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TOPOLOGIES GENERATED BY THE ψ-SPARSE SETS

Abstract

We study the notion of ψ-sparse point and ψ-sparse topology for nondecreasing continuous function ψ. We show that ψ-sparse topology is stronger then the ψ-density topology and weaker than the density topology.

1 Introduction

We shall use the following notations: $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{N}$ - the set of all positive integers, m^* - the outer Lebesgue measure, $\mathcal{L}$ - the σ-algebra of Lebesgue measurable sets, m - the Lebesgue measure and $\mathcal{C}$ - the family of all continuous, nondecreasing functions $\psi : (0, \infty) \to (0, 1)$ such that $\lim_{x \to 0^+} \psi(x) = 0$.

For $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, we let

$$\underline{d}(E, x) = \liminf_{h \to 0^+} \frac{m^*(E \cap [x - h, x + h])}{2h}$$

and

$$\overline{d}(E, x) = \limsup_{h \to 0^+} \frac{m^*(E \cap [x - h, x + h])}{2h}$$

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as the lower and upper outer density of a set E at a point x, respectively.

Analogously, for any $\psi \in \mathcal{C}$, $E \subset \mathbb{R}$ and $x \in \mathbb{R}$ let

$$\psi - \underline{d}(E,x) = \liminf_{h \to 0^+} \frac{m^*(E \cap [x-h, x+h])}{2h\psi(2h)}$$

and

$$\psi - \overline{d}(E,x) = \limsup_{h \to 0^+} \frac{m^*(E \cap [x-h, x+h])}{2h\psi(2h)}$$

denote the lower and upper outer ψ-density of a set E at a point x, respectively.

Definition 1.1. [1] *We say that $x \in \mathbb{R}$ is a density point of a set $E \in \mathcal{L}$ if $\underline{d}(E,x) = 1$. We say that $x \in \mathbb{R}$ is a dispersion point of a set $E \in \mathcal{L}$ if x is a density point of the set $\mathbb{R} \setminus E$.*

Set, for each $E \in \mathcal{L}$,

$$\Phi(E) = \{x \in \mathbb{R} : x \text{ is a density point of } E\}.$$

Then the family $d = \{E \in \mathcal{L} :\ E \subset \Phi(E)\}$ is a topology on the real line called the density topology [1].
Let $\psi \in \mathcal{C}$.

Definition 1.2. [3] *We say that $x \in \mathbb{R}$ is a ψ-dispersion point of a set $E \in \mathcal{L}$ if $\psi - \overline{d}(E,x) = 0$. We say that $x \in \mathbb{R}$ is a ψ-density point of a set $E \in \mathcal{L}$ if x is a ψ-dispersion point of the set $\mathbb{R} \setminus E$.*

For $E \in \mathcal{L}$, let

$$\Phi_\psi(E) = \{x \in \mathbb{R} :\ x \text{ is a } \psi - \text{density point of } E\}$$

and

$$\mathcal{T}_\psi = \{E \in \mathcal{L} :\ E \subset \Phi_\psi(E)\}.$$

Theorem 1.1. [3] *The family $\mathcal{T}_\psi$ is a topology on the real line, stronger than the Euclidean topology and weaker than the density topology d.*

Definition 1.3. [2] *We say that a set $E \subset \mathbb{R}$ is sparse at a point $x \in \mathbb{R}$ on the right if there exists, for every $\varepsilon > 0$, $\delta > 0$ such that every interval $(a,b) \subset (x, x+\delta)$, with $m((x,a)) < \delta m((x,b))$, contains at least one point y such that $m^*(E \cap (x,y)) < \varepsilon m((x,y))$.*

The family of sets sparse at x on the right is denoted by $\mathcal{S}(x+)$, and E is said to be sparse at x if $E \in \mathcal{S}(x) = \mathcal{S}(x+) \cap \mathcal{S}(x-)$, where $\mathcal{S}(x-)$ denotes, by convention, the family of sets sparse at x on the left.

Let $\mathcal{S}_0(x) = \{E \subset \mathbb{R} :\ \overline{d}(E,x) = 0\}$. Then by [2], for each $x \in \mathbb{R}$ $S_0(x) \subset S(x)$.

Theorem 1.2. [2] *Let $x \in \mathbb{R}$ and $E \subset \mathbb{R}$. The following conditions are equivalent:*

(i) $E \in \mathcal{S}(x)$,

(ii) *for each $F \subset \mathbb{R}$, if $\underline{d}(F,x) = 0$ then $\underline{d}(E \cup F, x) = 0$.*

2 ψ-sparse sets

In this chapter ψ will be an arbitrary fixed function from $\mathcal{C}$ and

$$g(x) = \begin{cases} 2x\psi(2x) & \text{if} \quad x \in (0,1], \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Then the function g is continuous and increasing. Moreover, $g(x) < 2x$ and $g(ax) \leq ag(x)$ for any $x \in (0,1]$ and $a \in (0,1)$.

Definition 2.1. *We say that a set $E \subset \mathbb{R}$ is ψ-sparse at a point $x \in \mathbb{R}$ if for each $F \subset \mathbb{R}$, the following holds:*

$$\textit{if } \ \psi - \underline{d}(F,x) = 0 \ \ \textit{then} \ \ \psi - \underline{d}(E \cup F, x) = 0.$$

For each $x \in \mathbb{R}$, we denote by $\psi - \mathcal{S}(x)$ the family of all sets which are ψ-sparse at x. Put, for each $x \in \mathbb{R}$, $\psi - \mathcal{S}_0(x) = \{E \subset \mathbb{R} : \ \psi - \overline{d}(E,x) = 0\}$. Then the following proposition and two theorems are obvious.

Proposition 2.1. *Let $A \subset \mathbb{R}$, $B \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then*

1. *if $A \in \psi - S(x)$ and $B \in \psi - S(x)$, then $A \cup B \in \psi - S(x)$,*

2. *if $A \in \psi - S(x)$ and $B \subset A$, then $B \in \psi - S(x)$.*

Theorem 2.1. *For each $x \in \mathbb{R}$, $\psi - \mathcal{S}_0(x) \subset \psi - \mathcal{S}(x)$.*

Theorem 2.2. *For any $E \subset \mathbb{R}$ and $x \in \mathbb{R}$, $E \in \psi - \mathcal{S}(x)$ if and only if $\{y - x : \ y \in E\} \in \psi - \mathcal{S}(0)$.*

Theorem 2.3. *Let $E \subset \mathbb{R}$ and let A be a measurable cover of E. Then the following conditions are equivalent:*

(i) $E \in \psi - \mathcal{S}(0)$.

(ii) *For each $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that, for each interval $[a,b] \subset (0,\delta)$, if $g(a) < \delta g\left(x - \frac{\varepsilon}{2}g(x)\right)$ for each $x \in [b,1]$, then there exists $y \in (a,b)$ such that $m^*(E \cap (-y,y)) < \varepsilon g(y)$.*

(iii) $A \in \psi - \mathcal{S}(0)$.

Proof. $(i) \Rightarrow (ii)$ For any $\varepsilon, \delta \in (0,1)$, denote by $W(\varepsilon, \delta)$ the family of all intervals $[a,b] \subset (0,\delta)$ such that, for each $x \in [b,1]$, $g(a) < \delta g\left(x - \frac{\varepsilon}{2} g(x)\right)$ and, for each $y \in (a,b)$, $m^*(E \cap (-y,y)) \geq \varepsilon g(y)$.

Proceeding by contradiction, assume that $E \in \psi - \mathcal{S}(0)$ and (ii) is false. From our assumption it follows that there is $\varepsilon \in (0,1)$ such that $W(\varepsilon, \delta) \neq \emptyset$, for each $\delta \in (0,1)$.

Let $\delta_1 \in (0,1)$ be such that $\psi(2\delta_1) < \frac{1}{4}$ and $[a_1, b_1] \in W(\varepsilon, \delta_1)$. For each $n \in \mathbb{N}$, let $0 < \delta_{n+1} < \min\left\{\frac{1}{n+1}, \frac{1}{2} g(a_n)\right\}$ and $[a_{n+1}, b_{n+1}] \in W(\varepsilon, \delta_{n+1})$.

By the above we have defined the sequence of disjoint intervals $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ and the sequence of real positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that

(1) $\psi(2\delta_1) < \frac{1}{4}$ and $\delta_1 \in (0,1)$,

(2) for each $n \in \mathbb{N}$, $[a_n, b_n] \in W(\varepsilon, \delta_n)$,

(3) for each $n \in \mathbb{N}$, $0 < \delta_{n+1} < \min\left\{\frac{1}{n+1}, \frac{1}{2} g(a_n)\right\}$ and $0 < \delta_{n+1} < a_n < \delta_1$,

(4) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$.

Let $n \in \mathbb{N}$ and $x_n \in [b_{n+1}, a_n]$ be such that

$$x_n - \frac{\varepsilon}{2} g(x_n) = \min\left\{x - \frac{\varepsilon}{2} g(x) : \ x \in [b_{n+1}, a_n]\right\}.$$

Set $y_{n+1} = x_n - \frac{\varepsilon}{2} g(x_n)$ and $z_{n+1} = y_{n+1} + g(a_n)$. We shall show that

(5) $g(a_{n+1}) < \frac{1}{n+1} g(y_{n+1})$,

(6) $a_{n+1} < y_{n+1} < b_{n+1} < z_{n+1} < 2g(a_n) < a_n$.

By (2) and (3), we have that $g(a_{n+1}) < \delta_{n+1} g\left(x_n - \frac{\varepsilon}{2} g(x_n)\right) < \frac{1}{n+1} g(y_{n+1})$. Therefore the monotonicity of function g implies $a_{n+1} < y_{n+1}$. By the definition of the point x_n, we have $y_{n+1} = x_n - \frac{\varepsilon}{2} g(x_n) \leq b_{n+1} - \frac{\varepsilon}{2} g(b_{n+1}) < b_{n+1}$. By the above and (2),(3) we get $z_{n+1} < b_{n+1} + g(a_n) < \delta_{n+1} + g(a_n) < 2g(a_n)$. Additionally, by (3) and (1) $2g(a_n) = 4a_n\psi(2a_n) \leq 4a_n\psi(2\delta_1) < a_n$. Besides that $z_{n+1} \geq y_{n+1} + g(x_n) = x_n - \frac{\varepsilon}{2} g(x_n) + g(x_n) > x_n \geq b_{n+1}$. Therefore the conditions (5) and (6) are satisfied.

Let $F = \bigcup\limits_{n=1}^{\infty} [y_{n+1}, z_{n+1}]$. By (6) and (5), we observe that, for each $n \in \mathbb{N}$,

$$m(F \cap [-y_{n+1}, y_{n+1}]) < z_{n+2} < 2g(a_{n+1}) < \frac{2}{n+1} g(y_{n+1}).$$

Hence,

$$\psi - \underline{d}(F,0) \leq \lim_{n\to\infty} \frac{m(F \cap [-y_{n+1}, y_{n+1}])}{2y_{n+1}\psi(2y_{n+1})} = 0.$$

Now, we shall show that $\psi - \underline{d}(E \cup F, 0) > 0$. Let $h \in (0, b_1)$. Then there exists $n \in \mathbb{N}$ such that $h \in [b_{n+1}, b_n)$. There are three cases to consider.

(α) $h \in (a_n, b_n)$. Then by (2),

$$m^*((E \cup F) \cap [-h,h]) \geq m^*(E \cap [-h,h]) \geq \varepsilon g(h).$$

(β) $h \in [z_{n+1}, a_n]$. Then by the definition of z_{n+1},

$$m^*((E \cup F) \cap [-h,h]) \geq m(F \cap [0,h]) > z_{n+1} - y_{n+1} = g(a_n) \geq g(h).$$

(γ) $h \in [b_{n+1}, z_{n+1})$. Then by $h - \frac{\varepsilon}{2}g(h) \geq x_n - \frac{\varepsilon}{2}g(x_n) = y_{n+1}$, we have that $h - y_{n+1} \geq \frac{\varepsilon}{2}g(h)$. Hence

$$m^*((E \cup F) \cap [-h,h]) \geq m(F \cap [0,h]) > h - y_{n+1} \geq \frac{\varepsilon}{2}g(h).$$

Therefore

$$\liminf_{h\to 0^+} \frac{m^*((E \cup F) \cap [-h,h])}{2h\psi(2h)} \geq \frac{\varepsilon}{2}.$$

We have shown that there exists a set $F \subset \mathbb{R}$ such that $\psi - \underline{d}(F,0) = 0$ and $\psi - \underline{d}(E \cup F, 0) > 0$. Thus $E \notin \psi - \mathcal{S}(0)$, a contradiction.

(ii) $\Rightarrow$ *(iii)* Suppose that (ii) is fulfilled. First we show that $\psi - \underline{d}(A,0) = 0$. Let $n \in \mathbb{N}$. By our assumption, there exists $\delta_n \in (0,1)$ such that, for each interval $[a,b] \subset (0, \delta_n)$, if $g(a) < \delta_n g\left(x - \frac{1}{2(n+1)}g(x)\right)$ for each $x \in [b,1]$, then there exists $y \in (a,b)$ such that

$$m(A \cap (-y,y)) = m^*(E \cap (-y,y)) < \frac{1}{n+1}g(y).$$

Let $0 < b_n < \min\left\{\delta_n, \frac{1}{n}\right\}$ and $z_n = \min\left\{\delta_n g\left(x - \frac{1}{2(n+1)}g(x)\right) :\ x \in [b_n, 1]\right\}$. By the continuity of g at 0, there exists $a_n \in (0, b_n)$ such that $g(a_n) < z_n$. Therefore, $g(a_n) < \delta_n g\left(x - \frac{1}{2(n+1)}g(x)\right)$ for each $x \in [b_n, 1]$ and by our assumption there exists $y_n \in (a_n, b_n)$ such that $m(A \cap (-y_n, y_n)) < \frac{1}{n+1}g(y_n)$. Thus

$$\psi - \underline{d}(A,0) \leq \lim_{n\to\infty} \frac{m(A \cap [-y_n, y_n])}{2y_n\psi(2y_n)} = 0.$$

Let $F \subset \mathbb{R}$ be such that $\psi - \underline{d}(F,0) = 0$. It is sufficient to show that for each $n \in \mathbb{N} \setminus \{1\}$ there exists $v_n \in (0, \frac{1}{n})$ such that

$$m^*((A \cup F) \cap [-v_n, v_n]) \leq \frac{4}{n} g(v_n).$$

Let $n \in \mathbb{N} \setminus \{1\}$ and

$$A_n = \left\{ h \in (0,1) : \ m(A \cap [-h,h]) > \frac{1}{n} g(h) \right\}.$$

Observe that $(0, \frac{1}{n}) \setminus A_n \neq \emptyset$. If there exists $v_n \in (0, \frac{1}{n}) \setminus A_n$ such that $m^*(F \cap [-v_n, v_n]) \leq \frac{1}{n} g(v_n)$, then

$$m^*((A \cup F) \cap [-v_n, v_n]) \leq \frac{2}{n} g(v_n).$$

We assume that $m^*(F \cap (-x,x)) > \frac{1}{n} g(x)$ for each $x \in (0, \frac{1}{n}) \setminus A_n$.

By our assumption there exists $\delta \in (0,1)$ such that for each closed interval $[a,b] \subset (0,\delta)$, if $g(a) < \delta g\left(x - \frac{1}{2n} g(x)\right)$ for each $x \in [b,1]$, then there exists $y \in (a,b)$ such that $m(A \cap (-y,y)) < \frac{1}{n} g(y)$. Let $\delta_1 = min\left\{\delta, \frac{1}{n}\right\}$. By $\psi - \underline{d}(A,0) = 0$ there exists $y_0 \in (0, \delta_1)$ such that

$$m(A \cap (-y_0, y_0)) < \frac{1}{n} g(y_0)$$

and, by $\psi - \underline{d}(F,0) = 0$, there exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ tending to zero, such that

$$m^*(F \cap [-t_k, t_k]) < \delta \frac{1}{n} g(t_k)$$

and $t_k < y_0$ for each $k \in \mathbb{N}$. Then $t_k \in (0, \frac{1}{n})$ and $m^*(F \cap [-t_k, t_k]) < \frac{1}{n} g(t_k)$ so we have $t_k \in A_n$ for each $k \in \mathbb{N}$.

Let k be a fixed positive integer number. Since $t_k \in A_n$ it follows that there exists a component (a_k, b_k) of the open set A_n, such that $t_k \in (a_k, b_k)$. We observe that

$$m(A \cap [-x,x]) > \frac{1}{n} g(x)$$

for each $x \in (a_k, b_k)$,

$$m(A \cap [-a_k, a_k]) = \frac{1}{n} g(a_k)$$

and

$$m(A \cap [-b_k, b_k]) = \frac{1}{n} g(b_k).$$

So $y_0 \notin [a_k, b_k]$ and as $t_k < y_0$, we have $b_k < y_0 < \delta$.

We have proven that $[a_k, b_k] \subset (0, \delta)$ and $m(A \cap [-x, x]) > \frac{1}{n} g(x)$ for each $x \in (a_k, b_k)$. Therefore, there exists $x_k \in [b_k, 1]$ such that

$$g(a_k) \geq \delta g\left(x_k - \frac{1}{2n} g(x_k)\right).$$

Moreover, $a_k \notin A_n$ and $a_k \in (0, \frac{1}{n})$, hence $m^*(F \cap [-a_k, a_k]) > \frac{1}{n} g(a_k)$. Therefore

$$\begin{aligned} \frac{1}{n}\delta g\left(x_k - \frac{1}{2n} g(x_k)\right) &\leq \frac{1}{n} g(a_k) < m^*(F \cap [-a_k, a_k]) \\ &\leq m^*(F \cap [-t_k, t_k]) < \delta \frac{1}{n} g(t_k) \end{aligned}$$

and, by the monotonicity of the function g, we have

$$x_k - \frac{1}{2n} g(x_k) < t_k < b_k \leq x_k.$$

Thus

$$\begin{aligned} m(A \cap [-x_k, x_k]) &\leq m(A \cap [-b_k, b_k]) + 2(x_k - b_k) \\ &\leq \frac{1}{n} g(b_k) + \frac{1}{n} g(x_k) \leq \frac{2}{n} g(x_k), \end{aligned}$$

and

$$\begin{aligned} m^*(F \cap [-x_k, x_k]) &\leq m^*(F \cap [-t_k, t_k]) + 2(x_k - t_k) \\ &< \delta \frac{1}{n} g(t_k) + \frac{1}{n} g(x_k) < \frac{2}{n} g(x_k). \end{aligned}$$

Hence, $m^*((A \cup F) \cap [-x_k, x_k]) < \frac{4}{n} g(x_k)$.

Moreover, $\limsup\limits_{k\to\infty} \left(x_k - \frac{1}{2n} g(x_k)\right) \leq \lim\limits_{k\to\infty} t_k = 0$, so $\lim\limits_{k\to\infty} x_k = 0$. Now we put $v_n = x_k$, where $x_k \in (0, \frac{1}{n})$.

$(iii) \Rightarrow (i)$ Assume that (iii) is fulfilled. Let $F \subset \mathbb{R}$ be a set such that $\psi - \underline{d}(F, 0) = 0$. Then

$$\liminf_{h\to 0^+} \frac{m^*((E \cup F) \cap [-h, h])}{g(h)} \leq \liminf_{h\to 0^+} \frac{m^*((A \cup F) \cap [-h, h])}{g(h)} = 0.$$

Hence $E \in \psi - \mathcal{S}(0)$.

Lemma 2.1. *For each real number $\alpha \in (0, 1)$, there exists an open interval $(a, b) \subset (0, \alpha)$ such that $b - a = 2b\psi(2b)$ and $2a\psi(2a) \geq b\psi(2b)$.*

PROOF. Let $\alpha \in (0,1)$ and $\delta > 0$ be such that, for each $x \in (0,\delta)$, we have $\psi(2x) < \frac{1}{4}$. Put $\gamma = min\{\alpha,\delta\}$, $b_1 \in (0,\gamma)$ and, for each $n \in \mathbb{N}$, b_{n+1} be such that $b_{n+1}\psi(2b_{n+1}) = \frac{1}{2^n} b_1\psi(2b_1)$. Then $\lim\limits_{n\to\infty} b_n = 0$.

Suppose that $b_n - b_{n+1} < 2b_n\psi(2b_n)$ for each $n \in \mathbb{N}$. Then

$$b_1 = \sum_{n=1}^{\infty}(b_n - b_{n+1}) \leq \sum_{n=1}^{\infty} 2b_n\psi(2b_n) = 4b_1\psi(2b_1) < b_1,$$

which is impossible. Thus there exists $n \in \mathbb{N}$ such that $b_n - b_{n+1} \geq 2b_n\psi(2b_n)$.

Let $b = b_n$ and $a = b_n - 2b_n\psi(2b_n)$. Then $b - a = 2b\psi(2b)$, $a \geq b_{n+1}$ and $2a\psi(2a) \geq 2b_{n+1}\psi(2b_{n+1}) = b\psi(2b)$. □

Theorem 2.4. *There exists an open set H such that $H \in \psi - \mathcal{S}(0) \setminus \psi - \mathcal{S}_0(0)$.*

PROOF. By Lemma 2.1, we can defined a sequence of disjoint open intervals $\{(c_n,d_n)\}_{n\in\mathbb{N}} \subset (0,1)$ such that for each $n \in \mathbb{N}$,

1. $d_n - c_n = g(d_n)$,
2. $g(c_n) \geq \frac{1}{2}g(d_n)$,
3. $d_{n+1} < \min\left\{\frac{1}{n}, \frac{1}{2^n}g(c_n)\right\}$.

Put $H = \bigcup\limits_{n\in\mathbb{N}}(c_n,d_n)$. Then $m(H \cap [-d_n,d_n]) \geq d_n - c_n = g(d_n)$ for each $n \in \mathbb{N}$. Therefore $H \notin \psi - \mathcal{S}_0(0)$.

We shall show that $H \in \psi - \mathcal{S}(0)$. Let $\varepsilon \in (0,1)$. Choose $n_0 \in \mathbb{N}$ such that $\max\left\{c_{n_0}, \frac{1}{2^{n_0}}\right\} < \frac{\varepsilon}{4}$. Then, for each $n > n_0$, the inequality

$$\frac{\varepsilon}{2}g(d_{n+1}) < g(d_{n+1}) < 2d_{n+1} < \frac{\varepsilon}{2}g(c_n)$$

implies that there exists $y_n \in (d_{n+1},c_n)$ such that $g(d_{n+1}) = \frac{\varepsilon}{2}g(y_n)$.

Let $x_0 \in [0,1]$ be such that

$$m = x_0 - \frac{\varepsilon}{2}g(x_0) = \sup\left\{x - \frac{\varepsilon}{2}g(x) :\ x \in [0,1]\right\}.$$

It is easily seen that $x_0 \neq 0$ and $m > 0$. Choose $n_1 > n_0$ such that $c_{n_1} < m$ and $c_{n_1} < x_0$. Put $\delta = c_{n_1}$. Let $[a,b] \subset (0,\delta)$ be an interval such that, for each $x \in [b,1]$, $g(a) < \delta g(x - \frac{\varepsilon}{2}g(x))$. If there exists $n \geq n_1$ such that $c_n \in (a,b)$, then

$$m(H \cap [-c_n,c_n]) < d_{n+1} < \frac{1}{2^n}g(c_n) < \varepsilon g(c_n).$$

Now let as assume that for each $n \geq n_1$ $c_n \notin (a,b)$. Then there exists $n \geq n_1$ such that $(a,b) \subset (c_{n+1}, c_n)$. Suppose $(a,b) \subset (c_{n+1}, y_n)$. Then, by $0 < y_n - \frac{\varepsilon}{2}g(y_n) < y_n < c_{n_1} < m$ and $y_n < x_0$, there exists $x \in (y_n, x_0) \subset [b,1]$ such that $x - \frac{\varepsilon}{2}g(x) = y_n$. Therefore, by 2

$$g(a) \geq g(c_{n+1}) \geq \frac{1}{2}g(d_{n+1}) = \frac{\varepsilon}{4}g(y_n) \geq \delta g(x - \frac{\varepsilon}{2}g(x)).$$

But this contradicts the definition of the interval $[a,b]$, so $(a,b) \cap (y_n, c_n) \neq \emptyset$. Let $h \in (a,b) \cap (y_n, c_n)$. Then

$$m(H \cap [-h,h]) < \frac{1}{2^{n+1}}g(c_{n+1}) + g(d_{n+1}) < \frac{\varepsilon}{2}g(h) + \frac{\varepsilon}{2}g(y_n) < \varepsilon g(h).$$

We have shown that, for each $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that, for each interval $[a,b] \subset (0,\delta)$, if $g(a) < \delta g(x - \frac{\varepsilon}{2}g(x))$ for each $x \in [b,1]$, then there exists $y \in (a,b)$ such that $m(H \cap (-y,y)) < \varepsilon g(y)$. Thus, by Theorem 2.3, $H \in \psi - \mathcal{S}(0)$. □

Theorem 2.5. *For each $x \in \mathbb{R}$, $\psi - \mathcal{S}(x) \cap \mathcal{L} \subset \mathcal{S}_0(x)$.*

PROOF. We may assume that $x = 0$. We suppose that there exists a set $A \in \psi - \mathcal{S}(0) \cap \mathcal{L} \setminus \mathcal{S}_0(0)$. Then there exists a real number $\alpha \in (0,1)$ such that

(7) $$\limsup_{x \to 0^+} \frac{m^*(A \cap [-x,x])}{2x} > \alpha$$

and, by Theorem 2.3, there exists a real number $\delta \in (0,1)$ such that

(8) for each interval $[a,b] \subset (0,\delta)$, if $g(a) < \delta g\left(x - \frac{1}{4}g(x)\right)$ for each $x \in [b,1]$, then there exists $y \in (a,b)$ such that $m(A \cap (-y,y)) < \frac{1}{2}g(y)$.

Let γ be a real positive number such that $\gamma < \delta$ and, for each $x \in (0,\gamma)$, $\psi(2x) < \alpha$. By the continuity of the function $x - \frac{1}{4}g(x)$, for each $b \in (0,1)$, there exists a point $t(b) \in [b,1]$ such that

$$t(b) - \frac{1}{4}g(t(b)) = \min\left\{x - \frac{1}{4}g(x) :\ x \in [b,1]\right\} \leq b - \frac{1}{4}g(b).$$

Then, by $\lim_{b \to 0^+}\left(t(b) - \frac{1}{4}g(t(b)\right) = 0$ and by the definition of the function g, we see that

$$\lim_{b \to 0^+} \frac{g(t(b))}{t(b) - \frac{1}{4}g(t(b))} = 0.$$

Thus there exists a real positive number $\delta_1 < \gamma$ such that, for any $b \in (0, \delta_1)$ and $x \in [b, 1]$,

$$g(b) \leq g(t(b)) < 2\alpha\delta \left(t(b) - \frac{1}{4} g(t(b)) \right) \leq 2\alpha\delta \left(x - \frac{1}{4} g(x) \right).$$

Consequently,

(9) for any $b \in (0, \delta_1)$ and $x \in [b, 1]$,

$$g\left(\frac{1}{2\alpha} g(b) \right) < g\left(\delta \left(x - \frac{1}{4} g(x) \right) \right) \leq \delta g \left(x - \frac{1}{4} g(x) \right).$$

By $A \in \psi - \mathcal{S}(0)$, there exists $x_1 \in (0, \delta_1)$ such that $m(A \cap [-x_1, x_1]) < g(x_1)$ and, by (7), there exists $x_2 \in (0, x_1)$ such that $m(A \cap [-x_2, x_2]) > 2\alpha x_2$. Put

$$E = \{ x \in [x_2, 1] : \ m(A \cap [-x, x]) \leq g(x) \ \}.$$

Then $x_1 \in E$. Set $b = \min E$. Since $\psi(2x_2) < \alpha$, we have that

$$m(A \cap [-x_2, x_2]) > 2\alpha x_2 > g(x_2)$$

and $x_2 < b < x_1$. Put $a = x_2$. Then

$$\textbf{(10)} \qquad g(b) = m(A \cap [-b, b]) \geq m(A \cap [-a, a]) > 2\alpha a$$

and

(11) for each $t \in (a, b)$, $m(A \cap (-t, t)) > g(t)$.

Let $x \in [b, 1]$. By (10) and (9),

$$g(a) < g\left(\frac{1}{2\alpha} g(b) \right) < \delta g \left(x - \frac{1}{4} g(x) \right),$$

for each $x \in [b, 1]$. Therefore, by (8), there exists $y \in (a, b)$ such that

$$m(A \cap [-y, y]) < \frac{1}{2} g(y),$$

contrary to (11). □

Theorem 2.6. *There exists an open set H such that $H \in \mathcal{S}_0(0) \setminus \psi - \mathcal{S}(0)$.*

PROOF. Let $b_0 \in (0, 1)$ be such that $\psi(2b_0) < \frac{1}{16}$ and, for each $n \in \mathbb{N}$, $b_n = \frac{1}{2^n} b_0$. We choose a_1 as an arbitrary point of an interval (b_2, b_1) and, for each $n \geq 2$, put $a_n = b_n - g(b_{n-2})$. We observe that, for each $n \geq 2$,

$$a_n = b_n - g(b_{n-2}) = 2b_{n+1}(1 - 8\psi(2b_{n-2})) > b_{n+1}.$$

Put $H = \bigcup_{n=2}^{\infty}(a_n, b_n)$. We shall show that $H \notin \psi - S(0)$. Let $h \in (0, a_2]$. Then there exists $n \geq 3$ such that $h \in (a_n, a_{n-1}]$. Therefore

$$m(H \cap [-h, h]) > b_{n+1} - a_{n+1} = g(b_{n-1}) > g(h).$$

Now we shall show that $H \in \mathcal{S}_0(0)$. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$, $\psi(2b_n) < \frac{\varepsilon}{16}$. Put $\delta = a_{n_0+1}$ and let $h \in (0, \delta)$. Then there exists $n > n_0 + 1$ such that $h \in [a_n, a_{n-1})$, and

$$m(H \cap [-h, h]) \leq \sum_{k=n}^{\infty}(b_k - a_k) = \sum_{k=n}^{\infty} g(b_{k-2}) < \frac{\varepsilon}{8}\sum_{k=n}^{\infty} b_{k-2} = \varepsilon 2b_{n+1} < \varepsilon 2h.$$

Thus

$$\lim_{h \to 0^+} \frac{m(H \cap [-h, h])}{2h} = 0.$$

□

3 ψ-sparse topology

Let $\psi \in \mathcal{C}$. For $E \in \mathcal{L}$, put

$$\Gamma_\psi(E) = \{x \in \mathbb{R} : \ x \text{ is a } \psi - \text{sparse point of } \mathbb{R} \setminus E\}.$$

Let $A \in \mathcal{L}$ and $B \in \mathcal{L}$. We denote $A \sim B$, if $m(A \triangle B) = 0$, where $A \triangle B$ is the symmetric difference of A and B.

It is easy to see that the following theorem is true.

Theorem 3.1. *Let $\psi \in \mathcal{C}$. Then for each $A, B \in \mathcal{L}$*

1. *if $A \subset B$, then $\Gamma_\psi(A) \subset \Gamma_\psi(B)$;*

2. *if $A \sim B$, then $\Gamma_\psi(A) = \Gamma_\psi(B)$;*

3. *$\Gamma_\psi(\emptyset) = \emptyset, \quad \Gamma_\psi(\mathbb{R}) = \mathbb{R}$;*

4. *$\Gamma_\psi(A \cap B) = \Gamma_\psi(A) \cap \Gamma_\psi(B)$.*

By theorems 3.1, 2.1, 2.4, 2.5 and 2.6, we have the following

Theorem 3.2. *Let $\psi \in \mathcal{C}$ and*

$$\tau_\psi = \{E \in \mathcal{L} : \ E \subset \Gamma_\psi(E)\}.$$

Then τ_ψ is a topology on the real line, stronger than the ψ-density topology $\mathcal{T}_\psi$ and weaker than the density topology d.

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