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ON DIFFERENTIABILITY OF FUNCTIONS OF TWO VARIABLES

Abstract

Some special conditions (equidifferentiability or absolute equicontinuity) implying (or not) the differentiability of functions of two variables are considered.

1 Equidifferentiability

Let $\mathbb{R}$ be the set of all reals. We denote by $|x|$ the absolute value of $x \in \mathbb{R}$, by $|(y,z)|$ the Euclidean norm of $(y,z) \in \mathbb{R}^2$, and by $|I|$ the length of the interval $I \subset \mathbb{R}$. Let

$$\mathcal{A} = \{f_s : \mathbb{R} \to \mathbb{R}; s \in S\},$$

where S denotes a set of indexes. We say that the functions of the family $\mathcal{A}$ are equidifferentiable at a point $x \in \mathbb{R}$ if they are differentiable at x and for every positive real η there is a positive real δ such that for each function $f \in \mathcal{A}$ and for all points t such that $0 < |t - x| < \delta$ the inequality

$$\left| \frac{f(t) - f(x)}{x - t} - f'(x) \right| < \eta.$$

holds. Now, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables. It is well known that the differentiability of all sections $F_x(t) = F(x,t)$ and all sections $F^y(t) = F(t,y)$, $x, y, t \in \mathbb{R}$, need not imply the differentiability of F.

Theorem 1. *Let a function $F : \mathbb{R}^2 \to \mathbb{R}$ be given and let $(x,y) \in \mathbb{R}^2$ be a point such that the section F_x is differentiable at y and there is a positive real r such that the sections F^v, $v \in (y-r, y+r)$, are equidifferentiable at x. If*

$$(1), \qquad \lim_{v \to y} \frac{\partial F}{\partial x}(x,v) = \frac{\partial F}{\partial x}(x,y)$$

then the function F is differentiable at the point (x,y).

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PROOF. Let $a(v) = \frac{\partial F}{\partial x}(x, v)$, $v \in (y-r, y+r)$ and $b = \frac{\partial F}{\partial y}(x, y)$. Fix a positive real η. Since the sections F^v, $v \in (y-r, y+r)$, are equidifferentiable at the point x, there is a positive real δ_1 such that for each point $u \in (x-\delta_1, x+\delta_1)$ and for each $v \in (y-r, y+r)$ we have

$$\left| \frac{F(u,v) - F(x,v)}{u-x} - a(v) \right| < \frac{\eta}{2} \tag{2.}$$

By (1) and by the differentiability of the section F_x at y there is a positive real $\delta_2 < r$ such that for each point $v \in (y-\delta_2, y+\delta_2)$ the inequalities

$$\left| \frac{F(x,v) - F(x,y)}{v-y} - b \right| < \frac{\eta}{4} \tag{3}$$

and

$$|a(v) - a(y)| < \frac{\eta}{4} \tag{4}$$

are valid. Let

$$\delta = \min(\delta_1, \delta_2), \quad I = (x-\delta, x+\delta) \times (y-\delta, y+\delta).$$

Fix a point $(u,v) \in I$. Then, by (2), (3) and (4) we obtain

$$\begin{aligned} &\left| \frac{F(u,v) - F(x,y) - a(y)(u-x) - b(v-y)}{|(u,v)-(x,y)|} \right| \leq \\ &\left| \frac{F(u,v) - F(x,v)}{u-x} - a(v) \right| + |a(v) - a(y)| + \\ &\left| \frac{F(x,v) - F(x,y)}{v-y} - b \right| < \frac{\eta}{2} + \frac{\eta}{4} + \frac{\eta}{4} = \eta. \end{aligned}$$

So,

$$\lim_{(u,v)\to(x,y)} \frac{F(u,v) - F(x,y) - a(x)(u-x) - b(v-y)}{|(u,v)-(x,y)|} = 0,$$

and F is differentiable at (x,y). $\square$

Observe that the function

$$F(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = 0,0) \end{cases}$$

satisfies the condition (1) for $(x,y) = (0,0)$, but it is not differentiable at the point $(0,0)$. So, for each $r > 0$ the sections F^v, $v \in (-r,r)$, are not equidifferentiable at 0. The next example shows that the condition (1) in Theorem 1 is essential.

Example 1. For $n = 1, 2, \ldots$ let $I_n = [\frac{1}{n} - \frac{1}{4n^2}, \frac{1}{n} + \frac{1}{4n^2}] = [a_n, b_n]$, and

$$f_n(y) = \begin{cases} 0 & \text{if} \quad y = a_n \ \textit{or} \ y = b_n \\ 1 & \text{if} \quad y = \frac{a_n + b_n}{2} = c_n \\ \text{linear} & \text{on} \quad [a_n, c_n], \ \ [c_n, b_n]. \end{cases}$$

We let

$$f(y) = \begin{cases} f_n(y) & \text{for} \quad y \in I_n, n = 1, 2, \ldots \\ 0 & \text{on} \quad \mathbb{R} \setminus \bigcup_n I_n \end{cases}$$

and $F(x, y) = xf(y)$ for $(x, y) \in \mathbb{R}^2$. Then F is continuous on $\mathbb{R}^2 \setminus \{(x, 0); x \neq 0\}$, the section F_0 is everywhere differentiable and the sections F^y, $y \in \mathbb{R}$, are equidifferentiable at 0. Now we will show that F is not differentiable at the point $(0, 0)$. Observe that $F(0, 0) = 0$, $F(\frac{1}{n}, \frac{1}{n}) = \frac{1}{n}$ for $n \geq 1$, $|(\frac{1}{n}, \frac{1}{n})| = \frac{\sqrt{2}}{n}$ for $n \geq 1$ and $\frac{\partial F}{\partial x}(0, 0) = \frac{\partial F}{\partial y}(0, 0) = 0$. So, for $n \geq 1$ we obtain

$$\frac{F(\frac{1}{n}, \frac{1}{n}) - F(0, 0)}{|(\frac{1}{n}, \frac{1}{n})|} = \frac{1}{\sqrt{2}},$$

and consequently the function F is not differentiable at the point $(0, 0)$.

Theorem 2. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function such that all sections F^v, $v \in \mathbb{R}$, are continuous and the section F_x is differentiable at a point y. Suppose that there is a positive real r and a linear set $A \subset (y - r, y + r)$ dense in the interval $(y - r, y + r)$ such the the sections F^y, $y \in A$, are equidifferentiable at x and*

$$\lim_{\substack{v \to y \\ v \in A}} \frac{\partial F}{\partial x}(x, v) = \frac{\partial F}{\partial x}(x, y).$$

Then F is differentiable at the point (x, y).

Proof. Same as in the proof of Theorem 1 we can show that

$$\lim_{\substack{u \to x, v \to y \\ v \in A}} \frac{F(u, v) - F(x, y) - a(y)(u - x) - b(v - y)}{|(u, v) - (x, y)|} = 0,$$

where $a(y)$ and b are the same as these in the proof of Theorem 1. By the continuity of the sections F_x, $x \in \mathbb{R}$, we obtain that the above limit is also equal to 0 if $(u, v) \to (x, y)$, so the function F is differentiable at (x, y). $\square$

Theorem 3. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the section F_x is differentiable at a point y. Suppose that there is a positive real r such that the partial derivative $\frac{\partial F}{\partial x}$ is continuous on the open circle $K((x, y), r)$. Then F is differentiable at the point (x, y).*

Proof. It suffices to prove that F satisfies the hypothesis of Theorem 1. Condition (1) follows from the continuity of partial derivative $\frac{\partial F}{\partial x}$ at (x, y). The existence and the boundedness of the partial derivative $\frac{\partial F}{\partial x}$ on some neighborhood $V \subset K((x, y), r)$ implies, by the LaGrange theorem, the equidifferentiability of the sections F^v, $v \in V_x = \{t : (x, t) \in V\}$ at the point x. □

2 Absolute Equicontinuity

Now, let $\mathcal{B}$ be a family of functions $f_s : I \to \mathbb{R}$, $s \in S$ and $I = [0, 1]$. We say that the functions of the family $\mathcal{B}$ are absolutely equicontinuous if for every positive real η there is a positive real δ such that for each index $s \in S$ and for each family $\{I_i = [a_i, b_i]; i \leq k\}$ of closed subintervals of I with $\operatorname{int} I_i \cap int I_j = \emptyset$ for $i \neq j$, $i, j \leq k$, ($\operatorname{int} I_i$ denotes the interior of I_i) and $\sum_{i \leq k}(b_i - a_i) < \delta$ the inequality $\sum_{i \leq k} |f_s(b_i) - f_s(a_i)| < \eta$ holds.

Theorem 4. *Let $F : I^2 \to \mathbb{R}$ be a function such that the sections F_x, $x \in I$, are absolutely equicontinuous and the sections F^y, $y \in I$, are absolutely equicontinuous. Then F has the following property.*

(P) For each positive real η there is a positive real δ such that for every family of closed intervals $I_1, \dots, I_k; J_1, \dots, J_k \subset I$ such that

$$\operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset, \wedge \operatorname{int} J_i \cap \operatorname{int} J_j = \emptyset, \quad i \neq j, \quad i, j \leq k, \tag{5}$$

and

$$\sum_{i \leq k;} (|I_i| + |J_i|) < \delta \tag{6}$$

the inequality $\sum_{i \leq k} \operatorname{diam}(F(I_i \times J_i)) < \eta$ holds ($\operatorname{diam}(X)$ denotes the diameter of the set X).

Proof. Since the sections F_x, $x \in I$, and F^y, $y \in I$, are equicontinuous, the function F is continuous. Fix a positive real η. There is a positive real δ such that for every point $(x, y) \in I^2$ and for each family of closed intervals

$$K_1, K_2, \dots, K_l \subset I$$

with $\operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset$ for $i \neq j\, i, j \leq l$, and $\sum_{i \leq l} |K_i| < \delta$ the inequalities

$$\sum_{i \leq l} \operatorname{diam}(F_x(K_i)) < \frac{\eta}{2} \text{ and } \sum_{i \leq l} diam(F^y(K_i)) < \frac{\eta}{2}$$

hold.

Let $I_1, \ldots, I_k; J_1, \ldots, J_k \subset I$ be closed intervals satisfying conditions (5) and (6). Let

$$F(a_i, b_i) = \max_{(x,y)\in I_i\times J_i} F(x,y), \wedge F(c_i, d_i) = \min_{(x,y)\in I_i\times J_i} F(x,y), \text{ for } i \leq k.$$

Then

$$\big|\sum_{i\leq k} \operatorname{diam}(F(I_i \times J_i))\big| = \sum_{i\leq k}(F(a_i, b_i) - F(c_i, d_i)) \leq$$
$$\sum_{i\leq k} |F(a_i, b_i) - F(c_i, b_i)| + \sum_{i\leq k} |F(c_i, b_i) - F(c_i, d_i)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

and the proof is completed. □

Example 2. For $n = 1, 2, \ldots$ let $I_n\left[\frac{1}{n} - \frac{1}{4n^2}\right] = [a_n, b_n]$, $J_n = \left[-\frac{1}{n}, \frac{1}{n}\right] = [c_n, d_n]$ and

$$f_n(y) = \begin{cases} 0 & \text{if } y = a_n \text{ or } y = b_n \\ 1 & \text{if } y = \frac{a_n+b_n}{2} = c_n \\ \text{linear} & \text{on } [a_n, c_n], [c_n, b_n]. \end{cases}$$

We let

$$f(y) = \begin{cases} f_n(y) & \text{for } y \in I_n,\ n = 1, 2, \ldots \\ 0 & \text{on } \mathbb{R} \setminus \cup_n I_n \end{cases}$$

and

$$F(x,y) = \begin{cases} 0 = xf(y) & \text{if } y \in \mathbb{R} \setminus \cup_n I_n \\ xf(y) & \text{if } (x,y) \in J_n \times I_n,\ n = 1, 2, \ldots \\ e_n f(y) & \text{if } x \leq e_n \wedge y \in I_n,\ n = 1, 2, \ldots \\ d_n f(y) & \text{if } x \geq d_n \wedge y \in I_n,\ n = 1, 2, \ldots . \end{cases}$$

Put

$$G(x,y) = \begin{cases} F(x,y) & \text{if} \quad y \geq 0,\ x \leq -2y \\ F(x,y) & \text{if} \quad y \geq 0,\ x \geq 2y \\ 0 & \text{if} \quad y < 0 \\ F(-y,y) + \frac{F(y,y) - F(-y,y)}{x+y} & if \quad y > 0,\ x \in (-y, y). \end{cases}$$

Observe that the restricted function $g = G|I^2$ does not satisfy condition (P), but the sections g^y, $y \in I$, are absolutely equicontinuous on I and the sections g_x, $x \in I$, are absolutely continuous.

The next example shows that there is a function $F : I^2 \to \mathbb{R}$ having absolutely equicontinuous sections F_x and F^y, $x, y \in I$, such that the set of points where F is not differentiable is of positive measure.

Example 3. Let $C \subset I$ be a Cantor set of positive measure and let $A = C \times C$. Since the set A is compact and the set C is nowhere dense, for each positive integer n there are points $(x_{n,i}, y_{n,i}) \subset (I \setminus C)^2$ for $i \le k(n)$, such that

$$\begin{aligned} 0 &< \operatorname{dist}((x_{n,j}, y_{n,j}), A) = \inf\{|(x_{n,j}, y_{n,j}) - (x,y)|; (x,y) \in A\} \\ &< \min\{\operatorname{dist}(x_{n-1,i}, y_{n-1,i}), A); i \le k(n-1)\}, \quad j \le k(n) \text{ for } n > 1, \end{aligned}$$

$$x_{n_1,i_1} \ne x_{n_1,i_2} \quad \wedge \quad y_{n_1,i_1} \ne y_{n_1,i_1}$$

for $(n_1, i_1) \ne (n_2, i_2)$, $n_1, n_2 = 1, 2, \ldots$, $i_j \le k(n_j)$, $j = 1, 2$, and

$$\forall_{(x,y)\in A} \forall_n \exists_{i \le k(n)} |(x,y) - (x_{n,i}, y_{n,i})| < \frac{1}{n^2}.$$

For each pair (n, i), $n \ge 1$, $i \le k(n)$, we can find a positive real $r_{n,i}$ such that

$$[x_{n_1,i_1} - r(n_1,i_1), x_{n_1,i_1} + r(n_1,i_1)] \cap [x_{n_2,i_2} - r(n_2,i_2), x_{n_2,i_2} + r(n_2,i_2)] = \emptyset$$

and

$$[y_{n_1,i_1} - r(n_1,i_1), y_{n_1,i_1} + r(n_1,i_1)] \cap [y_{n_2,i_2} - r(n_2,i_2), y_{n_2,i_2} + r(n_2,i_2)] = \emptyset$$

for $(n_1, i_1) \ne (n_2, i_2)$, $n_1, n_2 \ge 1$, $i_j \le k_{n_j}$, $j = 1, 2$. Let $K_{n,i}$ be the circle with center $(x_{n,i}, y_{n,i})$ and radius $r(n,i)$, let $S_{n,i}$ be the boundary of the circle $K_{n,i}$ and let $F_{n,i} : K_{n,i} \to [0, \frac{1}{n^2}]$, $n \ge 1$, $i \le k(n)$, be the continuous function defined by

$$F_{n,i}(x,y) = \frac{\operatorname{dist}((x,y), S_{n,i})}{n^2 r(n,i)}.$$

Let

$$F(x,y) = \begin{cases} F_{n,i}(x,y) & \text{for } (x,y) \in K_{n,i},\ n \ge 1,\ i \le k(n), \\ 0 & \text{on } I^2 \setminus \bigcup_{n,i} K_{n,i}. \end{cases}$$

For each point $(x,y) \in A$ and for each positive integer n there is a point $(x_{n,i}, y_{n,i})$ with $|(x_{n,i}, y_{n,i}) - (x,y)| < \frac{1}{n^2}$. Clearly, $\frac{\partial F}{\partial x}(x,y) = \frac{\partial F}{\partial y}(x,y) = 0$. So,

$$\left| \frac{F(x_{n,i}, y_{n,i}) - F(x,y)}{|(x_{n,i}, y_{n,i}) - (x,y)|} \right| \ge \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1,$$

and thus F is not differentiable at (x,y).

Now we will show that the sections F_x, $x \in I$, are absolutely equicontinuous. Fix a positive real η. There is an positive integer k with $\sum_{n>k} \frac{1}{n^2} < \frac{\eta}{2}$. Let δ be a positive real such that

$$\delta < \frac{\eta}{2 \max_{n \le k; i \le k(n)} (\frac{1}{r(n,i)n^2})}.$$

Suppose that closed intervals $\{I_i = [a_i, b_i]; i \le j\}$ satisfy the following conditions:

$$\operatorname{int}(I_i) \cap \operatorname{int}(I_m) = \emptyset \text{ for } i \ne m \text{ and } i, m \le j,$$

and $\sum_{i \le j} |I_i| < \delta$. Fix a point $x \in I$ and denote by K the set of all positive integers $l \le j$ for which there is a pair (n, i) with $n \le k$ such that $I_i \cap (K_{n,i})_x \ne \emptyset$ and by L the set $\{1, \dots, j\} \setminus K$. Then

$$\sum_{i \le j} |F(x, b_i) - F(x, a_i)| = \sum_{i \in K} |F(x, b_i) - F(x, a_i)| +$$

$$\sum_{i \in L} |F(x, b_i) - F(x, a_i)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

and the sections F_x, $x \in I$, are absolutely equicontinuous. The proof that the sections F^y, $y \in I$, are absolutely equicontinuous is analogous. □

3 Other Results.

It is well known that if the sections F_x and F^y, $x, y \in \mathbb{R}$, of a function $F : \mathbb{R}^2 \to \mathbb{R}$ are polynomials of degree $\le n$, then F is a polynomial of degree $\le n$. By the Baire category method we can show that if the sections F_x and F^y, $x, y \in \mathbb{R}$, are polynomials, then F is also a polynomial. In this section we will solve some analogous problems concerning the differentiability of functions of two variables.

Let $\mathcal{F} = \{f_0, f_1, \dots, f_n, \dots\}$ be a family of differentiable real functions defined on $\mathbb{R}$ such that for every integer $n \ge 0$ and for each collection of different points $y_0, \dots, y_n \in \mathbb{R}$ we have

$$\det[f_i(y_j)]_{0 \le i,j \le n} \ne 0.$$

Furthermore, for $n = 0, 1, \dots$, let

$$\mathcal{W}_n(\mathcal{F}) = \{g = \sum_{0 \le k \le n} a_k f_k; a_0, \dots, a_n \in \mathbb{R}\}$$

and suppose that there is a finite collection of points $y_0, \dots, y_{k(n)}$ which is a determining set for the family $\mathcal{W}_n(\mathcal{F})$; **i.e.**, for each pair of functions $g_1, g_2 \in \mathcal{W}_n(\mathcal{F})$, if $g_1(y_i) = g_2(y_i)$ for $i = 0, \dots, k(n)$, then $g_1(y) = g_2(y)$ for $y \in \mathbb{R}$.

Remark 1. *Two important examples of such families $\mathcal{F}$ are the following.*

$$f_n(y) = y^n; n \geq 0, \quad y \in \mathbb{R}$$

and

$$f_{2n}(y) = \cos 2ny, \quad f_{2n+1}(y) = \sin(2n+1)y, \quad n \geq 0, \quad y \in \mathbb{R}.$$

We shall say that $F : \mathbb{R}^2 \to \mathbb{R}$ has locally differentiable sections F^y if for each $(x_0, y_0) \in \mathbb{R}^2$ there is an $\eta > 0$ such that the family $\{F^y; y \in (y_0 - \eta, y_0 + \eta)\}$ is equidifferentiable at x_0.

Theorem 5. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function with locally equidifferentiable sections F^y, $y \in \mathbb{R}$. If all sections F_x, $x \in \mathbb{R}$ are in $\mathcal{W}(\mathcal{F}) = \bigcup_{n \geq 0} \mathcal{W}_n(\mathcal{F})$, then for every nonempty perfect set $A \subset \mathbb{R}$ there is an open interval I such that $I \cap A \neq \emptyset$ and F is differentiable (as the function of two variables) at each point of the set $(A \cap I) \times \mathbb{R}$.*

PROOF. Let $A \subset \mathbb{R}$ be a nonempty perfect set. For each integer $n \geq 0$ let

$$A_n = \{x \in A; F_x \in \mathcal{W}_n(\mathcal{F})\}.$$

Since $A = \bigcup_{n \geq 0} A_n$ and since the set A is a complete space, by the Baire category theorem we obtain that there is an integer $i \geq 0$ such that the set A_i is of the second category in A. So there is an open interval I such that $I \cap A \neq \emptyset$ and $I \cap A_i$ is dense in $I \cap A$. There is a finite determining set $\{y_0, y_1, \ldots, y_{k(i)}\}$ for the family $\mathcal{W}_i(\mathcal{F})$. We can obviously assume that $k(i) \geq i$. Observe that for each point x the system of equations

$$\sum_{0 \leq n \leq k(i)} f_n(y_j) h_n(x) = F(x, y_j), \quad 0 \leq j \leq k(i),$$

is a Cramer's system and it has unique solution $\{h_0(x), \ldots, h_{k(i)}(x)\}$. Since the functions $x \to F(x, y_j)$, $j = 0, 1, \ldots, k(i)$, are differentiable, the functions $x \to h_n(x)$ for $n = 0, 1, \ldots, k(i)$, are also differentiable.

For $(x, y) \in \mathbb{R}^2$ let $G(x, y) = \sum_{0 \leq n \leq k(i)} h_n(x) f_n(y)$. Fix a point $x \in A_i$ and observe that we have $G_x, F_x \in \mathcal{W}_{k(i)}(\mathcal{F})$ and $G(x, y_n) = F(x, y_n)$ for $n = 0, 1, \ldots, k(i)$. Since $\{y_n; 0 \leq n \leq k(i)\}$ is a determining set for the family $\mathcal{W}_{k(i)}(\mathcal{F})$, the equality $F(x, y) = G(x, y)$ holds for all points $(x, y) \in A_i \times \mathbb{R}$. But since the sections F^y and G^y, $y \in \mathbb{R}$, are differentiable (and hence continuous) and the set A_i is dense in $I \cap A$, the equality $F(x, y) = G(x, y)$ holds for all points $(x, y) \in (I \cap A) \times \mathbb{R}$. Let $H(x, y) = F(x, y) - G(x, y)$, $(x, y) \in \mathbb{R}^2$. Clearly $H(x, y) = 0$, $(x, y) \in (A \cap I) \times \mathbb{R}$, and all sections H^y, $y \in \mathbb{R}$, are differentiable. So, for each point $(x, y) \in (A \cap I) \times \mathbb{R}$ we have

$$\frac{\partial H}{\partial x}(x, y) = \frac{\partial H}{\partial y}(x, y) = 0.$$

Since G obviously has locally equidifferentiable sections G^y, the function H has that property too. Now the differentiability of H at $(x, y) \in (A \cap I) \times \mathbb{R}$ follows immediately by Theorem 1.

Since the function G is differentiable at each point $(x, y) \in \mathbb{R}^2$ and $F = G + H$, the function F is differentiable at each point $(x, y) \in (A \cap I) \times \mathbb{R}$. □

For a function $F : \mathbb{R}^2 \to \mathbb{R}$ let $S(F)$ denote the set of points (x, y) at which F is not differentiable.

Remark 2. *Let F be the function from Theorem 3. If there is an integer $n \geq 0$ such that $F_x \in \mathcal{W}_n(\mathcal{F})$, $x \in \mathbb{R}$, then $S(F) = \emptyset$*

Proof. We can repeat the proof of Theorem 3 taking $A = I = \mathbb{R}$. □

Corollary 1. *If a function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies the hypothesis of Theorem 3, then the projection $\Pr(S(F)) = \{x \in \mathbb{R}; \exists_y (x, y) \in S(F)\}$ of the set $S(F)$ is a countable set such that the closure of each nonempty subset of $\Pr(S(F))$ contains an isolated point.*

Proof. If there is a nonempty set $A \subset \Pr S(F))$ such that the closure $\mathrm{cl}(A)$ is a perfect set, then by Theorem 3 there is an open interval I such that $I \cap \mathrm{cl}(A) \neq \emptyset$ and $(I \cap \mathrm{cl}(A)) \times \mathbb{R} \subset \mathbb{R}^2 \setminus S(F)$. So, we obtain a contradiction with $A \cap I \neq \emptyset \wedge A \subset \Pr(S(F))$. If $\Pr(S(F))$ is not countable, then there is a nonempty set $A \subset \Pr(S(F))$ such that $\mathrm{cl}(A)$ is a perfect set. □

In the next example we will show that there are functions F satisfying the hypothesis of Theorem 3 for which the set $D(F)$ of discontinuity points of F is not countable.

Example 4. Let $C \subset [0, 1]$ be the ternary Cantor set and let $\{I_n = (a_n, b_n); n \geq 1\}$ be an enumeration of all components of the set $[0, 1] \setminus C$ such that $I_n \cap I_m = \emptyset$ for $n \neq m$ and $n, m \geq 1$. For each integer $n \geq 1$ we find squares

$$K_{n,m} = [a_{n,m}, b_{n,m}] \times [c_{n,m}, d_{n,m}], \quad m \leq n,$$

such that

- $[a_{n_1,m_1}, b_{n_1,m_1}] \cap [a_{n_2,m_2}, b_{n_2,m_2}] = \emptyset$,
- $[c_{n_1,m_1}, d_{n_1,m_1}] \cap [c_{n_2,m_2}, d_{n_2,m_2}] = \emptyset$, for $(n_1, m_1) \neq (n_2, m_2)$ satisfying $n_1, m_1, n_2, m_2 \geq 1$,
- $[c_{n,m}, d_{n,m}] \subset I_m$, $m \leq n$, $n \geq 1$,
- $\forall_m \lim_{n\to\infty} c_{n,m} = a_m$ and

- $\forall_n \forall_{m \le n} 0 < a_{n,m} < b_{n,m} < \frac{1}{n}$.

For $n \ge 1$ and $m \le n$ let $(x_{n,m}, y_{n,m})$ be the center of the square $K_{n,m}$, let

$$f_{n,m}(y) = \begin{cases} 0 & \text{if } y \in (-\infty, c_{n,m}] \cup [d_{n,m}, \infty) \\ 1 & \text{if } y = y_{n,m} \\ \text{linear} & \text{on } [c_{n,m}, y_{n,m}] \wedge [y_{n,m}, d_{n,m}], \end{cases}$$

and let

$$g_{n,m}(x) = \begin{cases} \frac{4(\min(|x-a_{n,m}|,|x-b_{n,m}|))^2}{|b_{n,m}-a_{n,m}|^2} & \text{on } [a_{n,m}, b_{n,m}] \\ 0 & \text{on } (-\infty, a_{n,m}] \cup [b_{n,m}, \infty). \end{cases}$$

By the well known Weierstrass theorem, for $m \le n$ and $n \ge 1$ there is a polynomial $h_{n,m}$ such that $\forall_{y \in [0,n]} |f_{n,m}(y) - h_{n,m}(y)| < \frac{1}{4^n}$. For $(x, y) \in \mathbb{R}^2$ define

$$F(x,y) = \begin{cases} g_{n,m}(x) h_{n,m}(y) & \text{for } x \in [a_{n,m}, b_{n,m}],\ n \ge 1,\ m \le n \\ 0 & \text{for } x \in \mathbb{R} \setminus \bigcup_{n \ge 1, m \le n} [a_{n,m}, b_{n,m}]. \end{cases}$$

Clearly, the sections F_x, $x \in \mathbb{R}$, are polynomials and the sections F^y, $y \in \mathbb{R}$, are differentiable. Since $F(0, y) = 0$, $y \in R$, and

$$F(x_{n,m}, y_{n,m}) > 1 - \frac{1}{4^n}, \quad n \ge 1, \quad m \le n,$$

the function F is not continuous at each point $(0, y)$ where $y \in C$.

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