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EXISTENCE OF MEASURES WITH DOMINATED TRANSLATES

Abstract

Let G be a topological group of second category, $\mathcal{B}_G$ be its Borel σ-algebra and $\mathcal{B}$ a σ-algebra of subsets of G such that $(G,\mathcal{B})$ is a measurable group. For a probability measure P on $(G,\mathcal{B})$, let $P_g(E) := P(gE)$ for $g \in G, E \in \mathcal{B}$. The aim of this note is to show that there exists an inner-regular probability measure P and a σ-finite measure μ on $(G,\mathcal{B})$ such that $P_g \ll \mu \ \forall\ g \in G$, iff G is locally-compact and in that case $P_g \ll \lambda_G \ll \mu \ \ \forall\ g \in G$ on the σ-algebra $\mathcal{B} \cap \mathcal{B}_G$, where λ_G denotes a Haar measure of G.

A measurable space $(X,\mathcal{B})$ is said to be *standard* if there exists a complete separable metric space Y and a bijective map $T : X \to Y$ such that both T and T^{-1} are measurable with natural Borel σ-algebra on Y. By a *measurable group* $(G,\mathcal{B})$ we mean that G is a group and $\mathcal{B}$ is a σ-algebra of subsets of G such that the map $(g,h) \mapsto gh^{-1}$ from $G\times G \to G$ is measurable when $G\times G$ is given the product σ-algebra $\mathcal{B}\times\mathcal{B}$. Let $(G,\mathcal{B})$ be a measurable group. A measure μ on $(G,\mathcal{B})$ is called *(left) quasi-invariant* if $\ \forall\ E \in \mathcal{B},\ \mu(E) = 0 \Leftrightarrow \mu_g(E) := \mu(gE) = 0\ \forall\ g \in G$. We say μ has *dominated translates* if there exists a σ-finite measure ν on $(G,\mathcal{B})$ such that $\mu_g \ll \nu\ \forall\ g \in G$. The well-known theorem due to G.W. Mackey (refer [2]) says that there exist a quasi-invariant measure on a standard measurable group $(G,\mathcal{B})$ iff G is a locally-compact group under some topology τ, and in that case the Borel σ-algebra given by τ is $\mathcal{B}$ with $\mu \equiv \lambda_G$, the (left) Haar measure on G. In [4], the following generalization of this result is proved :

Theorem 1. *Let G be a topological group of second category and having cardinality almost that of the continuum. If for some σ-algebra $\mathcal{B}$ of subsets of G, such that $(G,\mathcal{B})$ is a measurable group, there exists an inner-regular probability measure P on $(G,\mathcal{B})$ having dominated translates, then G is locally-compact.*

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Further, if $\mathcal{B}_G$ is the σ-algebra of Borel subsets of G and λ_G is a Haar measure of G, then $P_g \ll \lambda_G \ \forall\, g \in G$ on the σ-algebra $\mathcal{B} \cap \mathcal{B}_G$.

The proof of the above theorem assumed the continuum hypothesis. In the present note we give a simple argument that avoids the use of the continuum hypothesis and extends the above theorem to all groups of second category. The statement of the following lemma can be found in [1].

For this discussion, the key result is due to Halmos-Savage and is given in the next lemma. We include a proof which uses an argument similar to those used in [1].

Lemma 1. *Let $(X, \mathcal{B})$ be a measurable space and let $\mathcal{P} = \{P_\theta\}_{\theta \in \Theta}$ be any family of probability measures on $(X, \mathcal{B})$. If there exists a σ-finite measure μ on $(X, \mathcal{B})$ such that $P_\theta \ll \mu \ \forall\, \theta \in \Theta$, then there exists a countable sub-family $\mathcal{Q} = \{P_{\theta_n}\}_{n \geq 1}$ of $\mathcal{P}$ such that for $E \in \mathcal{B}, P_\theta(E) = 0 \Leftrightarrow P_{\theta_n}(E) = 0 \ \ \forall\, n \geq 1$ (we write this as $\mathcal{P} \equiv \mathcal{Q}$).*

Proof. Without loss of generality we may assume that μ is finite. For every $\theta \in \Theta$, let $S_\theta = \{x \in X | \frac{dP_\theta}{d\mu}(x) > 0\}$. Let $C \in \mathcal{B}$ be called a *chain* if C is a countable union of sets from $\mathcal{B}$, each of which is a subset of some S_θ. Let $\alpha = \sup\{\mu(C) | C \text{ a chain}\}$. Clearly α is finite and is in fact attained since countable union of chains is a chain. Let $\alpha = \mu(C)$, where $C = \bigcup_{n=1}^{\infty} B_n$ and $B_n \subseteq S_{\theta_n}$. Let $\mathcal{Q} = \{P_{\theta_n} | n = 1, 2, \ldots\}$. We show that $\mathcal{Q}$ is the required subfamily. Let $P_{\theta_n}(A) = 0 \,\forall\, n \geq 1$. Then $\mu(A \cap S_{\theta_n}) = 0$ and hence

$$\mu(A \cap C) = 0 \text{ as } C \subseteq \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} S_n.$$

This implies that $P_\theta(A \cap C) = 0$ for all $\theta \in \Theta$. Also if $P_\theta(A - C) > 0$ for some θ, then $P_\theta((A - C) \cap S_\theta) > 0$. Hence $C' = C \cup ((A - C) \cap S_\theta)$ is a chain with $\mu(C') > \mu(C)$, contradicting the maximality of $\mu(C)$. Hence $P_\theta(A - C) = 0 \,\forall\, \theta \in \Theta$. This proves that $\mathcal{Q} \equiv \mathcal{P}$. □

The following lemma can be found in [3].

Lemma 2. *Let G be a topological group of second category. Then there exists a locally-finite inner-regular quasi-invariant measure on a σ-field $\mathcal{B}$ of subsets of G such that $(G, \mathcal{B})$ is a measurable group, if and only if G is locally-compact.*

Using the results listed above, we prove the following :

Theorem 2. *Let G be a topological group of second category and $\mathcal{B}$ be a σ-field $\mathcal{B}$ of subsets of G such that $(G, \mathcal{B})$ is a measurable group. Then there exists* *an inner-regular probability measure P on $(G,\mathcal{B})$ with dominated translates if and only if G is locally-compact. In that case $\forall\ g \in G$, $P_g \ll \lambda_G$ on the σ-algebra $\mathcal{B}_G \cap \mathcal{B}$, where $\mathcal{B}_G$ is the σ-algebra of Borel subsets of G and λ_G is a Haar-measure on G.*

Proof. Obviously, if G is locally-compact and λ_G denotes a Haar measure, then with $\mathcal{B} = B_G$, the σ-algebra of Borel subsets of G, and $P = \mu$, some probability measure equivalent to λ_G, the required properties hold.

Conversely, let $\mathcal{B}$ be a σ-algebra of subsets of G such that $(G,\mathcal{B})$ is a measurable group. Further, let there exist a probability P having dominated translates, i.e., there exists a σ-finite measure μ on $(G,\mathcal{B})$ such that $P_g \ll \mu\ \ \forall\ g \in G$. We first show that there exists a quasi-invariant probability measure on $(G,\mathcal{B})$. By Lemma 1, there exists a countable sub-family $\mathcal{Q} = \{P_{g_i} \mid i = 1,2,\dots\}$ of $\mathcal{P}$ such that $\mathcal{P} \equiv \mathcal{Q}$. Define $\nu := \sum_{i=1}^{\infty} \frac{P_{g_i}}{2^i}$. It is to check that ν is an inner-regular probability measure on $(G,\mathcal{B})$. Further ν is quasi-invariant. Now by Lemma 2, G is locally-compact. Let $\mathcal{B}_G$ denote the Borel σ-algebra of G and λ_G denote a Haar measure on G. Then λ_G and ν are two quasi-invariant measures on $(\mathcal{B}_G \cap \mathcal{B})$. Hence $\lambda_G \equiv \nu$. Clearly $\mu \gg \nu$, and hence $\mu \gg \lambda_G \gg P_g\ \ \forall\, g \in G$ on $\mathcal{S} \cap \mathcal{B}$. $\square$

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