (i_k)!}$ modulo 2.

2.1. Rattray type results. These results give sufficient conditions for a triple (n,m,k) to be in $\mathcal{R}_*^*$ and can be formulated in the following way.

THEOREM 2.1. *Let* $(n,m,k)\in\mathbb{N}^3$. *Then*

(a) $\prod_{1\le i<j\le k}(t_i+t_j)^{2m}\notin\langle t_1^n,\dots,t_k^n\rangle \implies (n,m,k)\in\mathcal{R}_{\mathrm{odd}}$,

(b) $\prod_{1\le i<j\le k}(t_i+t_j)^{m}\notin\langle t_1^n,\dots,t_k^n\rangle \implies (n,m,k)\in\mathcal{R}_{\mathrm{odd,sym}}$,

(c) $\prod_{1\le i<j\le k}(t_i+t_j)^{2m}\notin\langle \overline{w}_{n-k+1},\dots,\overline{w}_n\rangle \implies (n,m,k)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd}}$,

(d) $\prod_{1\le i<j\le k}(t_i+t_j)^{m}\notin\langle \overline{w}_{n-k+1},\dots,\overline{w}_n\rangle \implies (n,m,k)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$.

REMARK 2.2. The degree of the polynomial

$$\prod_{1\le i<j\le k}(t_i+t_j)=\det\big(t_i^{j-1}\big)_{i,j=1}^k$$

is at most $k(k-1)/2$ and degree of each variable is at most $k-1$. Therefore,

$$\begin{aligned}(k-1)m<n \implies &\prod_{1\le i<j\le k}(t_i+t_j)^m\notin\langle t_1^n,\dots,t_k^n\rangle\\ \implies &(n,m,k)\in\mathcal{R}_{\mathrm{odd,sym}}.\end{aligned}\tag{2.1}$$

Similarly, $2(k-1)m<n$ implies $(n,m,k)\in\mathcal{R}_{\mathrm{odd}}$.

REMARK 2.3. Direct application of the criterion (d) of the theorem, for example, implies that $(3,2,2)$, $(4,1,2)$, $(4,2,2)$, $(5,m,2)$ for $1\le m\le 6$ and $(5,1,3)$ are elements of $\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$. The most striking example is that $(5,6,2)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$ since the triple does not fulfill even the inequality bound from the previous remark for being element of $\mathcal{R}_{\mathrm{odd,sym}}$. The fact $(5,6,2)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$ is the consequence of

$$(t_1+t_2)^6=t_1^6+t_1^4t_2^2+t_1^2t_2^4+t_2^6\notin\langle\overline{w}_4,\overline{w}_5\rangle$$

where

$$\begin{aligned}\overline{w}_4&=w_1^4+w_1^2w_2+w_2^2=t_1^4+t_1^3t_2+t_1^2t_2^2+t_1t_2^3+t_2^4,\\ \overline{w}_5&=w_1^5+w_1w_2^2=t_1^5+t_1^4t_2+t_1^3t_2^2+t_1^2t_2^3+t_2^5,\end{aligned}$$

and $w_1=t_1+t_2$, $w_2=t_1t_2$.

Let us present some immediate consequences of Theorem 2.1 that generalize results from [18].

Corollary 2.4. *Let* $(n,k,m)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$.

(a) *For every collection* $\phi_1,\dots,\phi_m$ *of* m *odd maps* $S^{n-1}\to S^{n-1}$ *there exists an orthonormal* k*-frame* $(e_1,\dots,e_k)\in V_n^k$ *such that for any* $1\le l\le m$ *the set* $(\phi_l(e_1),\dots,\phi_l(e_k))$ *is an orthonormal frame too.*

(b) *For every collection* $g_1,\dots,g_m$ *of* m *continuous even functions* $\mathbb{R}^n\to\mathbb{R}$ *there exists an orthonormal* k*-frame* $(e_1,\dots,e_k)\in V_n^k$ *such that for any* $1\le l\le m$ *and* $1\le i<j\le k$

$$g_l(e_i+e_j)=g_l(e_i-e_j).$$

Proof. For the first claim take $f_l(x,y)=(\phi_l(x),\phi_l(y))$ and apply Theorem 2.1, while for the second one take $f_l(x,y)=g_l(x+y)-g_l(x-y)$. $\square$

In some particular cases the obvious inequality bound (2.1) can be substantially improved by more precise cohomology computations.

Theorem 2.5. *Let* $n\in\mathbb{N}$ *and* $P(n)=\min\{2^s\mid s\in\mathbb{N},2^s\ge n\}$. *Then*

$$P(n)\ge m+2 \iff n\ge \tfrac12 P(m+2)+1 \implies (n,m,2)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}.$$

A further improvement of this result is possible, relating the Rattray problem for 2-frames to the famous problem of embedding of projective spaces into a Euclidean space.

Theorem 2.6. *If* $\mathbb{R}P^{n-1}$ *cannot be embedded into* $\mathbb{R}^m$ *because of the "deleted square obstruction", then* $(n,m,2)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,symm}}$.

Remark 2.7. The *deleted square obstruction* for an embedding $M\to\mathbb{R}^m$ is the obstruction to the existence of a $\mathbb{Z}_2$-equivariant map $(M\times M)\setminus\Delta(M)\to S^{m-1}$. Here $\mathbb{Z}_2$ acts on the deleted square $(M\times M)\setminus\Delta(M)$ by interchanging coordinates and on S^{m-1} antipodally. The Haefliger theory [13] states that in the range $m\ge 3n/2$ (the *metastable range*) this is the only obstruction for embedding. The results in [9] (see also the table [8] for some low-dimensional cases) show that asymptotically the required inequality for embedding of the projective space has the form $m\ge 2n-O(\log n)$, i.e. falls into the metastable range. It follows that for sufficiently large n the condition $(n,m,2)\in\mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,symm}}$ also has the asymptotic form $m\le 2n-O(\log n)$.

Let us state more results in case $k=3$. If we want to calculate in mod 2 equivariant cohomology, we may consider the Sylow subgroup $W_3^{(2)}=D_8\times\mathbb{Z}_2$ (D_8 is the square group). We obtain the following algebraic criterion.

Theorem 2.8. *Consider the graded algebra* $\mathbb{F}_2[x,y,w,t]$ *with* $\dim x=\dim y=\dim t=1$, $\dim w=2$, *and relation* $xy=0$. *Put*

(a) $w_*=(1+x+y+w)(1+t)$;

(b) $\overline{w}_*=(w_*)^{-1}$.

In the above notation, if $y^m(t^2 + t(x+y) + w)^m \notin \langle \overline{w}_{n-2}, \overline{w}_{n-1}, \overline{w}_n \rangle$ *then* $(n, m, 3) \in \mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,symm}}$.

Remark 2.9. It can be checked "by hand" than $(3, 1, 3) \in \mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,symm}}$, i.e. the Rattray theorem for $n = 3$ follows from this theorem.

The results of Rattray type can be extended also in the following direction. It can be asked in addition for the "diagonal" values $f_l(e_i, e_i)$ to be equal.

Theorem 2.10. *Let k and m be positive integers. There exists a function $n\colon \mathbb{N}^2 \to \mathbb{N}$ such that for every $n \geq n(k, m)$ and any collection $f_1, \dots, f_m$ of m odd functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$ there exists an orthonormal k-frame $(e_1, \dots, e_k) \in V_n^k$ such that for any $1 \leq l \leq m$ and $1 \leq i < j \leq k$*

$$f_l(e_i, e_j) = 0 \quad \textit{and} \quad f_l(e_1, e_1) = \ldots = f_l(e_k, e_k).$$

Remark 2.11. Description of the function $n(k, m)$ remains a challenging open problem.

The final result of Rattray type we present is the following theorem.

Theorem 2.12. *Let $\psi\colon S^{n-1} \to S^{m-1}$ be an odd continuous map and $1 \leq k \leq n$. For any linear subspace $L \subseteq \mathbb{R}^m$ of codimension $n - k$ there exists an orthonormal k-frame $(e_1, \dots, e_k)$ in $\mathbb{R}^n$ such that $(\psi(e_1), \dots, \psi(e_k))$ is an orthonormal k-frame in L.*

Remark 2.13. This theorem implies that m must be at least n (when considered $k = n$), i.e. it implies the Borsuk–Ulam theorem.

2.2. Makeev type results. The following theorem gives sufficient conditions for (n, m, k, l) to be in $\mathcal{M}^*$.

Theorem 2.14. *Let $(n, m, k, l) \in \mathbb{N}^4$. Then*

(a) $$\prod_{\substack{s_1, \dots, s_k \in \mathbb{Z}_2 \\ 1 \leq s_1 + \ldots + s_k \leq l}} (s_1 t_1 + \ldots + s_k t_k)^m \notin \langle t_1^{n+1}, \dots, t_k^{n+1} \rangle \implies (n, m, k, l) \in \mathcal{M},$$

(b) $$\frac{1}{t_1 \cdots t_k} \prod_{\substack{s_1, \dots, s_k \in \mathbb{Z}_2 \\ 1 \leq s_1 + \ldots + s_k \leq l}} (s_1 t_1 + \ldots + s_k t_k)^m \notin \langle \overline{w}_{n-k+1}, \dots, \overline{w}_n \rangle \implies (n, m, k, l) \in \mathcal{M}^{\mathrm{orth}}.$$

Remark 2.15. By considering the maximal degree of the test polynomial in every variable we can get a rough bound

$$n \geq m\left(\sum_{i=0}^{l} \binom{k-1}{i}\right) \implies (n, m, k, l) \in \mathcal{M}.$$

Remark 2.16. Notice that for $m = 1$ and $l = 2$ algebraic conditions of Theorem 2.14(b) and Theorem 2.1(d) coincide.

Remark 2.17. For $l = k$, the case (a) is equivalent to the main result of the paper by Mani-Levitska, S. Vrećica, R. Živaljević [16, Theorem 39]. They obtained that

$$n \geq 2^{q+k-1} + r \implies (n, 2^q + r, k, k) \in \mathcal{M}$$

where $m = 2^q + r$ and $0 \leq r \leq 2^q - 1$.

Similar to Theorem 2.6, we prove another particular result on partitioning measures by pairs of hyperplanes. This result is a projective analogue of the "ham sandwich" theorem [22], [21], the concept of "projective measure partitions" is due to Benjamin Matschke (private communication).

Theorem 2.18. *Suppose $\mathbb{R}P^{n-1}$ cannot be embedded into $\mathbb{R}^m$ because of the "deleted square obstruction". Let $\mu_0, \dots, \mu_m$ be $m + 1$ absolutely continuous probabilistic measures on $\mathbb{R}P^{n-1}$. Then there exists a pair of hyperplanes $H_1, H_2 \subseteq \mathbb{R}P^{n-1}$, partitioning every measure μ_i into two equal parts.*

Remark 2.19. A single hyperplane does not partition a projective space, but two hyperplanes partition it into two parts.

Remark 2.20. The condition is asymptotically $m \leq 2n - O(\log n)$, as in Theorem 2.6.

3. Equivariant cohomology of the Stiefel manifold

Let V_n^k denote the Stiefel manifold of all orthonormal k-frames in $\mathbb{R}^n$. Any subgroup $G \subseteq \mathrm{O}(k)$ acts naturally on k-frames by

$$(e_1, \dots, e_k) \cdot g = \left(\sum_j e_j s_{j1}, \dots, \sum_j e_j s_{jk} \right)$$

where $(e_1, \dots, e_k) \in V_n^k$ and $g = (s_{ij})_{i,j=1}^k \in \mathrm{O}(k)$. The action is right, but it transforms in a left action in the usual way $g \cdot (e_1, \dots, e_k) := (e_1, \dots, e_k) \cdot g^{-1}$.

In this section we compute the Fadell–Husseini index of the Stiefel manifold V_n^k with the respect to the action of any subgroup $G \subseteq \mathrm{O}(k)$ and coefficients $\mathbb{F}_2$, i.e. we determine the generators of the following ideal

$$\mathrm{Index}_{G,\mathbb{F}_2} V_n^k = \ker(H^*(G; \mathbb{F}_2) \longrightarrow H^*(\mathrm{E}G \times_G V_n^k; \mathbb{F}_2)).$$

In particular, we determine explicitly the index with respect to the subgroup $\mathbb{Z}_2^k$ of diagonal matrices with $\{-1, 1\}$ entries on diagonal. One description of the index $\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} V_n^k$ is given in the paper of Fadell and Husseini [10, Theorem 3.16, p. 78].

3.1. The cohomology of the Stiefel manifold V_n^k with $\mathbb{F}_2$ coefficients is the quotient algebra (consult [6])

$$H^*(V_n^k;\mathbb{F}_2) = \mathbb{F}_2[e_{n-k},\dots,e_{n-1}]/\mathcal{J}_n^k$$

where $\deg e_i = i$ and $\mathcal{J}_n^k$ is the ideal generated by the relations

$$e_i^2 = e_{2i} \quad \text{for } 2i \le n-1, \qquad e_i^2 = 0 \quad \text{for } 2i \ge n.$$

In what follows, for a vector bundle $F \to \xi \to B$ we denote by $w_i(\xi) \in H^i(B;\mathbb{F}_2)$ the associated Stiefel–Whitney classes, by $\overline{w}_i(\xi) \in H^i(B;\mathbb{F}_2)$ its dual Stiefel–Whitney classes, $i \ge 0$. There is a relation between these classes expressed via the total class by $w \cdot \overline{w} = 1$ or particularly, for $l \ge 1$ by

$$\overline{w}_l(\xi) = \sum_{\substack{i_1,\dots,i_k \ge 0 \\ i_1+\dots+ki_k = l}} \binom{i_1+\dots+i_k}{i_1 \;\dots\dots\; i_k} w_1^{i_1}(\xi)\dots w_k^{i_k}(\xi).$$

Let us recall that:

(a) the Grassmann manifold $G^k(\mathbb{R}^\infty)$ of all k-flats in $\mathbb{R}^\infty$ is the classifying space of the group $\mathrm{O}(k)$ and we denote $G^k(\mathbb{R}^\infty)$ also by $\mathrm{BO}(k)$,

(b) the Stiefel manifold V_∞^k of all k-frames in $\mathbb{R}^\infty$ as a contractible free $\mathrm{O}(k)$ space serves as a model for $\mathrm{EO}(k)$,

(c) the associated canonical bundle:

$$\mathbb{R}^k \longrightarrow \gamma^k \longrightarrow G^k(\mathbb{R}^\infty)$$

can be seen as a Borel construction of the $\mathrm{O}(k)$-space $\mathbb{R}^k$ (where the action is given by the matrix multiplication from the left):

$$\mathbb{R}^k \longrightarrow \mathrm{EO}(k) \times_{\mathrm{O}(k)} \mathbb{R}^k \longrightarrow \mathrm{BO}(k),$$

(d) the cohomology of the Grassmannian $G^k(\mathbb{R}^\infty) \approx \mathrm{BO}(k)$ with coefficients in $\mathbb{F}_2$ is the polynomial algebra generated by the Stiefel–Whithey classes $w_1,\dots,w_k$ of the canonical vector bundle γ^k:

$$H^*(\mathrm{BO}(k);\mathbb{F}_2) = \mathbb{F}_2[w_1,\dots,w_k].$$

Now we state a very useful result from [6] (see also [15, Theorem 3.3]).

PROPOSITION 3.1. *Let $(E_i^{*,*}, d_i)_{i\ge 2}$ denote the Leray–Serre spectral sequence associated with the Borel construction*

$$\mathbb{R}^k \longrightarrow \mathrm{EO}(k) \times_{\mathrm{O}(k)} \mathbb{R}^k \longrightarrow \mathrm{BO}(k).$$

Then

$$\mathrm{Index}_{\mathrm{O}(k),\mathbb{F}_2} V_n^k = \langle \overline{w}_{n-k+1},\dots,\overline{w}_n\rangle \subset \mathbb{F}_2[w_1,\dots,w_k]$$

where $\overline{w}_i = \overline{w}_i(\gamma^k) = d_{i-1}(e_{i-1})$.

3.2. The Borel construction is a functorial construction and therefore there is a morphism of fiber bundles induced by the inclusion $\iota\colon G\subseteq \mathrm{O}(k)$:

$$\begin{tikzcd} \mathrm{EO}(k)\times_G V_n^k \arrow[r] \arrow[d, "\pi"'] & \mathrm{EO}(k)\times_{\mathrm{O}(k)} V_n^k \arrow[d, "\mu"] \\ \mathrm{B}G \arrow[r, "B\iota"'] & \mathrm{BO}(k) \end{tikzcd}$$

In the bundle on the left, $\mathrm{EO}(k)$ is used as a model for $\mathrm{E}G$. The action of $\mathrm{O}(k)$ on the Stiefel manifold V_n^k is free. Therefore, the $E_\infty^{p,q}$-term of the Leray–Serre spectral sequence for the fibration $\mathrm{EO}(k)\times_{\mathrm{O}(k)} V_n^k\to \mathrm{BO}(k)$ has to vanish for $p+q>\dim V_n^k$. Furthermore, $\mathrm{O}(k)$ acts trivially on the cohomology $H^*(V_n^k;\mathbb{F}_2)$ and so by Proposition 3.1 we have that $d_i(e_i)=\overline{w}_{i+1}$ for $n-k\le i\le n-1$. Here d_i denotes the i-th differential of the Leray–Serre spectral sequence. The morphism of the bundles we considered induces a morphism of the associated Leray–Serre spectral sequences as well. The morphism in the E_2-term on the 0-column is the identity and on the 0-row determines the restriction morphism $\iota^*=\mathrm{res}_G^{\mathrm{O}(k)}$. Thus,

$$\begin{aligned} \mathrm{Index}_{G,\mathbb{F}_2}V_n^k &= \ker\pi^*=\mathrm{res}_G^{\mathrm{O}(k)}(\ker\mu^*)=\mathrm{res}_G^{\mathrm{O}(k)}(\langle\overline{w}_{n-k+1},\dots,\overline{w}_n\rangle)\\ &=\langle \mathrm{res}_G^{\mathrm{O}(k)}(\overline{w}_{n-k+1}),\dots,\mathrm{res}_G^{\mathrm{O}(k)}(\overline{w}_n)\rangle. \end{aligned}$$

We have proved the following claim:

Proposition 3.2. $\mathrm{Index}_{G,\mathbb{F}_2}V_n^k=\langle \mathrm{res}_G^{\mathrm{O}(k)}(\overline{w}_{n-k+1}),\dots,\mathrm{res}_G^{\mathrm{O}(k)}(\overline{w}_n)\rangle$.

3.3. In the final step we identify the restriction morphism $\mathrm{res}_G^{\mathrm{O}(k)}$. Consider $\mathbb{R}^k$ as an $\mathrm{O}(k)$-space where the action is given by the left matrix multiplication. The inclusion $\iota\colon G\subseteq\mathrm{O}(k)$ gives to $\mathbb{R}^k$ the structure of a G-space. Again, there is a morphism of associated Borel constructions, which in this case is also a morphism of vector bundles:

$$\begin{tikzcd} \mathrm{EO}(k)\times_G \mathbb{R}^k \arrow[r] \arrow[d, "\phi"'] & \mathrm{EO}(k)\times_{\mathrm{O}(k)} \mathbb{R}^k \arrow[d, "\psi"] \\ \mathrm{B}G \arrow[r, "B\iota"'] & \mathrm{BO}(k) \end{tikzcd}$$

The naturality of the Stiefel–Whitney classes implies that

$$w_i(\mathrm{EO}(k)\times_G\mathbb{R}^k)=\iota^*(w_i)=\mathrm{res}_G^{\mathrm{O}(k)}(w_i)$$

and consequently

$$\overline{w}_i(\mathrm{EO}(k)\times_G\mathbb{R}^k)=\mathrm{res}_G^{\mathrm{O}(k)}(\overline{w}_i).$$

Thus we have proved the following fact:

Proposition 3.3.

$$\mathrm{Index}_{G,\mathbb{F}_2} V_n^k = \langle \overline{w}_{n-k+1}(\mathrm{EO}(k) \times_G \mathbb{R}^k), \dots , \overline{w}_n(\mathrm{EO}(k) \times_G \mathbb{R}^k)\rangle.$$

3.4. Let $G = \mathbb{Z}_2^k$ be the subgroup of diagonal matrices with $\{-1, 1\}$ entries. Let $A := H^*(\mathbb{Z}_2^k; \mathbb{F}_2) = \mathbb{F}_2[t_1, \dots , t_k]$ be the polynomial algebra with variables $t_1, \dots , t_k$ of degree 1.

It is well known that the k-dimensional real $\mathbb{Z}_2^k$-representation $\mathbb{R}^k$ can be decomposed into the sum of 1-dimensional irreducible real $\mathbb{Z}_2^k$-representation. The total Stiefel–Whitey class of $\mathrm{EO}(k) \times_{\mathbb{Z}_2^k} \mathbb{R}^k$ is given by

$$w(\mathrm{EO}(k) \times_{\mathbb{Z}_2^k} \mathbb{R}^k) = \prod_{i=1}^{k}(1 + t_i) = 1 + \omega_1 + \ldots + \omega_k$$

where ω_i denotes both: the elementary symmetric polynomial of degree i in variables $t_1, \dots , t_k$ and the i-th Stiefel–Whitney class of $w_i(\mathrm{EO}(k) \times_{\mathbb{Z}_2^k} \mathbb{R}^k)$. For example, $\omega_1 = t_1 + \ldots + t_k$ while $\omega_k = t_1 \ldots t_k$. Finally, we obtain the following result.

Proposition 3.4. *Let*

$$\overline{\omega}_l = \sum_{\substack{i_1,\dots,i_k \geq 0 \\ i_1+\ldots+ki_k=l}} \binom{i_1 + \ldots + i_k}{i_1 \;\ldots\ldots\; i_k} \omega_1^{i_1} \ldots \omega_k^{i_k},$$

for $l \geq 1$, then

$$\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} V_n^k = \langle \overline{\omega}_{n-k+1}, \dots , \overline{\omega}_n\rangle \subset A.$$

4. Proof of Rattray type results

4.1. The proofs of these results will be done via the configuration space/test map method. There are two different natural configuration spaces of interest:

$$\begin{aligned} X &= (S^{n-1})^k = \text{the space of all collections of } k \text{ vectors on the sphere } S^{n-1}, \\ Y &= V_n^k \qquad = \text{the space of all orthogonal } k\text{-frames in } \mathbb{R}^n. \end{aligned}$$

The group $W_k = (\mathbb{Z}_2)^k \rtimes \Sigma_k \subset \mathrm{O}(k)$ acts naturally on both configurations spaces. For the generators $\varepsilon_1, \dots , \varepsilon_n$ of the component $(\mathbb{Z}_2)^n$ and $(e_1, \dots , e_k) \in X$ or Y the action is given by

$$\varepsilon_i \cdot (e_1, \dots , e_k) = (e'_1, \dots , e'_k) \quad \text{ where } e'_i = -e_i \text{ and } e'_j = e_j \text{ for } j \neq i,$$

and for the permutation $\pi \in \Sigma_k$ by

$$\pi \cdot (e_1, \dots , e_k) = (e_{\pi(1)}, \dots , e_{\pi(k)}).$$

Let us consider the space M_k of all real $k \times k$-matrices as a real $\mathrm{O}(k)$-representation with respect to the action $m \mapsto gmg^{-1}$ where $m \in M_k$ and g

is $k \times k$-matrix representing an element of $\mathrm{O}(k)$. Then M_k has a structure of a real W_k-representation via the inclusion map $W_k \hookrightarrow \mathrm{O}(k)$. Consider the following real vector subspaces of M_k:

$$\begin{aligned} &R_k \text{ of all } k \times k \text{ symmetric matrices with zeros on the diagonal,} \\ (4.1)\quad &U_k \text{ of all } k \times k \text{ matrices with zeros on the diagonal,} \\ &I_k \text{ of all } k \times k \text{ matrices with zeros outside the diagonal and trace zero.} \end{aligned}$$

These are all real W_k-subrepresentations of M_k. Moreover, when we consider only the subgroup $(\mathbb{Z}_2)^k$ there is a decomposition $U_k \cong R_k \oplus R_k$ of $(\mathbb{Z}_2)^k$-representation.

For an odd (and symmetric) function $f\colon S^{n-1} \times S^{n-1} \to \mathbb{R}$ and k-vectors (k-frame) $(e_1, \ldots, e_k)$, we denote by:

- $\mu_f(e_1, \ldots, e_k) \in U_k$ $[\mu_f(e_1, \ldots, e_k) \in R_k]$ the matrix given by entries
$$(\mu_f(e_1, \ldots, e_k))_{ij} = \begin{cases} f(e_i, e_j) & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$
- $\eta_f(e_1, \ldots, e_k) \in I_k$ the matrix given by entries
$$(\eta_f(e_1, \ldots, e_k))_{ij} = \begin{cases} f(e_i, e_i) - c & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$
where $c = \frac{1}{k}(f(e_1, e_1) + \ldots + f(e_k, e_k))$.

4.2. Proof of Theorem 2.1. Let $(n, m, k) \in \mathbb{N}^3$ and $f_1, \ldots, f_m$ be a collection of m odd (and symmetric) functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$. Let us introduce the test maps for the Rattray problems:

$$\tau_{\mathrm{odd}}\colon X \to U_k^{\oplus m}, \quad \tau_{\mathrm{odd,sym}}\colon X \to R_k^{\oplus m}, \quad \tau_{\mathrm{odd}}^{\mathrm{orth}}\colon Y \to U_k^{\oplus m}, \quad \tau_{\mathrm{odd,sym}}^{\mathrm{orth}}\colon Y \to R_k^{\oplus m}.$$

All four test maps are defined by the same formula

$$(e_1, \ldots, e_k) \overset{\tau_*^*}{\longmapsto} (\mu_{f_r}(e_1, \ldots, e_k))_{r=1}^m$$

assuming appropriate domains and codomains. Have in mind that the test maps are functions of the collection $f_1, \ldots, f_m$, even we abbreviate this from notation. The test maps are all W_k-equivariant maps and moreover have the following obvious but very important properties: If for every collection $f_1, \ldots, f_m$ of m odd (and symmetric) functions $S^{n-1} \times S^{n-1} \to \mathbb{R}$

- $\{\mathbf{0} \in U_k^{\oplus m}\} \in \tau_{\mathrm{odd}}(X)$, then $(n, m, k) \in \mathcal{R}_{\mathrm{odd}}$,
- $\{\mathbf{0} \in U_k^{\oplus m}\} \in \tau_{\mathrm{odd}}(X)$, then $(n, m, k) \in \mathcal{R}_{\mathrm{odd}}$,
- $\{\mathbf{0} \in R_k^{\oplus m}\} \in \tau_{\mathrm{odd,sym}}(X)$, then $(n, m, k) \in \mathcal{R}_{\mathrm{odd,sym}}$,
- $\{\mathbf{0} \in U_k^{\oplus m}\} \in \tau_{\mathrm{odd}}^{\mathrm{orth}}(Y)$, then $(n, m, k) \in \mathcal{R}_{\mathrm{odd}}^{\mathrm{orth}}$,
- $\{\mathbf{0} \in R_k^{\oplus m}\} \in \tau_{\mathrm{odd,sym}}^{\mathrm{orth}}(Y)$, then $(n, m, k) \in \mathcal{R}_{\mathrm{odd,sym}}^{\mathrm{orth}}$.

Let us assume that Theorem 2.1 fails in each case. This means that for a specific collection $f_1, \dots, f_m$ of m odd (and symmetric) functions $\mathbf{0} \in U_k^{\oplus m}$ or $\mathbf{0} \in R_k^{\oplus m}$ is not in the image of any of the test maps. Therefore, we have constructed the following W_k-equivariant maps

$$X \to U_k^{\oplus m} \backslash \{\mathbf{0}\}, \quad X \to R_k^{\oplus m} \backslash \{\mathbf{0}\}, \quad Y \to U_k^{\oplus m} \backslash \{\mathbf{0}\}, \quad Y \to R_k^{\oplus m} \backslash \{\mathbf{0}\}, \tag{4.2}$$

i.e. after W_k-equivariant homotopy, the W_k-equivariant maps

$$X \to S(U_k^{\oplus m}), \quad X \to S(R_k^{\oplus m}), \quad Y \to S(U_k^{\oplus m}), \quad Y \to S(R_k^{\oplus m}). \tag{4.3}$$

Obviously all these maps are $\mathbb{Z}_2^k$-equivariant maps, where $\mathbb{Z}_2^k$ is the diagonal subgroup of W_k.

The basic monotonicity property of the Fadell–Husseini index theory [10] states that when there is a G map $A \to B$ between G-spaces A and B there has to be an inclusion of associated indexes $\mathrm{Index}_{G,*} A \supseteq \mathrm{Index}_{G,*} B$. Using the subgroup $\mathbb{Z}_2^k$ of W_k the maps (4.3) induce the following inclusions

$$\begin{aligned} \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} X &\supseteq \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(U_k^{\oplus m}, \quad & \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} X &\supseteq \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(R_k^{\oplus m}), \\ \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} Y &\supseteq \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(U_k^{\oplus m}), \quad & \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} Y &\supseteq \mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(R_k^{\oplus m}). \end{aligned} \tag{4.4}$$

We determine all Fadell–Husseini indexes appearing in (4.4).

Claim 4.1. *With notation already introduced:*

(a) $\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} X = \langle t_1^n, \dots, t_k^n \rangle$,
(b) $\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} Y = \langle \overline{\omega}_{n-k+1}, \dots, \overline{\omega}_n \rangle$,
(c) $\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(R_k^{\oplus m}) = \Big\langle \prod_{1 \le a < b \le k} (t_a + t_b)^m \Big\rangle$,
(d) $\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(U_k^{\oplus m}) = \Big\langle \prod_{1 \le a < b \le k} (t_a + t_b)^{2m} \Big\rangle$.

Proof. (a) Since the $\mathbb{Z}_2^k$-action on X is component-wise antipodal the index of X is computed in the paper of Fadell and Husseini [10, Example 3.3, p. 76].
(b) This fact is established in Proposition 3.4.
(c) Let us denote by R_{ab}, for $1 \le a < b \le k$, the 1-dimension real vector subspace of R_k described by

$$R_{ab} = \{m \in R_k \mid m_{ij} = 0 \text{ for } (i,j) \notin \{(a,b),(b,a)\} \text{ and } m_{ab} = m_{ba} \in \mathbb{R}\}.$$

The subspace R_{ab} is $\mathbb{Z}_2^k$-invariant and

$$\varepsilon_i \cdot m = \begin{cases} -m & \text{for } i \in \{a,b\}, \\ m & \text{for } i \in \{1,\dots,k\} \setminus \{a,b\}. \end{cases}$$

Moreover, $R_k \cong \bigoplus_{1\le a<b\le k} R_{ab}$ as a $\mathbb{Z}_2^k$-module. Since the Fadell–Husseini index of a sphere in this case is a principal ideal generated by the Euler class (= the top Stiefel–Whitney class) of the vector bundle

$$R_k \longrightarrow \mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} R_k \longrightarrow \mathrm{B}\mathbb{Z}_2^k$$

then

$$\mathfrak{e}(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} R_k) = \prod_{1\le a<b\le k} \mathfrak{e}(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} R_{ab}) = \prod_{1\le a<b\le k} (t_a + t_b).$$

For details consult [5, Proof of Proposition 3.11]. It follows directly that

$$\mathfrak{e}(\mathrm{E}\mathbb{Z}_2^k \times_{\mathbb{Z}_2^k} R_k^{\oplus m}) = \prod_{1\le a<b\le k} (t_a + t_b)^m$$

and consequently

$$\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S(R_k^{\oplus m}) = \Big\langle \prod_{1\le a<b\le k} (t_a + t_b)^m \Big\rangle.$$

(d) Follows from the decomposition $U_k \cong R_k \oplus R_k$ of $\mathbb{Z}_2^k$-module. $\square$

Now, the inclusions (4.4) with just determined indexes imply that:

$$\begin{aligned}
\prod_{1\le a<b\le k} (t_a + t_b)^m &\in \langle t_1, \dots, t_k\rangle,\\
\prod_{1\le a<b\le k} (t_a + t_b)^m &\in \langle t_1, \dots, t_k\rangle,\\
\prod_{1\le a<b\le k} (t_a + t_b)^m &\in \langle \overline{\omega}_{n-k+1}, \dots, \overline{\omega}_n\rangle,\\
\prod_{1\le a<b\le k} (t_a + t_b)^m &\in \langle \overline{\omega}_{n-k+1}, \dots, \overline{\omega}_n\rangle.
\end{aligned}$$

This gives a *contradiction* with the assumptions of Theorem 2.1. Therefore, all claims of Theorem 2.1 hold.

4.3. Proof of Theorem 2.5. Before starting the proof let us once more isolate an important property of Stiefel–Whitney classes already used in the proof of Theorem 2.1. Let H be a subgroup of a group G and V a real G-representation. Then the following equality between the total Stiefel–Whitney classes holds:

$$\begin{aligned}
&w(\mathrm{E}H \times_H V) = \mathrm{res}_H^G(w(\mathrm{E}G \times_G V))\\
&\qquad\iff w_i(\mathrm{E}H \times_H V) = \mathrm{res}_H^G(w_i(\mathrm{E}G \times_G V)) \quad \text{for all } i \ge 1,
\end{aligned}$$

where V inherits the H-representation structure from the inclusion map $H \hookrightarrow G$.

In the proof we use the complete group of symmetries $W_2 = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_2 = (\langle\varepsilon_1\rangle \times \langle\varepsilon_2\rangle) \rtimes \langle\sigma\rangle$ which is isomorphic to the dihedral group D_8. The cohomology of the dihedral group D_8 with $\mathbb{F}_2$ coefficients is given by

$$H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy\rangle,$$

where $\deg x = \deg y = 1$ and $\deg w = 2$. Consult [1, Section IV.1, p. 116] or [5, Section 4.2]. In what follows we use the notations introduced in the paper [5, Section 4.3.2]. For example subgroup $(\mathbb{Z}_2)^2$ is denoted by H_1, while subgroup $\langle\sigma\rangle$ is either K_4 or K_5. Let us assume for clarity that $K_5 = \langle\sigma\rangle$.

Let us consider $W_2 = D_8$ and its already introduced representations R_2 and $\mathbb{R}^2$. Computation of the total Stiefel–Whitney class $w(\mathrm{E}(\mathbb{Z}_2)^2 \times_{(\mathbb{Z}_2)^2} R_2)$ conducted in Section 4.2, when translated into the notation of [5, Section 4.3.2], gives us that

$$w(\mathrm{E}H_1 \times_{H_1} R_2) = 1 + (a + a + b) = 1 + b$$

Moreover, since $\mathrm{E}K_5 \times_{K_5} R_2$ is a trivial vector bundle

$$w(\mathrm{E}K_5 \times_{K_5} R_2) = 1.$$

Thus, the restriction diagram presented in [5, Section 4.3.2, (26) and (27)] implies that

$$w(\mathrm{E}D_8 \times_{D_8} R_2) = 1 + y. \tag{4.5}$$

On the other hand, presented in the new notation

$$w(\mathrm{E}H_1 \times_{H_1} \mathbb{R}^2) = (1 + a)(1 + a + b) = 1 + b + a(a + b).$$

The 2-dimensional real K_5-representation $\mathbb{R}^2$ can be decomposed into the direct sum $\mathbb{R}^2 \cong V_0 \oplus V_1$ of the trivial 1-dimensional real K_5-representation V_0 and the 1-dimensional real K_5-representation V_1 where the action of generator $\sigma \in K_5$ is given by $\sigma \cdot v = -v$, for $v \in V_1$. Then the total Stiefel–Whitney class is

$$w(\mathrm{E}K_5 \times_{K_5} \mathbb{R}^2) = 1 + t_5.$$

Again the restriction diagram [5, Section 4.3.2, (26) and (27)] implies that

$$w(\mathrm{E}D_8 \times_{D_8} \mathbb{R}^2) = 1 + (y + x) + w. \tag{4.6}$$

Proposition 4.2. *With notation already introduced:*

(a) $\mathrm{Index}_{D_8,\mathbb{F}_2} V_n^2 = \langle \overline{w}_{n-1}(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), \overline{w}_n(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2)\rangle \subseteq H^*(D_8, \mathbb{F}_2)$ *where*

$$(1 + \overline{w}_1(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2) + \overline{w}_2(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2) + \dots)(1 + (y + x) + w) = 1.$$

(b) $\mathrm{Index}_{D_8,\mathbb{F}_2} S(R_2^{\oplus m}) = \langle y^m\rangle$.

(c) $y^m \notin \langle \overline{w}_{n-1}(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), \overline{w}_n(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2)\rangle \Longrightarrow (n, m, 2) \in \mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$.

(d) $y^m \notin \langle \overline{w}_{n-1}(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), \overline{w}_n(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), x\rangle \implies (n, m, 2) \in \mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}$.

PROOF. (a) Proposition 3.3 together with the evaluated total Stiefel–Whitney class (4.6) implies the claim.

(b) From (4.5) it follows that $\mathfrak{e}(\mathrm{E}D_8 \times_{D_8} R_2) = y$ and consequently $\mathfrak{e}(\mathrm{E}D_8 \times_{D_8} R_2^{\oplus m}) = y^m$. Since the Fadell–Husseini index of a sphere in this case is a principal ideal generated by the Euler class [5, Proof of Proposition 3.11] the claim is proved.

(c) This is a direct consequence of the configuration test map construction presented at the beginning of Section 4.2.

(d) If y^m is not an element of the bigger ideal

$$\langle \overline{w}_{n-1}(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), \overline{w}_n(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), x\rangle$$

it certainly can not belong to the smaller ideal

$$\langle \overline{w}_{n-1}(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), \overline{w}_n(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2)\rangle.$$

The statement follows from (c). □

Hence, the final effort is to determine a condition on the integer m such that

$$y^m \notin \langle \overline{w}_{n-1}(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), \overline{w}_n(\mathrm{EO}(2) \times_{D_8} \mathbb{R}^2), x\rangle$$

or $0 \neq y^m \in \mathbb{F}_2[y, w]/\langle \overline{w}_{n-1}, \overline{w}_n\rangle$ where $(1 + y + w)(1 + \overline{w}_1 + \overline{w}_2 + \dots) = 1$.

If y and w are interpreted as the first and the second Stiefel–Whitney class in the cohomology of the Grassmannian $G^2(\mathbb{R}^n)$ we can identify $\mathbb{F}_2[y, w]/\langle \overline{w}_{n-1}, \overline{w}_n\rangle$ with $H^*(G^2(\mathbb{R}^n); \mathbb{F}_2)$. Then our final step coincides with the well known problem of *determining the height* (*maximal nonzero power*) *of the first Stiefel–Whitney class in the cohomology of the Grassmannian* $G^2(\mathbb{R}^n)$. In [14, Proposition 2.6, p. 525] the following statement is proved:

LEMMA 4.3. *Let $n \geq 2$ and let $P(n) := 2^s$ be the minimal power of two, satisfying $2^s \geq n$. For the first Stiefel–Whitney class w_1 of the Grassmannian $G^2(\mathbb{R}^n)$ holds*

$$w_1^{2^s-2} \neq 0 \quad \text{and} \quad w_1^{2^s-1} = 0.$$

Therefore,

$$P(n) \geq m + 2 \iff n \geq \frac{1}{2}P(m + 2) + 1 \implies (n, m, 2) \in \mathcal{R}^{\mathrm{orth}}_{\mathrm{odd,sym}}.$$

4.4. Proof of Theorem 2.6. Consider the Stiefel manifold V_n^2 with D_8 action on it. We want to know whether V_n^2 can be mapped D_8-equivariantly to $(R_2)^m \setminus \{\mathbf{0}\}$.

Denote by σ_1, σ_2, τ the generators of D_8, where σ_1 and σ_2 reflect the base vectors in $\mathbb{R}^2$, and τ transposes the base vectors. R_2 is the one-dimensional real D_8-representation on which σ_1 and σ_2 act antipodaly, and τ acts trivially.

Now consider an automorphism of D_8, defined by

$$\sigma_1' = \sigma_1\sigma_2\tau. \qquad \sigma_2' = \tau, \qquad \tau' = \sigma_1.$$

Under this automorphism the representation of D_8 on $\mathbb{R}^2$ remains the same (it is sufficient to change the base $e_1' = e_1+e_2$, $e_2' = -e_1+e_2$). The representation R_2 is now given by trivial action of σ_1' and σ_2' and by antipodal action of τ'. Thus, we pass to the space $X_n = V_n^2/(\sigma_1', \sigma_2')$ of all ordered pairs of orthogonal lines through the origin in $\mathbb{R}^n$. This space has the action of $\mathbb{Z}_2 = (\tau')$ which permutes the lines. We want to know whether X can be mapped $\mathbb{Z}_2$-equivariantly to $\gamma^m \setminus \{\mathbf{0}\}$, where γ is the unique non-trivial one-dimensional representation of $\mathbb{Z}_2$. It is well known that X is homotopy equivalent to the deleted square of the projective space $\mathbb{R}P^{n-1}$, i.e.

$$X \simeq (\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}) \setminus \Delta(\mathbb{R}P^{n-1}).$$

The existence of a $\mathbb{Z}_2$-equivariant map $X \to S(\gamma^m)$ is exactly the "deleted square obstruction" for the embedding of $\mathbb{R}P^{n-1}$ to $\mathbb{R}^m$.

The idea of considering the same automorphism of D_8 was used by González and Landweber in [11], where the deleted square obstruction is related to another problem of finding the symmetric topological complexity of the projective space.

4.5. Proof of Theorem 2.8. We consider the group $G := W_3^{(2)} = D_8 \times \mathbb{Z}_2$. We already know that

$$H^*(D_8, \mathbb{F}_2) = \mathbb{F}_2[x, y, w]/\langle xy\rangle, \quad H^*(\mathbb{Z}_2, \mathbb{F}_2) = \mathbb{F}_2[t],$$

and therefore $H^*(G, \mathbb{F}_2) = \mathbb{F}_2[x, y, w, t]/\langle xy\rangle$ by the Künneth formula. The Stiefel–Whitney class of the standard G-representation on $\mathbb{R}^3$ is

$$w(\mathbb{R}^3) = (1 + x + y + w)(1 + t),$$

and the Euler class of the representation R_3 is

$$\mathfrak{e}(R_3) = y(t^2 + t(x + y) + w),$$

because $\mathbb{R}^3(G) = \mathbb{R}^2(D_8) \oplus \mathbb{R}^1(\mathbb{Z}_2)$ and $R_3(G) = R_2(D_8) \oplus \mathbb{R}^2(D_8) \otimes \mathbb{R}^1(\mathbb{Z}_2)$ in the obvious notation. The rest of the proof proceeds in the footsteps of the proof of Theorem 2.1.

4.6. Proof of Theorem 2.10. Before proving Theorem 2.10 we recall some basic facts and results on the following Borsuk–Ulam type problem (consult the book [3]).

Problem 4.4. Let G be a finite group and V its real representation such that $V^G = \{0\}$. Determine the conditions for the vector bundle $EG \times V \to EG$ to have a G-equivariant nonzero section.

The following result for p-groups will be used, consult [2]–[4], [7].

Lemma 4.5. *Let G be a p-group and V its real representation such that $V^G = \{\mathbf{0}\}$. Then the image of an equivariant map $f\colon EG \to V$ intersects $V^G = \mathbf{0}$. Moreover, there exists an integer $n(G,V)$ such that for every free G-space X is $(n-1)$-connected where $n \geq n(G,V)$, the image of an equivariant map $f\colon X \to V$ meets $V^G = \mathbf{0}$.*

In order to prove Theorem 2.10 we slightly change the configuration test map construction given at the beginning of this chapter. Let us fix positive integers k and m, and consider a collection of m odd functions $f_1, \dots, f_m$. The test map in this case is the W_k-equivariant map $\upsilon\colon Y \to R_k^{\oplus m} \oplus I_k^{\oplus m}$ defined by

$$(e_1, \dots, e_k) \xmapsto{\upsilon} (\mu_{f_r}(e_1, \dots, e_k))_{r=1}^m \oplus (\eta_{f_r}(e_1, \dots, e_k))_{r=1}^m$$

where Y stands for the Stiefel manifold V_n^k as before. If there exists a positive integer $n = n(k,m)$ such that there is no W_k-equivariant map

$$Y \to (R_k^{\oplus m} \oplus I_k^{\oplus m}) \setminus \{\mathbf{0}\} \to S(R_k^{\oplus m} \oplus I_k^{\oplus m})$$

then Theorem 2.10 is proved.

Without loss of generality we may increase n and k in such a way that k becomes power of 2. This can be done since we do not need an optimal n and moreover proving the theorem for bigger k and fixed n and m yields the same result for smaller k. Now consider the 2-Sylow subgroup $W_k^{(2)}$ of W_k. Since the $W_k^{(2)}$-fixed point set of the representation $R_k^{\oplus m} \oplus I_k^{\oplus m}$ is trivial, i.e. $(R_k^{\oplus m} \oplus I_k^{\oplus m})^{W_k^{(2)}} = \{\mathbf{0}\}$ the previously presented lemma implies that every map $Y \to R_k^{\oplus m} \oplus I_k^{\oplus m}$ must meet origin. Thus there cannot be any $W_k^{(2)}$-equivariant (and consequently W_k-equivariant) map $Y \to S\left(R_k^{\oplus m} \oplus I_k^{\oplus m}\right)$. This completes the proof of the theorem.

4.7. Proof of Theorem 2.12. Let $\lambda_1, \dots, \lambda_{n-k}$ be independent linear forms defining the subspace L in $\mathbb{R}^m$. In this proof we take $\mathbb{R}^k$ to be an $\mathrm{O}(k)$-representation where the action is given by the left matrix multiplication. The inclusion $W_k \subseteq \mathrm{O}(k)$ gives to $\mathbb{R}^k$ also the structure of a W_k-representation. Let us denote this W_k-representation by P_k. Consider the following W_k-equivariant maps

- $\phi_0\colon V_n^k \to R_k$ given by
$$\phi_0(e_1, \dots , e_k) = (\psi(e_i), \psi(e_j))_{1\le i<j\le k},$$
- $\phi_r\colon V_n^k \to P_k$, for $1 \le r \le n-k$, given by
$$\phi_r(e_1, \dots , e_k) = (\lambda_r(\psi(e_1)), \dots , \lambda_r(\psi(e_k))) \quad \text{for } 1 \le i \le k.$$

The sum of these maps, the W_k-equivariant map,
$$\phi = \phi_0 \oplus \phi_1 \oplus \dots \oplus \phi_{n-k}\colon V_n^k \to R_k \oplus (P_k)^{n-k}$$
has the property that if the image of ϕ meets the zero in $R_k \oplus P_k^{n-k}$ then the theorem follows. It is sufficient to show that the Euler class
$$\mathfrak{e}(R_k \oplus P_k^{n-k}) \in H^*(\mathrm{B}W_k; \mathbb{F}_2)$$
has nonzero image in $H^*_{W_k}(V_n^k; \mathbb{F}_2)$, i.e.
$$\mathfrak{e}(R_k \oplus P_k^{n-k}) \notin \mathrm{Index}_{\mathbb{W}_k,\mathbb{F}_2} V_n^k.$$

Let us prove non-vanishing of the Euler class by counting zeroes of a generic map. We construct another W_k-equivariant map:
$$\tau\colon V_n^k \to R_k \oplus P_k^{n-k}$$
with the unique (up to W_k-action) non-degenerated zero. This will imply that $\mathfrak{e}(R_k \oplus P_k^{n-k}) \neq 0$ as an element of $H^*_{W_k}(V_n^k; \mathbb{F}_2)$.

Let $M = \mathbb{R}^k \subseteq \mathbb{R}^n$ be a standard inclusion, and let $f(x, y)$ be a symmetric quadratic form, such that $f|_{M\times M}$ is generic. Put
$$\tau_0(e_1, \dots , e_k) = (f(e_i, e_j))_{1\le i<j\le k},$$
and for $1 \le r \le n - k$
$$\tau_r(e_1, \dots , e_k) = (x_{k+r}(e_1), \dots , x_{k+r}(e_k)),$$
where x_{k+r} are coordinate functions in $\mathbb{R}^n$. Then a unique (up to W_k-action) basis in M is mapped by τ to zero; because the conditions $\tau_r(e_1, \dots , e_k) = 0$ (for $1 \le r \le n - k$) imply $e_1, \dots , e_k \in M$ and condition $\tau_0(e_1, \dots , e_k) = 0$ implies that $f|_{M\times M}$ is diagonal in the basis $(e_1, \dots , e_k)$ of M. This zero is non-degenerate, because the image of the differential $d\tau$ at $(e_1, \dots , e_k)$

- contains R_k, similar to the proof of the Rattray theorem;
- surjects onto P_k^{n-k}, because in the first order approximation the frame $(e_1 + \delta_1, \dots , e_k + \delta_k)$ is orthonormal for any $\delta_1, \dots , \delta_k \in M^\perp$.

Thus $0 \neq \mathfrak{e}(R_k \oplus P_k^{n-k}) \in H^*_{W_k}(V_n^k; \mathbb{F}_2)$ and the proof is complete.

5. Proof of Makeev type results

5.1. Proof of Theorem 2.14. Makeev type results will be considered via the classical configuration space/test map scheme used for mass partition problems by hyperplanes, consult [16] or [5] for more details. We consider two different configuration spaces depending whether we considerconfigurations of orthogonal hyperplanes or not.

Let $\mathbb{R}^n$ be embedded in $\mathbb{R}^{n+1}$ by $(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n, 1)$. Every oriented affine hyperplane H in $\mathbb{R}^n$ determines a unique oriented hyperplane H' through the origin in $\mathbb{R}^{n+1}$ by $H' \cap \mathbb{R}^n = H$. Converse is also true if the hyperplane $x_{n+1} = 0$ is excluded. Any oriented hyperplane H in $\mathbb{R}^{n+1}$ passing through the origin is uniquely determined by the unit vector $v \in S^d$ pointing inside the halfspace H^+. Such a hyperplane we denote also by H_v. Notice that $H^-_{-v} = H^+_v$. Thus, the space of all oriented affine hyperplanes in $\mathbb{R}^n$ (including two hyperplanes at "infinity") can be considered to be the sphere S^n. The first configuration space we consider is

$$X = (S^n)^k = \text{ the space of all collections of } k \text{ oriented affine hyperplanes in } \mathbb{R}^n.$$

Let μ be an absolutely continuous probabilistic measure on $\mathbb{R}^n$ with connected support. Then the second configuration space $Y_\mu = V^k_n$ is shaped by μ in the following way: every orthonormal k-frame $(e_1, \dots, e_k) \in V^k_n$ determines a unique collection of k oriented affine hyperplanes $(H_1, \dots, H_k)$ in $\mathbb{R}^n$ with the property that $e_i \perp H_i$ and $\mu(H^+_i) = \mu(H^-_i)$ for all $1 \leq i \leq k$. This is because for every given direction e_i there is a unique hyperplane orthogonal to e_i that partitions μ into equal halves. In case μ has disconnected support, we may approximate μ by a sequence of measures with connected support, prove the theorem in this case, and then go to the limit using the compactness of the following space: for a given $0 < \varepsilon < 1$ consider the space of hyperplanes H that partition μ into parts H^+, H^- with difference $|\mu(H^+) - \mu(H^-)| \leq \epsilon$.

The group $W_k = (\mathbb{Z}_2)^k \rtimes \Sigma_k \subset \mathrm{O}(k)$ acts on both configuration spaces X and Y in the same way as in Section 4.

Before defining the test maps let us introduce a particular W_k and $(\mathbb{Z}_2)^k$-representation on the vector space $\mathbb{R}^{2^k}$and study its structure. If we assume that the coordinate functions $x_{(a_1,\dots,a_k)}$ on $\mathbb{R}^{2^k}$ are indexed by the elements $(a_1, \dots, a_k)$ of the group $(\mathbb{Z}_2)^k$, then the W_k-action we consider is given by

$$((b_1, \dots, b_k) \rtimes \pi) \cdot x_{(a_1,\dots,a_k)} = x_{(b_1 a_{\pi^{-1}(1)}, \dots, b_k a_{\pi^{-1}(k)})}$$

where $(b_1, \dots, b_k) \in (\mathbb{Z}_2)^k$ and $\pi \in \Sigma_k$. The inclusion $(\mathbb{Z}_2)^k \subset W_k$ induces also the structure of $(\mathbb{Z}_2)^k$-representation on $\mathbb{R}^{2^k}$.

All real irreducible representations of the group $(\mathbb{Z}_2)^k$ are all 1-dimensional. They are completely determined by characters $\chi\colon (\mathbb{Z}_2)^k \to \mathbb{Z}_2$.

For $(a_1, \dots, a_k) \in (\mathbb{Z}_2)^k = \{+1, -1\}^{2^k}$, let

$$V_{a_1 \dots a_k} = \operatorname{span}\{v_{a_1,\dots,a_k}\} \subset \mathbb{R}^{2^k}$$

denotes the 1-dimensional representation given by

$$\varepsilon_i \cdot v_{a_1 \dots a_k} = a_i \; v_{a_1 \dots a_k}.$$

Then there is a decomposition of the real $(\mathbb{Z}_2)^k$-representation

$$\mathbb{R}^{2^k} \cong \sum_{a_1,\dots,a_k \in (\mathbb{Z}_2)^k} V_{a_1 \dots a_k} \cong V_{+\dots+} \oplus \sum_{a_1,\dots,a_k \in (\mathbb{Z}_2)^k \setminus \{+\dots+\}} V_{a_1,\dots,a_k}.$$

Observe that $V_{+\dots+}$ is the trivial 1-dimensional real $(\mathbb{Z}_2)^k$-representation. In order to simplify further notation let us define for $1 \le i \le j \le k$ the following $(\mathbb{Z}_2)^k$-representation

$$S_{ij} = \sum_{\substack{a_1,\dots,a_k \in (\mathbb{Z}_2)^k \setminus \{+\dots+\} \\ i \le s(a_1,\dots,a_k) \le j}} V_{a_1 \dots a_k}$$

where $s(a_1, \dots, a_k)$ denotes the number of -1 in the sequence $(a_1, \dots, a_k)$.

Let $\mu_1, \dots, \mu_m$ be a collection of m absolutely continuous probabilistic measures on $\mathbb{R}^n$. The test maps we consider

$$\tau \colon X \to S_{1l}^{\oplus m} \quad \text{and} \quad \tau^{\mathrm{orth}} \colon Y_{\mu_1} \to S_{1l}^{\oplus m}$$

are defined by

$$(v_1, \dots, v_k) \overset{\tau}{\longmapsto} \left(\left(\mu_i(H_{v_1}^{a_1} \cap \dots \cap H_{v_k}^{a_k}) - \frac{1}{2^k} \mu_i(\mathbb{R}^d) \right)_{(a_1,\dots,a_k) \in (\mathbb{Z}_2)^k} \right)_{i \in \{1,\dots,m\}},$$

$$(e_1, \dots, e_k) \overset{\tau^{\mathrm{orth}}}{\longmapsto} \left(\left(\mu_i(H_{e_1}^{a_1} \cap \dots \cap H_{e_k}^{a_k}) - \frac{1}{2^k} \mu_i(\mathbb{R}^d) \right)_{(a_1,\dots,a_k) \in (\mathbb{Z}_2)^k} \right)_{i \in \{1,\dots,m\}},$$

for $(v_1, \dots, v_k) \in X$ and $(e_1, \dots, e_k) \in Y_{\mu_1}$. Since the configuration space Y_{μ_1} is chosen in such a way that each hyperplane equipartitions the measure μ_1 the test map τ^{orth} factors

$$Y_{\mu_1} \xrightarrow{\rho} S_{2l} \oplus S_{1l}^{\oplus(m-1)} \xrightarrow{\iota} S_{1l}^{\oplus m}$$

so that $\tau^{\mathrm{orth}} = \iota \circ \rho$ and ι is induced by the inclusion $S_{2l} \to S_{1l}$.

All test maps τ, τ^{orth} and ρ are W_k-equivariant maps, when the introduced actions on the spaces are assumed. The key property of these test maps is that: For every collection $\mu_1, \dots, \mu_m$ of m absolutely continuous probabilistic measures on $\mathbb{R}^n$:

- if $\{\mathbf{0} \in S_{1l}^{\oplus m}\} \in \tau(X)$, then $(n, m, k, l) \in \mathcal{M}$,
- if $\{\mathbf{0} \in S_{2l} \oplus S_{1l}^{\oplus(m-1)}\} \in \rho(Y_{\mu_1})$, then $(n, m, k, l) \in \mathcal{M}^{\mathrm{orth}}$.

Using the contraposition we get that

- $(n,m,k,l) \notin \mathcal{M}$
  $\Longrightarrow$ there exists a collection of m absolutely continuous probabilistic measures on $\mathbb{R}^n$ such that $\{\mathbf{0} \in S_{1l}^{\oplus m}\} \notin \tau(X)$
  $\Longrightarrow$ there exists a W_k-equivariant map
$$X = (S^n)^k \to S_{1l}^{\oplus m} \setminus \{\mathbf{0}\} \to S(S_{1l}^{\oplus m}),$$
- $(n,m,k,l) \in \mathcal{M}^{\mathrm{orth}}$
  $\Longrightarrow$ there exists a collection of m absolutely continuous probabilistic measures on $\mathbb{R}^n$ such that $\{\mathbf{0} \in S_{2l} \oplus S_{1l}^{\oplus(m-1)}\} \notin \rho(Y_{\mu_1})$
  $\Longrightarrow$ there exists a W_k-equivariant map
$$Y_{\mu_1} = V_n^k \to S_{2l} \oplus S_{1l}^{\oplus(m-1)} \setminus \{\mathbf{0}\} \to S(S_{2l} \oplus S_{1l}^{\oplus(m-1)}).$$

This implies that

- if there is no W_k-equivariant map $X = (S^n)^k \to S(S_{1l}^{\oplus m})$, then $(n,m,k,l) \in \mathcal{M}$,
- if there is no W_k-equivariant map $Y_{\mu_1} = V_n^k \to S(S_{2l} \oplus S_{1l}^{\oplus(m-1)})$, then $(n,m,k,l) \in \mathcal{M}^{\mathrm{orth}}$.

Therefore, by proving the following statement we conclude the proof of Theorem 2.14.

PROPOSITION 5.1.

(a) *If*
$$\prod_{\substack{s_1,\dots,s_k \in \mathbb{Z}_2 \\ 1 \le s_1+\dots+s_k \le l}} (s_1t_1 + s_2t_2 + \dots + s_kt_k)^m \notin \langle t_1^{n+1}, \dots, t_k^{n+1} \rangle$$
then there is no W_k-equivariant map $X = (S^n)^k \to S(S_{1l}^{\oplus m})$,

(b) *If*
$$\frac{1}{t_1 \dots t_k} \prod_{\substack{s_1,\dots,s_k \in \mathbb{Z}_2 \\ 1 \le s_1+\dots+s_k \le l}} (s_1t_1 + s_2t_2 + \dots + s_kt_k)^m \notin \langle \overline{w}_{n-k+1}, \dots, \overline{w}_n \rangle$$
then there is no W_k-equivariant map $Y_{\mu_1} = V_n^k \to S(S_{2l} \oplus S_{1l}^{\oplus(m-1)})$.

PROOF. Both statements follow from the Fadell–Husseini index computations:
$$\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2}(S^n)^k = \langle t_1^{n+1}, \dots, t_k^{n+1} \rangle,$$
$$\mathrm{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2}\ S_{1l}^{\oplus m} = \Big\langle \prod_{\substack{s_1,\dots,s_k \in \mathbb{Z}_2 \\ 1 \le s_1+\dots+s_k \le l}} (s_1t_1 + s_2t_2 + \dots + s_kt_k)^m \Big\rangle,$$

$$\operatorname{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} V_n^k = \langle \overline{\omega}_{n-k+1}, \dots, \overline{\omega}_n\rangle,$$

$$\operatorname{Index}_{\mathbb{Z}_2^k,\mathbb{F}_2} S_{2l} \oplus S_{1l}^{\oplus(m-1)} = \left\langle \frac{1}{t_1 \dots t_k} \prod_{\substack{s_1,\dots,s_k \in \mathbb{Z}_2 \\ 1 \le s_1+\dots+s_k \le l}} (s_1t_1 + s_2t_2 + \dots + s_kt_k)^m \right\rangle,$$

and its basic property that if there is a G-equivariant map $X \to Y$ then

$$\operatorname{Index}_{G,*} X \supseteq \operatorname{Index}_{G,*} Y. \qquad \square$$

5.2. Proof of Theorem 2.18. Let us lift the measures to $S^{n-1} \subseteq \mathbb{R}^n$; we obtain $m+1$ centrally symmetric measures on the sphere. It is sufficient to find a pair of oriented hyperplanes through the origin H_1, H_2 such that for every $i = 0, \dots, m$

$$\mu_i(H_1^+ \cap H_2^+) = \mu_i(H_1^+ \cap H_2^-) = \mu_i(H_1^- \cap H_2^+) = \mu_i(H_1^- \cap H_2^-).$$

Since the conditions $\mu_i(H_1^+ \cap H_2^+) = \mu_i(H_1^- \cap H_2^-)$ and $\mu_i(H_1^+ \cap H_2^-) = \mu_i(H_1^- \cap H_2^+)$ hold always (because of the central symmetry), we may select the components of the test map to be

$$f_i(H_1, H_2) = \mu_i(H_1^+ \cap H_2^+) - \mu_i(H_1^+ \cap H_2^-) - \mu_i(H_1^- \cap H_2^+) + \mu_i(H_1^- \cap H_2^-).$$

The rest of the proof would follow directly from the proof of Theorem 2.6 (see Section 4.4), if we had m measures. We are going to provide an additional argument to partition $m+1$ measures.

Take the measure μ_0 and assume that its support equals S^{n-1}. Any measure can be approximated by such a measure, and the standard compactness argument (the configuration space of all pairs (H_1, H_2) is compact) extends the solution to arbitrary measures. We are going to show the following:

Proposition 5.2. *If the support of μ_0 is the whole S^{n-1}, then the configuration space X of pairs (H_1, H_2) that equipartition μ_0 (i.e. $f_0(H_1, H_2) = 0$) is D_8-equivariantly homeomorphic to V_n^2.*

Proof. Take an orthogonal 2-frame (e_1, e_2). Denote the orthogonal complement of (e_1, e_2) by $L^\perp(e_1, e_2)$, and denote the reflections

$$\sigma_1 \colon x \mapsto x - 2(x, e_1)e_1, \qquad \sigma_2 \colon x \mapsto x - 2(x, e_2)e_2.$$

Note that the hyperplane H_1 is uniquely defined by the following conditions:

- $H_1 \supseteq L^\perp(e_1, e_2)$,
- $e_1, e_2 \in H_1^+$,
- $H_2 = \sigma_1(H_1) = -\sigma_2(H_1)$,
- $f_0(H_1, H_2) = 0$.

The dependence of H_1 on $(e_1, e_2) \in V_n^2$ is continuous, and therefore we obtain a homeomorphism between X and V_n^2, if the action of D_8 on V_n^2 is chosen properly. $\square$

Now we continue the proof of Theorem 2.18. The functions $f_1, \dots, f_m$ may be considered as functions on V_n^2. If we consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset D_8$, generated by σ_1, σ_2, then the functions f_i are invariant under this group action. Therefore they define the $\mathbb{Z}_2 = D_8/(\mathbb{Z}_2 \times \mathbb{Z}_2)$-equivariant map

$$\widetilde{f}\colon V_n^2/(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq (\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}) \setminus \Delta(\mathbb{R}P^{n-1}) \to \mathbb{R}^m,$$

where the action on $(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}) \setminus \Delta(\mathbb{R}P^{n-1})$ is given by interchanging factors in the product while the action on $\mathbb{R}^m$ is antipodal. This map has a zero, because the "deleted square obstruction" guarantees its existence.

References

[1] A. Adem and R.J. Milgram, *Cohomology of Finite Groups*, Second Edition, Grundlehren der Mathematischen Wissenschaften, vol. 309, Springer–Verlag, Berlin, 2004.

[2] T. Bartsch, *On the existence of Borsuk–Ulam theorems*, Topology **31** (1992), 533–543.

[3] ———, *Topological Methods for Variational Problems with Symmetries*, Springer–Verlag, Berlin–Heidelberg, 1993.

[4] T. Bartsch, M. Clapp and D. Puppe, *A mountain pass theorem for actions of compact Lie groups*, J. Reine Angew. Math. **419** (1991), 55–66.

[5] P. Blagojević and G. Ziegler, *The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes*, Topology Appl. **158** (2011), 1326–1351, http://dx.doi.org/10.1016/j.topol.2011.05.008.

[6] A. Borel, *Sur la cohomology des espaces fibrés principaux et des espace homogènes de groupes de Lie compact*, Ann. Math. **57** (1953), 115–207.

[7] M. Clapp and W. Marzantowicz, *Essential equivariant maps and Borsuk–Ulam theorems*, J. London Math. Soc. **61** (2000), 950–960.

[8] D. Davis, *Table of immersions and embeddings of real projective spaces*, http://www.lehigh.edu/ dmd1/immtable, http://www.lehigh.edu/ dmd1/immtable.

[9] D.M. Davis, *A strong nonimmersion theorem for real projective spaces*, Ann. of Math. **120** (1984), 517–528.

[10] E. Fadell and S. Husseini, *An ideal-valued cohomological index, theory with applications to Borsuk–Ulam and Bourgin–Yang theorems*, Ergodic Theory Dynam. Systems **8** (1988), 73–85.

[11] J. González and P. Landweber, *Symmetric topological complexity of projective and lens spaces*, http://arxiv.org/abs/0809.0816, arXiv:0809.0816 (2008).

[12] B. Grünbaum, *Partition of mass-distributions and convex bodies by hyperplanes*, Pacific J. Math. **10** (1960), 1257–1261.

[13] A. Haefliger, *Plongements differentiables dans le domain stable*, Comment. Math. Helv. **36** (1962–63), 155–176.

[14] H.L. Hiller, *On the cohomology of real grassmannians*, Trans. Amer. Math. Soc. **257** (1980), 521–533.

[15] J. Jaworowski, *Maps of Stiefel manifolds and a Borsuk–Ulam theorem*, Proc. Edinburgh Math. Soc. (Ser. 2) **32** (1989), 271–279.

[16] P. Mani-Levitska, S. Vrećica and R. Živaljević, *Topology and combinatorics of partitions of masses by hyperplanes*, Adv. Math. **207** (2006), 266–296.

[17] V.V. Makeev, *Equipartition of a continuous mass distribution*, J. Math. Sci. **140** (2007), 551–557.

[18] ———, *Geometry of finite-dimensional normed spaces, and continuous functions on the Euclidean sphere*, J. Math. Sci. **140** (2007), 558–563.

[19] E.A. Ramos, *Equipartiotion of mass distributions by hyperplanes*, Discrete Comput. Geom. **15** (1996), 147–167.

[20] B.A. Rattray, *An antipodal-point, orthogonal-point theorem*, Ann. of Math. **60** (1954), 502–512.

[21] H. Steinhaus, *Sur la division des ensembles de l'espaces par les plans et et des ensembles plans par les cercles*, Fund. Math. **33** (1945), 245–263.

[22] A.H. Stone and J.W. Tukey, *Generalized 'Sandwich' Theorems*, Duke Math. J. **9** (1942), 356–359.

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