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CORRIGENDUM IN: A GENERALIZATION OF DENSITY TOPOLOGY AND ON GENERALIZATION OF THE DENSITY TOPOLOGY ON THE REAL LINE

Abstract

The notion of $\mathcal{A}_d-$density point introduced in [1] leads to the operator $\Phi_{\mathcal{A}_d}(A)$ which is not a lower density operator. We present a counterexample and give a corrected definition which should be used in [1] and [2] to keep all results valid.

In [1] we introduced a notion of an $\mathcal{A}_d-$density density point of a measurable set in the following way.

Let $\mathcal{A}_d$ be a family of measurable subsets of $[-1,1]$ that have Lebesgue density one at 0.

Definition 1. *A point $x \in \mathbb{R}$ is an $\mathcal{A}_d-$density point of a measurable set $A \subset \mathbb{R}$ if for any sequence of real numbers $\{t_n\}_{n\in N}$ decreasing to zero, there is a subsequence $\{t_{n_m}\}_{m\in N}$ and a set $B \in \mathcal{A}_d$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$$

of characteristic functions converges almost everywhere on $[-1,1]$ to χ_B.

In contrast to what was incorrectly claimed in [1] the density operator $\Phi_{\mathcal{A}_d}(A)$ defined as the set of all $\mathcal{A}_d-$density points of A is not monotonic and thus is not a lower density. We shall present a counterexample and show how

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to modify the definition of an $\mathcal{A}_d-$density point so that the operator $\Phi_{\mathcal{A}_d}(A)$ is a lower density.

In our paper [3] we introduced a notion of a segment density point of a measurable set $A \subset \mathbb{R}$.

Definition 2. *[3] We say that x is a segment density point of a measurable set A, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a number α, $0<\alpha\le 1$, such that the sequence $\left\{\chi_{\left(\frac{1}{t_{n_m}}\cdot(A-x)\right)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ of characteristic functions converges almost everywhere on $[-\alpha,\alpha]$ to 1.*

In this definition, in contrast to Definition 1, we do not require any convergence of the sequence $\left\{\chi_{\left(\frac{1}{t_{n_m}}\cdot(A-x)\right)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$ on the set $[-1,1]\setminus[-\alpha,\alpha]$.

A Counterexample

Let $D=(0,\frac{1}{2})$. Then D is an open set such that $\lambda(D\cap(0,1))<1$. Let $\{c_n\}_{n\in N}$ be an arbitrary sequence of real numbers decreasing to 0, such that $c_1<1$ and $\lim_{n\to\infty}\frac{c_{n+1}}{c_n}=0$. We define a measurable set U as

$$U=\bigcup_{n=1}^{\infty}\left[(c_n\cdot D)\cap(c_{n+1},c_n)\right]$$

Let $A=-U\cup U$.

By Proposition 2 of [1], 0 is an $\mathcal{A}_d-$density point of A according to Definition 1. It is shown also in [1] that 0 fails to be a density point of A. Now let $D_1=[0,\frac{1}{2})\cup\left(\frac{3}{4},\frac{4}{4}\right)$, $D_2=[0,\frac{1}{2})\cup\left(\frac{5}{8},\frac{6}{8}\right)\cup\left(\frac{7}{8},\frac{8}{8}\right)$, and consecutively $D_n=[0,\frac{1}{2})\cup\left(\frac{1}{2}+\bigcup_{k=1}^{2^{n-1}}\left(\frac{2k-1}{2^{n+1}},\frac{2k}{2^{n+1}}\right)\right)$. Let $\{c_n\}_{n\in\mathbb{N}}$ be defined as above. We define a set $E\in S$ as

$$E=(-\infty,0)\cup\bigcup_{n=1}^{\infty}\left[(c_n\cdot D_n)\cap(c_{n+1},c_n)\right].$$

Clearly E is a superset of A.

We shall show now that 0 is not an $\mathcal{A}_d-$density point of E: On each interval $(a,b)\subset[\frac{1}{2},1]$ with $a<b$, and for every subsequence $\{c_{n_m}\}$ of the sequence $\{c_n\}$, there exists M so that $m>M$ implies $\lambda\left(\left(\frac{1}{c_{n_m}}E\right)\cap(a,b)\right)>\frac{3(b-a)}{8}$ and $\lambda\left((a,b)\setminus\left(\frac{1}{c_{n_m}}E\right)\right)>\frac{3(b-a)}{8}$. Now, suppose that for some $B\in\mathcal{A}_d$, and for some subsequence $\{c_{n_m}\}$, $\chi_{\left(\left(\frac{1}{c_{n_m}}E\right)\cap[\frac{1}{2},1]\right)}\xrightarrow{a.e.}\chi_B$ on $[\frac{1}{2},1]$. Either

$\lambda\left(B\cap[\frac{1}{2},1]\right)=0$ or $\lambda\left(B\cap[\frac{1}{2},1]\right)>0$. A contradiction ensues in either case. If $\lambda\left(B\cap[\frac{1}{2},1]\right)>0$, let a be a density point of $B\cap[\frac{1}{2},1]$. Then $a\in\left(\frac{1}{2},1\right)$ and there exists $h>0$ with $[a-h,a+h]\subset[\frac{1}{2},1]$ and $\lambda\left(B\cap[a-h,a+h]\right)>\frac{7}{4}h$. So there exists K such that for $m>K$, $\lambda\left(\left(\frac{1}{c_{n_m}}E\right)\cap[a-h,a+h]\right)>\frac{6}{4}h$. This contradics the fact that for $m>M$, $\lambda\left(\left(\frac{1}{c_{n_m}}E\right)\cap[a-h,a+h]\right)<\frac{5}{8}2h=\frac{5}{4}h$. A contradiction is similarly reached under the assumption that $\lambda\left(B\cap[\frac{1}{2},1]\right)=0$. Apparently, 0 can not be an $\mathcal{A}_d-$density point of E in the sense of $\mathcal{A}_d-$density point as defined in [1].

Finally we have $A\subset E$ but $0\in\Phi_{\mathcal{A}_d}(A)\setminus\Phi_{\mathcal{A}_d}(E)$, i.e. $\Phi_{\mathcal{A}_d}(E)$ is not monotonic. In particular part (4) of Theorem 1 in [1] is false.

A New Definition

Following the ideas from [3] we replace the Definition 1 in [1] with

Definition 3. *A point $x\in\mathbb{R}$ is an $\mathcal{A}_d-$density point of a measurable set $A\subset\mathbb{R}$ if for any sequence of real numbers $\{t_n\}_{n\in N}$ decreasing to zero there is a subsequence $\{t_{n_m}\}_{m\in N}$ and a set $B\in\mathcal{A}_d$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$$

of characteristic functions converges $I-$almost everywhere on B to 1.

The part (4) of Theorem 1 in [1] can be now proved as follows

Theorem 1. *Let S be the $\sigma-$algebra of all measurable subsets of $\mathbb{R}$. The mapping $\Phi_{\mathcal{A}_d}:S\to 2^{\mathbb{R}}$ has the following properties:*

(0) *for each $A\in S$, $\Phi_{\mathcal{A}_d}(A)\in S$,*

(1) *for each $A\in S$, $A\sim\Phi_{\mathcal{A}_d}(A)$,*

(2) *for each $A,B\in S$, if $A\sim B$ then $\Phi_{\mathcal{A}_d}(A)=\Phi_{\mathcal{A}_d}(B)$,*

(3) *$\Phi_{\mathcal{A}_d}(\emptyset)=\emptyset$, $\Phi_{\mathcal{A}_d}(\mathcal{R})=\mathcal{R}$,*

(4) *for each $A,B\in S$, $\Phi_{\mathcal{A}_d}(A\cap B)=\Phi_{\mathcal{A}_d}(A)\cap\Phi_{\mathcal{A}_d}(B)$.*

Proof. (4) Observe first that if $A\subset B$, $A,B\in S$, then $\Phi_{\mathcal{A}_d}(A)\subset\Phi_{\mathcal{A}_d}(B)$, so $\Phi_{\mathcal{A}_d}(A\cap B)\subset\Phi_{\mathcal{A}_d}(A)\cap\Phi_{\mathcal{A}_d}(B)$. To prove the opposite inclusion assume $x\in\Phi_{\mathcal{A}_d}(A)\cap\Phi_{\mathcal{A}_d}(B)$. Let $\{t_n\}_{n\in N}$ be an arbitrary sequence of real numbers decreasing to zero. From $x\in\Phi_{\mathcal{A}_d}(A)$ by definition there is its subsequence $\{t_{n_m}\}_{m\in N}$ and a set $A_1\in\mathcal{A}_d$ such that the sequence

$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$ of characteristic functions converges $I-$almost everywhere on A_1 to 1. Similarly for $\{t_{n_m}\}_{m\in N}$ from $x\in\Phi_{\mathcal{A}_d}(B)$, by definition there is a subsequence $\left\{t_{n_{m_k}}\right\}_{k\in N}$ and a set $B_1\in\mathcal{A}_d$ such that the sequence $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(A-x)\cap[-1,1]}\right\}_{k\in N}$ of characteristic functions converges $I-$almost everywhere on B_1 to 1. It is clear that the sequence $\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot((A\cap B)-x)\cap[-1,1]}\right\}_{k\in N}$ converges $I-$almost everywhere on $A_1\cap B_1$ to 1, i.e. x is a $\Phi_{\mathcal{A}_d}-$density point of $A\cap B$. $\square$

With the Definition 3 all results of [1] and [2] stay valid. Since we do not require any convergence of the sequence $\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in N}$ on the set $[-1,1]\setminus B$ some proofs may be even shorter, for example we may omit points a1) and a2) in proof of Proposition 2 in [1],

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