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# ON SUMS AND PRODUCTS OF PERIODIC FUNCTIONS

**Abstract**

The purpose of this work is to ascertain when arithmetic operations with periodic functions whose domains may not coincide with the whole real line preserve periodicity.

## 1 Introduction and Preliminaries.

The problem under research is when the arithmetic operations with periodic functions of one real variable whose domains may not coincide with the real line will give periodic functions. The answer is well known in the case when two nonconstant periodic functions are defined and continuous on the whole real line and the operation is addition. In this case the sum is periodic if and only if the periods of summands are commensurable. But it may be false if the domains of summands are proper subsets of reals.

In the following, the function $f$ defined on the set $D \subset \mathbb{R}$ is called *periodic* (or $T$-*periodic*) if $D+T=D$ and $f(x+T)=f(x)$ for all $x \in D$ hold for some real number $T \neq 0$. In this case $D$ is called $T$-*invariant* (or $T$-*periodic*), and $T$ is called a *period* of $f$ and $D$. The periods will be always assumed to be positive unless otherwise stated. The smallest positive period of $f$ and $D$ (if such exists) is called *fundamental*.

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If $D$ is $T$-invariant and $f(x+T) = f(x)$ for a.e. $x \in D$ only, we say that $f$ is *a.e. periodic* (with period $T$).

The function $f$ with domain $D(f)$ is called *not a.e. constant* if for every $c$ the set $\{x \in D(f) | f(x) \neq c\}$ has a positive Lebesgue measure. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ stand for sets of natural numbers, integers, rational numbers, and reals respectively, $\mu$ stands for Lebesgue measure on $\mathbb{R}$. Below we require that all functions considered have nonempty domains.

For our purposes the question on the commensurability of periods of periodic function is important. The following example shows that the answer to this question may be negative.

**Example 1.1.** Let $D := \mathbb{Z} + \sqrt{2}\mathbb{Z} + \sqrt{3}\mathbb{Z}$. The function on $D$ defined by the equality

$$f(k + l\sqrt{2} + m\sqrt{3}) = (-1)^m$$

is bounded and has two incommensurable periods 1 and $\sqrt{2}$.

At the same time, the next statement is well known (see, e.g., [3]).

**Theorem 1.1.** *If a periodic function $f$ is continuous and nonconstant on $D(f)$, then $f$ has fundamental period. In particular, every two periods off are commensurable.*

We mention two other conditions, which are sufficient for commensurability of the periods of periodic function.

**Theorem 1.2.** *Consider a set $D \neq \mathbb{R}$, $intD \neq \emptyset$. If $D$ is periodic then it has the fundamental period. In particular, if a periodic function $f$ is defined on $D$, then $f$ has the fundamental period, too. Thus every two periods of $f$ are commensurable.*

PROOF. The set $G$ of all periods of $D$ is an additive subgroup of $\mathbb{R}$ (we consider negative periods and zero as a period of $D$, too). Suppose that $G$ is not discrete. Then it is dense in $\mathbb{R}$ (see, e.g., [1]). Choose $a \notin D$. The set $-intD + a$ intersects $G$, so $-d + a = t$ for some $d$ in $D$ and $t$ in $G$; i.e. $a = d + t \in D$, a contradiction. Therefore $G = T_0\mathbb{Z}$ for some $T_0 \in \mathbb{R}$, $T_0 \neq 0$ (see ibid). This completes the proof. $\square$

**Theorem 1.3.** *If an a.e. periodic function $f$ is defined, measurable, and not a.e. constant on a set $D$ of positive measure, then every two periods of $f$ are commensurable.*

PROOF. Let $T_1$, $T_2$ be two periods of $f, S = \mathbb{R}(modT_1)$; i.e. $S$ is a circle with radius $r = T_1/2\pi$. The set $D_1 = D(modT_1)$ is the subset of $S$ of positive measure. One can assume that $f$ is defined on $D_1$. The rotation $R_O^\alpha$ of $S$ with the angle $\alpha = T_2/r = (T_2/T_1)2\pi$, which maps $D_1$ on itself, corresponds to the shift $x \mapsto x + T_2$ of the real line. If $T_1$ and $T_2$ are incommensurable, $R_O^\alpha$ is an ergodic transformation of $D_1$ by virtue of the equation $\alpha/2\pi = T_2/T_1$ (see, e.g., [7], Section II. 5). Since the function $f$ on $D_1$ is $R_O^\alpha$-invariant (that is $f(R_O^\alpha x) = f(x)$ for a.e. $x \in D_1$), it is an a.e. constant ([7], ibid), a contradiction. □

Note that Burtin's Theorem [2], [4] could be used to prove Theorem 1.3, too.

## 2 Sums of Several Periodic Functions with the Common Domain.

It is well known that the sum of two continuous periodic functions on $\mathbb{R}$ is periodic if and only if their periods are commensurable. In this section, we study the periodicity of sums of several periodic functions $f_i (i = 1, ..., n)$ in the case where $D(f_1) = \ldots = D(f_n)$ may not coincide with $\mathbb{R}$. The following example shows that the situation in this case is more complicated.

**Example 2.1.** Let $D := \mathbb{Z} + \sqrt{2}\mathbb{Z} + \sqrt{3}\mathbb{Z}$ as in Example 1.1. Two functions on $D$ defined by the equalities

$$f_1(k + l\sqrt{2} + m\sqrt{3}) = \frac{1}{|l| + 1} - \frac{1}{|m| + 1},$$
$$f_2(k + l\sqrt{2} + m\sqrt{3}) = \frac{1}{|k| + 1} + \frac{1}{|m| + 1}$$

are bounded and periodic, their periods are incommensurable, but the sum $f_1 + f_2$ is periodic.

If the periods $T_i$ of several periodic functions $f_i$ $(i = 1, ..., n)$ are commensurable, it is easy to prove that the sum $f_1 + \cdots + f_n$ is periodic. The converse is false, in general. If, say, $f_1 + f_2 = const$, the sum $f_1 + f_2 + f_3$ is periodic for incommensurable $T_1$ and $T_3$. So for converse we should assume that all the sums of $f_i$'s where the number of summands is less than $n$ are nonconstant.

**Theorem 2.1.** *Let $f_1, f_2, \ldots, f_n$ be continuous periodic functions, which are nonconstant on their common domain $D$. If all the sums of $f_i$'s where the* *number of summands is less than $n$ are nonconstant, then the sum $f_1+\ldots+f_n$ is periodic if and only if the periods of the summands are commensurable.*

Proof. We shall prove this theorem by induction with the following additional statement: in the case when the sum is nonconstant the periods of the summands are commensurable with the period of the sum. First we shall prove the conclusion of the theorem for $n = 2$.

Suppose that $T_1, T_2$, and $T$ are periods of $f_1, f_2$, and $f_1 + f_2$ respectively. Then we have for all $x \in D$

$$f_1(x+T) + f_2(x+T) = f_1(x) + f_2(x),$$

or

$$f_1(x+T) - f_1(x) = f_2(x) - f_2(x+T). \tag{1}$$

a) Suppose that both sides in (1) are nonconstant. Since the left-hand side and the right-hand one in (1) have periods $T_1$ and $T_2$ respectively, Theorem 1.1 implies that these periods are commensurable. Further since $T_1$ and $T_2$ are commensurable, the sum $f_1+f_2$ has certain period $T^*$ which is commensurable with $T_1$ and $T_2$. If $f_1 + f_2$ is nonconstant, then $T$ and $T^*$ are commensurable by Theorem 1.1, too.

b) Assume that both sides in (1) equal to a nonzero constant $c$. The iteration of the equation

$$f_1(x+T) - f_1(x) = c \tag{2}$$

implies $f_1(x + nT + mT_1) = f_1(x) + nc$ for all $m, n \in \mathbb{Z}$. We can find integers $n_k$ and $m_k$, with $n_k \to \infty$ such that $x + n_kT + m_kT_1 \to x$ and we have a contradiction with the continuity of $f_1$ if $c \neq 0$.

c) If both sides of (1) are zero, then $f_i(x + T) = f_i(x)$, and $T_i$ and $T$ are commensurable by Theorem 1.1 ($i = 1, 2$).

Now, let the conclusion of the theorem be true for all integers between 2 and $n$. We shall prove it for $n + 1$. Two cases are possible:

1) The sum $f_1 + \cdots + f_{n+1}$ is constant. Then $f_1(x) + \cdots + f_{n+1}(x) = c$ and $f_1(x) + \cdots + f_n(x) = c - f_{n+1}(x)$. Because the left-hand side is nonconstant, the inductive hypothesis implies that the periods of $f_1, \ldots, f_n$ and $T_{n+1}$ are pairwise commensurable.

2) This sum is nonconstant and $T$-periodic. If $g_i(x) := f_i(x + T) - f_i(x)$, then

$$g_1(x) + \cdots + g_{n+1}(x) = 0. \tag{3}$$

If some $g_i$ is a constant, then it equals 0 by b).

2.1) Let $g_i$'s be nonconstant for $i = 1, ..., n$.

d) If the sum $g_1 + \cdots + g_n (= -g_{n+1})$ has not proper subsums which are constant, the periods $T_1, \ldots, T_n$ are commensurable by inductive hypothesis. Then the first summand of the sum $(f_1 + \cdots + f_n) + f_{n+1}$ has period of the form $mT_1$, and again by inductive hypothesis $T_1, T_{n+1}$, and $T$ are commensurable.

e) If the sum $g_1 + \cdots + g_n (= -g_{n+1})$ has proper subsums which are constant, let us choose a minimal one, say, $g_1 + \cdots + g_k = const$ $(k > 1)$. Then by inductive hypothesis, $T_1, \ldots, T_k$ are commensurable. Like in d) the first summand of the sum $(f_1 + \cdots + f_k) + (f_{k+1} + \cdots + f_{n+1})$ has the period of the form $mT_1$, and by inductive hypothesis $T_1, T_{k+1}, \ldots, T_{n+1}$ and $T$ are commensurable.

2.2) If there exist constants among $g_i$'s (which are equal to 0), then let us re-index the functions such that $g_1, \ldots, g_k \neq 0$ and $g_{k+1} = \ldots = g_{n+1} = 0$ where $k < n+1$. Since for $i$ between $k+1$ and $n+1$ the difference $f_i(x+T) - f_i(x)$ equals 0, then by Theorem 1.1 the numbers $T_i$ and $T$ are commensurable. In addition we have $f_1(x+T) + \cdots + f_k(x+T) = f_1(x) + \cdots + f_k(x)$ where $k < n+1$. By the hypothesis of the theorem this sum is nonconstant, so by the inductive hypothesis the periods $T_1, \ldots, T_k$ are commensurable with $T$. Moreover, as we have shown numbers $T_{k+1}, \ldots, T_{n+1}$ are commensurable with $T$, too. □

We will employ the following lemma to prove Theorem 2.2. (As was mentioned by the referee, one can prove Theorem 2.2 using the Proposition 1 in [5] (see also [6]); we give an independent proof which seems to be more elementary).

**Lemma 2.1.** *Let the function $\psi$ be measurable on the segment $I$. There is a sequence $\xi_k \downarrow 0$ such that for every sequence $\delta_k, \delta_k \in (0, \xi_k)$*

$$\lim_{k\to\infty} \psi(x + \delta_k) = \psi(x) \tag{4}$$

*for a.e. $x \in I$.*

For the proof see, e.g., [8], proof of Theorem 1.4, especially formula (1.18).

**Theorem 2.2.** *Let a.e. $T_i$-periodic functions $f_i$ $(i = 1, \ldots, n)$ be defined, measurable, and not a.e. constant on the measurable set $D$ of positive measure.*

*Suppose that all the sums of $f_i$'s where the number of summands is less than $n$ are not a.e. constant. The sum $f_1 + \cdots + f_n$ is a.e. periodic if and only if the periods of the summands are commensurable.*

PROOF. As in proof of Theorem 2.1 we shall prove this theorem by induction with the following additional statement: in the case when the sum is not a.e. constant the periods of the summands are commensurable with the period of the sum. First we shall prove the conclusion of the theorem for $n = 2$.

Suppose that $T_1, T_2$, and $T$ are periods of $f_1, f_2$ and $f_1 + f_2$ respectively. Then (1) holds for a.e. $x \in D$.

a) Suppose that both sides in (1) are not a.e. constant. Since the left-hand side and the right-hand one in (1) have periods $T_1$ and $T_2$ respectively, Theorem 1.3 implies that these periods are commensurable. Further the sum $f_1+f_2$ is defined on the set of positive measure. Since $T_1$ and $T_2$ are commensurable, the sum has certain period $T^*$ which is commensurable with $T_1$ and $T_2$. If $f_1 + f_2$ is not a.e. constant, then $T$ and $T^*$ are commensurable by Theorem 1.3, too.

b) Suppose that both sides in (1) equal a constant $c$ a.e., so that (2) holds for a.e. $x \in D$ ($T$ is the period of $f_1 + f_2$). Then $D$ is $T$-invariant and $T_1$-invariant. Let $\psi(x) = f_1(x)$ for $x \in D$ and $\psi(x) = 0$ for $x \in \mathbb{R} \setminus D$. We have $\mu(D \cap I) > 0$ for some segment $I \subset \mathbb{R}$. Let $\xi_k \downarrow 0, \xi_k < T_1$ be as in Lemma 2.1. If $T$ and $T_1$ are incommensurable one can choose sequences $m_k, n_k \in \mathbb{Z}$ with the property $\delta_k := n_k T + m_k T_1 \in (0, \xi_k)$. Then $n_k \neq 0$. Choose $x \in D$ which satisfies the following three conditions: (4) holds, (2) holds for $y = x+iT+jT_1$ instead of $x$ for arbitrary integers $i, j$, and $f_1(y + T_1) = f_1(y)$ for the same $y$. Then $x + \delta_k \in D$ and the equation (4) implies that

$$\lim_{k\to\infty} f_1(x + n_k T + m_k T_1) = f_1(x). \tag{5}$$

On the other hand, (2) implies that for all $k$

$$f_1(x + n_k T + m_k T_1) = f_1(x) + n_k c.$$

It follows that $c = 0$ and therefore $f_1(x + T) = f_1(x)$ for a.e. $x \in D$. Now Theorem 1.3 implies that $T$ and $T_1$ are commensurable, a contradiction. The same is true for $T_2$.

Now, let the conclusion of the theorem be true for all integers between 2 and $n$. We shall prove it for $n + 1$. Two cases are possible:

1) The sum $f_1 + \cdots + f_{n+1}$ is a.e. constant. Then $f_1(x) + \cdots + f_{n+1}(x) = c$ and $f_1(x) + \cdots + f_n(x) = c - f_{n+1}(x)$ a.e. So, by the inductive hypothesis the

periods of $f_1, \ldots, f_n$ and the period $T_{n+1}$ of their sum are pairwise commensurable (their sum is not a.e. constant by the hypothesis of the theorem).

2) This sum is not a.e. constant. Let $g_i(x) := f_i(x+T) - f_i(x)$. Then

$$g_1(x) + \cdots + g_{n+1}(x) = 0.$$

2.1) Let $g_i$'s be not a.e. constant for $i = 1, ..., n+1$.

c) If the sum $g_1 + \cdots + g_n (= -g_{n+1})$ has not proper subsums which are a.e. constant, the periods $T_1, \ldots, T_{n+1}$ are commensurable by inductive hypothesis. Then the sum $f_1 + \cdots + f_{n+1}$ has period of the form $mT_1$, and again by inductive hypothesis $T_1$, and $T$ are commensurable.

d) If the sum $g_1 + \cdots + g_n (= -g_{n+1})$ has proper subsums which are a.e. constant, let us choose a minimal one, say, $g_1 + \cdots + g_k = const$ a.e. $(k > 1)$. Then by inductive hypothesis, $T_1, \ldots, T_k$ are commensurable. The first summand of the sum $(f_1 + \cdots + f_k) + (f_{k+1} + \cdots + f_{n+1})$ has the period of the form $mT_1$, and by inductive hypothesis $T_1, T_{k+1}, \ldots, T_{n+1}$ and $T$ are commensurable.

2.2) If there exist a.e. constants among $g_i$'s for $i = 1, ..., n+1$, say $g_1 = c$ a.e., like in b) it follows that $c = 0$ and $T_1$ is commensurable with $T$ by Theorem 1.3. So $mT_1 = lT$. Since the sum

$$f_2 + \cdots + f_{n+1} = \sum_{i=1}^{n+1} f_i - f_1$$

is $lT$-periodic and not a.e. constant by inductive hypothesis, $T_2, \ldots, T_{n+1}$ , and $T$ are commensurable by inductive hypothesis, too. □

## 3 The Product of Two Periodic Functions with Possibly Different Domains.

In this section, we assume, as usual, that the product (and the sum) of several functions with possibly different domains is defined on the intersection of the domains. First consider the following

**Example 3.1.** Let $D_1 := \mathbb{Z} + \sqrt{2}Z + \sqrt{3}\mathbb{Z} + \sqrt{5}\mathbb{Z}$, $D_2 := \mathbb{Z} + \sqrt{2}\mathbb{Z} + \sqrt{3}\mathbb{Z} + \sqrt{7}\mathbb{Z}$. The function $g_1$ on $D_1$ defined by the equality

$$g_1(k + l\sqrt{2} + m\sqrt{3} + n\sqrt{5}) = (|k|+1)(|m|+1)$$

has periods $a\sqrt{2} + b\sqrt{5}$ $(a, b \in \mathbb{Z})$, and the function $g_2$ on $D_2$ defined by the equality

$$g_2(k + l\sqrt{2} + m\sqrt{3} + n\sqrt{7}) = (|l| + 1)/(|m| + 1)$$

has periods $a + b\sqrt{7}$ $(a, b \in \mathbb{Z})$. But the product $g_1 g_2$ is defined on the set $D_1 \cap D_2 = \mathbb{Z} + \sqrt{2}\mathbb{Z} + \sqrt{3}\mathbb{Z}$ and has period $\sqrt{3}$.

At the same time for $D_i$ with nonempty interior there is a positive result.

**Theorem 3.1.** *Let $g_i$ be continuous $T_i$-periodic functions, and the restrictions $g_i|intD(g_i) \neq const$ $(i = 1, 2)$. The product $g_1 g_2$ is periodic if and only if the periods $T_1$ and $T_2$ are commensurable.*

We need several lemmas to prove the theorem.

**Lemma 3.1.** *Let $f_i$ be $T_i$-periodic continuous function $(i = 1, .., n)$, $D \subseteq \cap_{i=1}^n D(f_i)$, $\sum_{i=1}^n f_i \neq const$. If the restriction $\sum_{i=1}^n f_i|D$ is $T$-periodic, then the numbers $T_1^{-1}, \dots, T_n^{-1}$, and $T^{-1}$ are linearly dependent over $\mathbb{Q}$.*

Proof. Assume on the contrary that numbers $T_1^{-1}, \dots, T_n^{-1}$, and $T^{-1}$ are linearly independent over $\mathbb{Q}$. Since $T/T_1, \dots, T/T_n$ and 1 are linearly independent over $\mathbb{Q}$, too, Kronecker Theorem (see e.g. [1], Chapter 7, section 1, Corollary 2 of Proposition 7) implies, that for $x$ in $D$ , for every $y$ in $\cap_{i=1}^n D(f_i)$ and $k$ in $\mathbb{N}$ there exist such numbers $q_k$ and $p_{ik}$ in $\mathbb{Z}$, that

$$|q_k T/T_i - p_{ik} - (y - x)/T_i| < 1/(k \max T_i) \quad (i = 1, ..., n)$$

and so

$$|q_k T - p_{ik} T_i - (y - x)| < 1/k \quad (i = 1, ..., n).$$

Therefore

$$\lim_{k \to \infty} (q_k T - p_{ik} T_i) = y - x \quad (i = 1, ..., n).$$

Because for $x$ in $D$

$$f_1(x + q_k T - p_{1k} T_1) + \cdots + f_n(x + q_k T - p_{nk} T_n) = f_1(x) + \cdots + f_n(x)$$

and $f_i$'s are continuous, it follows that

$$f_1(y) + \cdots + f_n(y) = f_1(x) + \cdots + f_n(x)$$

and so $\sum_{i=1}^n f_i = const$, a contradiction. $\square$

**Corollary 3.1.** Let $f$ be nonconstant continuous $T_1$-periodic function on $D(f)$. If its restriction to a subset $D$ of $D(f)$ is $T$-periodic, then $T$ and $T_1$ are commensurable.

**Lemma 3.2.** *If the set $D_1 \neq \mathbb{R}$ is $T_1$-invariant and its subset $D, intD \neq \emptyset$, is $T$-invariant, then $T$ and $T_1$ are commensurable.*

PROOF. Let us suppose the contrary. Then the set $G = T_1\mathbb{Z}+T\mathbb{Z}$ is dense in $\mathbb{R}$ by Dirichlet Theorem. Note that every shift by the element of $G$ maps $D$ into $D_1$. Choose $a \notin D_1$. Since the open set $a - intD$ intersects $G$, $a - d = t$,where $d \in D, t \in G$. Then $a = d + t$ belongs to $D_1$, a contradiction. □

The following lemma is of intrinsic interest.

**Lemma 3.3.** *Let $f_i$ be $T_i$-periodic nonconstant continuous functions with open domains $D_i \quad (i = 1, 2)$. The sum $f_1 + f_2$ is periodic if and only if the periods of $f_i$'s are commensurable.*

PROOF. In view of Theorem 2.1 and Lemma 3.2 it remains to consider the case $D_1 \neq \mathbb{R}, D_2 = \mathbb{R}$. Let $T$ be the period of the sum $f_1 + f_2$, and suppose that $T$ and $T_2$ are incommensurable. By Lemma 3.2, $mT = kT_1$ for some $m, k$ from $\mathbb{Z}$. Replacing $mT$ by $kT_1$ in the first summand of the left-hand side of the equality

$$f_1(x + mT) + f_2(x + mT) = f_1(x) + f_2(x), x \in D_1$$

we have

$$f_2(x + mT) = f_2(x), x \in D_1.$$

It follows from Corollary 3.1 that $T_2$ and $T$ are commensurable, a contradiction. □

PROOF OF THEOREM 3.1. First note that the restrictions $g_i|intD(g_i)$ are $T_i$-periodic, too. So we can assume that $D(g_i)$ are open. Then the sets

$$D_i := \{x \in D(g_i)|g_i(x) \neq 0\} \quad (i = 1, 2)$$

are open and $T_i$-invariant. Several cases are possible.

1) $D_1 \cap D_2 \neq \emptyset$. Since $g_1g_2$ is periodic, the function on $D_1 \cap D_2$

$$\log |g_1g_2| = \log |g_1| + \log |g_2|$$

is periodic, too.

1.1). Let both functions $|g_i|$ be nonconstant. Then their periods are commensurable by Lemma 3.3.

1.2). Let both functions $|g_i|$ be constants. Then $D_i \neq \mathbb{R}$ for $i = 1, 2$ and one can use Lemma 3.2.

1.3). Let $|g_1|$ is nonconstant, and $|g_2|$ is constant (and so $g_2(x) = \pm c \neq 0$). It was noted above that $D_2 \neq \mathbb{R}$. In view of Lemma 3.2 we may assume that $D_1 = \mathbb{R}$; i.e. $g_1(x)$ has a fixed sign. Let $T$ be the period of $g_1 g_2$, so that for all $x$ in $D(g_2)$ we have

$$g_1(x+T)g_2(x+T) = g_1(x)g_2(x). \tag{6}$$

Thus the numbers $g_2(x+T)$ and $g_2(x)$ have the same sign, too, and therefore coincides. Now $T_2$ and $T$ are commensurable by Theorem 1.2. Then the equality (6) implies $g_1(x+T) = g_1(x)$ for all $x$ in $D(g_2)$, and the numbers $T_1$ and $T$ are commensurable by Corollary 3.1.

2) $D_1 \cap D_2 = \emptyset$. Suppose that $T_1$ and $T_2$ are incommensurable. Then for $d_2 \in D_2$ one can find two integers $m, n$ such that $mT_1 + nT_2 \in D_1 - d_2$. Therefore $d_2 + nT_2 = d_1 + (-m)T_1$ for some $d_1 \in D_1$. This is impossible because the left-hand side of the last equality belongs to $D_2$, but the right-hand one belongs to $D_1$. This completes the proof. $\square$

**Corollary 3.2.** Let $g_i$ be continuous $T_i$-periodic functions, and the restrictions $g_i|intD(g_i) \neq const$ $(i = 1, 2)$. The quotient $g_1/g_2$ is periodic if and only if the periods $T_1$ and $T_2$ are commensurable.

**Remark 1.** Let $f_i$ be periodic functions defined on the open subsets $D_i \subseteq \mathbb{R}, D_1 \neq \mathbb{R}$ and $E_i$ the range of $f_i$ $(i = 1, ..., n)$. If the function $F(y_1, \dots, y_n)$ on $E_1 \times \ldots \times E_n$ "really depends" on each $y_i$, the composition $F(f_1(x), \dots, f_n(x))$ is periodic if and only if the periods of $f_i$'s are commensurable. It follows from Lemma 3.2 immediately. In general the problem on the periodicity of the composition seems to be open.

**Remark 2.** The problems of generalization of Theorem 3.1 for $n > 2$ multipliers, for discontinuous multipliers and for general $D(g_i)$ seem to be open, too.

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