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LOCAL SYSTEMS AND TAYLOR'S THEOREM

Abstract

In this article we generalize Taylor's theorem, using the local systems introduced by B. S. Thomson in [8].

We shall denote by $\mathcal{C}$ the class of all continuous functions, by $\mathcal{D}$ the class of all Darboux functions, by $\mathcal{B}_1$ the class of all Baire one functions, and by $\mathcal{DB}_1$ the class of all Darboux Baire one functions.

Definition 1 (Thomson)**.** ([8], p. 3). A family $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ is said to be a local system if each $\mathcal{S}(x)$ is a collection of sets with the following properties:

(i) $\{x\} \notin \mathcal{S}(x)$;

(ii) If $\sigma_x \in \mathcal{S}(x)$ then $x \in \sigma_x$;

(iii) If $\sigma_x \in \mathcal{S}(x)$ and $\sigma_x \subset A$ then $A \in \mathcal{S}(x)$;

(iv) If $\sigma_x \in \mathcal{S}(x)$ and $\delta > 0$ then $\sigma_x \cap (x-\delta, x+\delta) \in \mathcal{S}(x)$.

Definition 2. Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ and $\mathcal{S}' = \{\mathcal{S}'(x)\}_{x\in\mathbb{R}}$ be local systems and let $x \in \mathbb{R}$, $A \subset \mathbb{R}$.

- (Thomson, [8], p. 5) We define the following local system: $\mathcal{S} \wedge \mathcal{S}' = \{(\mathcal{S} \wedge \mathcal{S}')(x)\}_{x\in\mathbb{R}}$, where $(\mathcal{S} \wedge \mathcal{S}')(x) = \mathcal{S}(x) \cap \mathcal{S}'(x)$ (it is easy to verify that this is a local system).
- (Thomson, [8], p. 37). $\mathcal{S}$ is said to be bilateral at x if σ_x has x as a bilateral accumulation point, whenever $\sigma_x \in \mathcal{S}(x)$. $\mathcal{S}$ is bilateral on A if it is bilateral at each point of A.

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- (Thomson, [8], p. 18). Let $\mathcal{S}_\infty = \{\mathcal{S}_\infty(x) : x \in \mathbb{R}\}$ denote the local system defined at each point x as $\mathcal{S}_\infty(x) = \{\sigma : \sigma$ contains x and has x as an accumulation point $\}$. We can define right and left versions of this, by writing: $\mathcal{S}_\infty^+(x) = \{\sigma : \sigma$ contains x and has x as a right accumulation point $\}$ and $\mathcal{S}_\infty^-(x) = \{\sigma : \sigma$ contains x and has x as a left accumulation point $\}$.

- Let $\mathcal{S}_{\infty,\infty} = \mathcal{S}_\infty^+ \wedge \mathcal{S}_\infty^-$. Clearly $\mathcal{S}_{\infty,\infty}(x) = \{\sigma : \sigma$ contains x and has x as a bilateral accumulation point $\}$.

- $\mathcal{S}$ is said to be $\mathcal{S}'$-filtering at x if $\sigma_x' \cap \sigma_x'' \in \mathcal{S}'(x)$ whenever $\sigma_x', \sigma_x'' \in \mathcal{S}(x)$. $\mathcal{S}$ is said to be $\mathcal{S}'$-filtering on A if it is so at each point of A.

- $\mathcal{S}$ is said to be filtering at x if $\mathcal{S}$ is $\mathcal{S}$-filtering at x (this is in fact Thomson's definition of [8], p. 10).

Remark 1. If $\mathcal{S}$ is $\mathcal{S}_{\infty,\infty}$-filtering on a set A then it is a bilateral local system on A.

Definition 3. Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system. Let $F : [a,b] \to \mathbb{R}$ and $t \in [a,b]$. F is said to be $\mathcal{S}$-continuous at t if for every $\epsilon > 0$ there exists $\sigma_t \in \mathcal{S}(t)$ such that $|F(x) - F(t)| < \epsilon$, whenever $x \in \sigma_t \cap [a,b]$. F is said to be $\mathcal{S}$-continuous on a set $A \subset [a,b]$ if it is so at each point $t \in A$.

Remark 2. For $t \in (a,b)$, Definition 3 is a reformulation of Thomson's Definition 31.1 of [8] (p. 70). However, our definition considers $t \in [a,b]$.

Lemma 1. *Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering. Let $F : [a,b] \to \mathbb{R}$ and $t \in [a,b]$. Suppose that there exists $c \in \overline{\mathbb{R}}$ with the following property: for every neighborhood U_c of c there is a set $\sigma_t \in \mathcal{S}(t)$ such that $(F(x) - F(t))/(x-t) \in U_c$, whenever $x \in \sigma_t \cap [a,b]$ and $x \neq t$. Then the number c is unique.*

PROOF. Suppose that there exists a number d, $d \neq c$, with the same properties as c. Let U_c and U_d be neighborhoods for c respectively d such that $U_c \cap U_d = \emptyset$. Let $\sigma_t', \sigma_t'' \in \mathcal{S}(t)$, such that $(F(x)-F(t))/(x-t) \in U_c$, whenever $x \in \sigma_t' \cap [a,b]$, $x \neq t$, and $(F(y) - F(t))/(y-t) \in U_d$, whenever $y \in \sigma_t'' \cap [a,b]$, $y \neq t$. Since $\mathcal{S}$ is $\mathcal{S}_{\infty,\infty}$-filtering it follows that $\sigma_t' \cap \sigma_t'' \setminus \{t\} \neq \emptyset$, a contradiction. $\square$

Definition 4. Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering. Let $F : [a,b] \to \mathbb{R}$ and $t \in [a,b]$.

(1) We denote the unique number c of Lemma 1 by $\mathcal{S}DF(t)$ (the $\mathcal{S}$-derivative of F at t).

(2) The function F is said to be $\mathcal{S}$-derivable on $[a,b]$ if $\mathcal{S}DF(t)$ exists and is finite at each $t \in [a,b]$.

(3) If F is $\mathcal{S}$-derivable on $[a,b]$ and the $\mathcal{S}$-derivative of $\mathcal{S}DF$ exists (finite or infinite) at t then we denote this derivative by $\mathcal{S}DF^{(2)}(t)$.

(4) F is said to be $\mathcal{S}^{(2)}$-derivable on $[a,b]$ if $\mathcal{S}DF^{(2)}(t)$ exists and is finite at each $t \in [a,b]$.

(5) Inductively we may define $\mathcal{S}DF^{(i)}(t)$ and the $\mathcal{S}^{(i)}$-derivability on $[a,b]$, $i = 1, 2, \ldots$. Let $\mathcal{S}DF^{(0)}(t) = F(t)$.

Remark 3. For $t \in (a,b)$, Definition 4, (1) is a reformulation of a part of Definition 7.1 of [8] (p. 14). Of course, Definition 4, (1) is less general, because Thomson's definition does not impose any conditions on the local system. However, our definition considers $t \in [a,b]$.

Lemma 2. *Let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering. Let $F : [a,b] \to \mathbb{R}$. If F is $\mathcal{S}^{(i)}$-derivable on $[a,b]$ then $\mathcal{S}DF^{(i-1)}$ is $\mathcal{S}$-continuous on $[a,b]$, $i = 1, 2, \ldots$.*

Definition 5. We define the following local systems:

- $\mathcal{S}_{1,1} = \{\mathcal{S}_{1,1}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{S}_{1,1}(x) = \{S : x \in S$ and $\underline{d}^i_+(S,x) = \underline{d}^i_-(S,x) = 1\}$. (Here $\underline{d}^i_+$ and $\underline{d}^i_-$ are the interior right respectively left densities of S at x – see for example [8], p. 22). Let $F^{(i)}_{ap}(x) = \mathcal{S}_{1,1}DF^{(i)}(x)$.
- For $\alpha, \beta \in (0,1)$, let $\mathcal{S}_{\alpha,\beta} = \{\mathcal{S}_{\alpha,\beta}(x)\}_{x\in\mathbb{R}}$, where $\mathcal{S}_{\alpha,\beta}(x) = \{S : x \in S$ and $\underline{d}^i_-(S,x) > \alpha$, $\underline{d}^i_+(S,x) > \beta\}$. Let $F^{(i)}_{pr}(x) = \mathcal{S}_{\frac{1}{2},\frac{1}{2}}DF^{(i)}(x)$.

Remark 4. The $\mathcal{S}_{1,1}$ and $\mathcal{S}_{\alpha,\beta}$ local systems are slight modifications of some systems introduced in [6] (pp. 81, 85), [7] (I, p. 75, 76) and [2] (p. 99).

Definition 6 (Preiss)**.** ([5] or [3], p. 35). Let $F : [a,b] \to \mathbb{R}$. F is said to be lower *internal**, if $F(x+) \ge F(x)$, whenever $x \in [a,b)$ and $F(x+)$ exists, and $F(x-) \le F(x)$, whenever $x \in (a,b]$ and $F(x-)$ exists. F is said to be upper *internal** if $-F$ is lower *internal**. F is said to be *internal** if it is simultaneously upper and lower *internal**.

Definition 7 (C.M.Lee)**.** ([4], [3], p. 35). Let $F : [a,b] \to \mathbb{R}$. F is said to be uCM if it is increasing on $[c,d] \subseteq [a,b]$, whenever it is so on (c,d). F is said to be ℓCM if $-F$ is uCM. Let $CM = \ell CM \cap uCM$ and $sCM = \{F : F(x) + \lambda x \in CM$ for each $\lambda \in \mathbb{R}\}$.

Remark 5. ([3], p. 36). Let $F : [a, b] \to \mathbb{R}$. Then we have:

(i) $\mathcal{C}$ + *internal** = *internal**;

(ii) $\mathcal{C} \subset \mathcal{DB}_1 \subset \mathcal{D} \subset$ *internal*$^* \subset sCM \subset CM \subset uCM$;

Theorem 1 (Thomson)**.** *(A special case of Theorem 33.1 of [8], p. 74). Let $\mathcal{S}$ be a local system satisfying an intersection condition of the form $\sigma_x \cap \sigma_y \neq \emptyset$, and let $F : [a, b] \to \overline{\mathbb{R}}$. If F is $\mathcal{S}$-continuous then $F \in \mathcal{B}_1$.*

Theorem 2 (Thomson)**.** *([8], p. 77). Let $\mathcal{S}$ be a bilateral local system, and let $F : [a, b] \to \mathbb{R}$. If F is $\mathcal{B}_1$ and $\mathcal{S}$-continuous on $[a, b]$ then $F \in \mathcal{D}$ on $[a, b]$.*

PROOF. See [1] (Theorem 1.1, (1), (2), pp. 8-9). □

Theorem 3. *([3], p. 30.) Let $F : [a, b] \to \mathbb{R}$ and let $\mathcal{S} = \{\mathcal{S}(x)\}_{x\in\mathbb{R}}$ be a local system satisfying the following conditions:*

- *$\mathcal{S}$ is $\mathcal{S}_{\infty,\infty}$-filtering on $[a, b]$;*
- *$\sigma_x \cap \sigma_y \cap (-\infty, x] \neq \emptyset$;*
- *$\sigma_x \cap \sigma_y \cap [y, +\infty) \neq \emptyset$;*
- *$\mathcal{S}\,DF(x)$ exists (finite or infinite) at each point $x \in [a, b]$.*

Then $\mathcal{S}\,DF(x)$ is $\mathcal{B}_1$ on $[a, b]$.

Theorem 4. *([3], p. 149-150). Let $\mathcal{S}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering, satisfying intersection condition $\sigma_x \cap \sigma_y \cap [x, y] \neq \emptyset$, and let $F : [a, b] \to \mathbb{R}$ be a function satisfying the following conditions:*

(1) $F \in sCM$ on $[a, b]$;

(2) $\mathcal{S}$-derivative $\mathcal{S}\,DF(x)$ exists (finite or infinite) at each $x \in [a, b]$ (respectively $x \in [a, b)$; $x \in (a, b)$);

(3) $\mathcal{S}\,DF(x)$ is $\mathcal{B}_1$ on $[a, b]$ (respectively $[a, b)$; (a, b)).

Then we have:

(i) $\mathcal{S}\,DF(x)$ is $\mathcal{D}$ and

(ii) F fulfills the Mean Value Theorem.

Lemma 3. *Let $\mathcal{S}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering, satisfying intersection condition $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$. Let $F, G, H : [a,b] \to \mathbb{R}$, $H(x) = (F(b) - F(a)) \cdot G(x) - (G(b) - G(a)) \cdot F(x)$ such that $\mathcal{S}DF(x)$ exists finite or infinite on (a,b), $G^{'}$ exists finite on (a,b) and $H \in sCM$ on $[a,b]$. Then there exists $\xi \in (a,b)$ such that*

$$(F(b) - F(a)) \cdot G^{'}(\xi) = (G(b) - G(a)) \cdot \mathcal{S}DF(\xi).$$

PROOF. We have $H(b) = H(a) = F(b)G(a) - G(b)F(a)$. Clearly $\mathcal{S}DH(x)$ exists finite or infinite on (a,b). By Theorem 4, (ii), there exists $\xi \in (a,b)$ such that $\mathcal{S}DH(\xi) = 0$. Now the conclusion of our lemma follows immediately. □

Corollary 1. *Let $\mathcal{S}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering, satisfying intersection condition $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$. Let $F, G : [a,b] \to \mathbb{R}$. If*

(i) $F \in internal^*$ and $G \in \mathcal{C}$ on $[a,b]$,

(ii) $\mathcal{S}$-derivative $\mathcal{S}DF(x)$ exists finite or infinite on (a,b) and $G^{'}(x)$ exists finite on (a,b),

then there exists $\xi \in (a,b)$ such that

$$(F(b) - F(a)) \cdot G^{'}(\xi) = (G(b) - G(a)) \cdot \mathcal{S}DF(\xi).$$

PROOF. Let H be the function defined in Lemma 3. Since $\mathcal{C} + internal^* = internal^* \subset sCM$ (see Remark 5) it follows that $H \in sCM$ on $[a,b]$. Now the proof follows by Lemma 3. □

Remark 6. In Lemma 3 and Corollary 1 we may put $\mathcal{S}DG$ instead of $G^{'}$ if $\mathcal{S}$ is supposed to be filtering.

Theorem 5. *(A strong form of Taylor's Theorem). Let $\mathcal{S}$ be a local system $\mathcal{S}_{\infty,\infty}$-filtering, satisfying the following intersection conditions:*

- $\sigma_x \cap \sigma_y \cap [x,y] \neq \emptyset$;
- $\sigma_x \cap \sigma_y \cap (-\infty, x] \neq \emptyset$;
- $\sigma_x \cap \sigma_y \cap [y, +\infty) \neq \emptyset$.

Let $F : [a,b] \to \mathbb{R}$ such that $F(b-) = F(b)$ if $F(b-)$ exists, and let $n > 1$ be an integer. If

(i) F is $\mathcal{S}^{(i)}$-derivable on $[a,b)$, $i = 1, 2, \ldots, n$ and

(ii) $\mathcal{S}DF^{(n+1)}(x)$ exists finite or infinite on (a,b),

then there exists $\xi \in (a,b)$ *such that*

$$F(b) = \sum_{i=0}^{n} \frac{\mathcal{S}DF^{(i)}(a)}{i!}(b-a)^i + \frac{\mathcal{S}DF^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

PROOF. Let

$$R(x) = F(x) - \sum_{i=0}^{n} \frac{\mathcal{S}DF^{(i)}(a)}{i!}(x-a)^i \text{ and } G(x) = (x-a)^{n+1}.$$

Clearly $R(a) = \mathcal{S}DR(a) = \ldots = \mathcal{S}DR^{(n)}(a) = 0$ and $\mathcal{S}DR^{(n+1)}(x) = \mathcal{S}DF^{(n+1)}(x)$ for each $x \in (a,b)$. But $G(a) = G^{'}(a) = \ldots = G^{(n)}(a) = 0$ and $G^{(n+1)}(x) = (n+1)!$ on (a,b). By Theorem 3, $\mathcal{S}DF^{(i)}$ is $\mathcal{B}_1$ on $[a,b)$, $i = 1,2,\ldots,n$ and $\mathcal{S}DF^{(n+1)}$ is $\mathcal{B}_1$ on (a,b). By Theorem 4, (i) it follows that $\mathcal{S}DF^{(i)} \in \mathcal{D}$ on $[a,b)$, $i = 1,2,\ldots,n$, and $\mathcal{S}DF^{(n+1)} \in \mathcal{D}$ on (a,b). By Lemma 2, F is $\mathcal{S}$-continuous on $[a,b)$, so by Theorem 1, $F \in \mathcal{B}_1$ on $[a,b)$. By Theorem 2, $F \in \mathcal{D}$ on $[a,b)$. By Remark 5, (ii) and the fact that $F(b-) = F(b)$ if $F(b-)$ exists, it follows that $F \in internal^*$ on $[a,b]$. Then $R \in internal^*$ on $[a,b]$ (see Remark 5, (i)). Applying Corollary 1, it follows that there exists $c_1 \in (a,b)$ such that $R(b)/G(b) = \mathcal{S}DF(c_1)/G^{'}(c_1)$. Since $\mathcal{S}DF \in \mathcal{DB}_1 \subset internal^*$ on $[a,c_1]$ (see Remark 5), applying Corollary 1 again, it follows that there exists $c_2 \in (a,c_1)$ such that $\mathcal{S}DF(c_1)/G^{'}(c_1) = \mathcal{S}DF^{(2)}(c_2)/G^{(2)}(c_2)$. Continuing, we obtain $b > c_1 > c_2 > \ldots > c_n > c_{n+1} > a$ such that

$$\frac{R(b)}{G(b)} = \frac{\mathcal{S}DR(c_1)}{G^{'}(c_1)} = \ldots = \frac{\mathcal{S}DR^{(n)}(c_n)}{G^{(n)}(c_n)} =$$

$$\frac{\mathcal{S}DR^{(n+1)}(c_{n+1})}{(n+1)!} = \frac{\mathcal{S}DF^{(n+1)}(c_{n+1})}{(n+1)!}.$$

Putting $\xi = c_{n+1}$ the assertion of the theorem follows. □

Corollary 2. *Let* $F : [a,b] \to \mathbb{R}$ *and let* $n \geq 1$ *be an integer. Suppose that*

(1) $F(b-) = F(b)$ *if* $F(b-)$ *exists;*

(2) $F_{ap}^{(i)}(x)$ *(respectively* $F_{pr}^{(i)}(x)$*) exists and is finite on* $[a,b)$*, for each* $i = 1,2,\ldots,n$ *and*

(3) $F_{ap}^{(n+1)}(x)$ *(respectively* $F_{pr}^{(n+1)}(x)$*) exists finite or infinite on* (a,b)*.*

Then there exists $\xi \in (a, b)$ *such that*

$$F(b) = \sum_{i=0}^{n} \frac{F_{ap}^{(i)}(a)}{i!}(b-a)^i + \frac{F_{ap}^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

(respectively

$$F(b) = \sum_{i=0}^{n} \frac{F_{pr}^{(i)}(a)}{i!}(b-a)^i + \frac{F_{pr}^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1} \ ).$$

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