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A REPRESENTATION OF MARTINGALE DIFFERENCES AND ORTHONORMAL SYSTEMS OF UNCONDITIONAL CONVERGENCE ALMOST EVERYWHERE

Abstract

We establish a special representation of uniformly bounded martingale differences and give some examples of its application. In particular, we prove that any uniformly bounded ONS $\{f_k\}$ contains a subsystem $\{f_{k_n}\}$ $(k_n \leq 2^n)$ of unconditional convergence with exponential estimation of majorants.

1 Introduction

Let $\{\Omega, F, P\}$ be a probability space and $\{F_k\}$ be a sequence of increasing sub-σ-algebras of F. In what follows, φ_k, $k = 1, 2, \ldots$,are martingale differences; that is, φ_k is F_k-measurable and $E\{\varphi_n | F_k\} = 0$ almost everywhere (a.e.) if $n > k$. We may assume that $E\varphi_1 = 0$.

We say that random variables η_k, $1 \leq k \leq n$ are independent on a set $Q \subset \Omega$ $(P(Q > 0))$ if for arbitrary sets $U_k \subset \mathbb{R}$, $1 \leq k \leq n$

$$P\left(Q \cap \bigcap_{k=1}^{n} \{\eta_k \in U_k\}\right) = (P(Q))^{1-n} \prod_{k=1}^{n} P(\{\eta_k \in U_k\} \cap Q).$$

In other words, the restrictions of the random variables η_k, $1 \leq k \leq n$ to the set Q are independent on the probability space $\{Q, F^{'}, P^{'}\}$ where $F^{'}$ is the restriction of F to the set Q and $P^{'}(*) = \dfrac{P(*)}{P(Q)}$. In Section 2, we prove the following.

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Theorem 1. *Let $\varphi_k,\ k = 1, 2, \ldots$ be martingale differences on a nonatomic probability space $\{\Omega, F, P\}$, $|\varphi_k| \le M_k$, and each φ_k assume a finite number of values. Then for an arbitrary natural number K there exist independent random variables $\xi_k,\ k = 1, 2, \ldots, K$ with $|\xi_k| = M_k$ and $E\xi_k = 0$, a partition of Ω into N disjoint sets $Q_n,\ n = 1, 2, \ldots, N$, and random variables $\eta_{k,n},\ k = 1, 2, \ldots, K$ independent on Q_n and equal to zero outside Q_n such that*

$$\varphi_k = \xi_k + \sum_{n=1}^{N} \eta_{k,n}, \quad k = 1, 2, \ldots, K. \tag{1.1}$$

This representation lets one extend some results on convergence of uniformly bounded independent random variables to uniformly bounded martingale differences. In this paper, we limit ourselves to the problem of unconditional convergence. Let $\psi = \{\psi_n(x)\}_{n=1}^{\infty}$ be an orthonormal system of functions (ONS) on a space X with a finite measure, say, $\mu X = 1$ and $a = \{a_n\}_{n=1}^{\infty}$ be a number sequence, $\|a\|^2 = \sum\limits_{n=1}^{\infty} a_n^2 < \infty$.

The series

$$\sum_{n=1}^{\infty} a_n \psi_n(x) \tag{1.2}$$

is called *unconditionally convergent* a.e. if for every rearrangement of the terms the resulting series converges a.e. The exceptional set of measure zero where divergence is allowed may depend on the order of the terms.

If the series (1.2) converges unconditionally a.e. for every $a, \|a\| < \infty$, the system ψ is said to be a *system of unconditional convergence (s.u.c.).*

For a permutation σ of the set of natural numbers, let $s^*(\sigma)$ denote the majorant of the series $\sum\limits_{n=1}^{\infty} a_n \psi_{\sigma(n)}$; that is,

$$s^*(\sigma) \equiv s^*(\psi, \sigma, a, x) \sup\nolimits_{1 \le N < \infty} \left| \sum_{n=1}^{N} a_n \psi_{\sigma(n)}(x) \right|$$

where $\sigma(n)$ is the number corresponding n.

As the first application of the representation (1.1), we prove the following.

Theorem 2. *Let $\varphi_k,\ k = 1, 2, \ldots,$ be martingale differences with $|\varphi_k| \le M_k$ and $M^2 = \sum\limits_{k=1}^{\infty} M_k^2 < \infty$. Then there exist absolute constants $C,\ \gamma \equiv \gamma(M) > 0$ such that for any permutation σ and for any number $\lambda > 0$ the majorant $s^*(\sigma)$*

of the series $\sum_{k=1}^{\infty} \varphi_{\sigma(k)}$ *satisfies the inequality*

$$P\left(s^*(\sigma) > \lambda\right) \leq Ce^{-\gamma\lambda^2}. \tag{1.3}$$

The next result is a simple consequence of the exponential estimate in (1.3).

Corollary. . *Let* $\{\varphi_k\}$ *be martingale differences,* $|\varphi_k| \leq M$, $1 \leq k < \infty$. *Then* $\{\varphi_k\}_{k=1}^{\infty}$ *is a system of unconditional convergence a.e., and for any permutation* σ *and for every* a, $\|a\| \leq 1$ *the majorant* $s^*(\sigma)$ *of the series* $\sum_{k=1}^{\infty} a_k \varphi_{\sigma(k)}$ *satisfies the inequality*

$$E\left(e^{\rho(s^*(\sigma))^2}\right) < \infty \text{ for some } \rho > 0. \tag{1.4}$$

In section 3, we apply the representation (1.1) to establish one property of uniformly bounded ONS. In the theory of orthogonal series the following questions are known. (We put them here in general form.)

1 Given an ONS $f = \{f_k\}$ belonging to a certain class, is there a subsequence $\psi = \{f_{k_n}\}$ of the ONS, which satisfies a certain property (A) ? (For example, is ψ a system of convergence?)

2 Does there exist a number sequence $\{l_n\}$ such that for any ONS in the class one can find a subsequence $\psi = \{f_{k_n}\}$ with that property (A) and with indices k_n growing not faster than l_n ?

The first question goes back to the thirties, to classical works of Menshov and Marcinkiewicz. The second appeared in the eighties. The history of the problems and some results can be found in [1]-[4] .

As is known every ONS contains a subsequence ψ that is *s.u.c.* (Komlos, [5]). The best estimation of $s^*(\psi, \sigma)$ one can expect in general case is $s^*(\psi, \sigma) \in L^2$ [2] , but, to our knowledge, it has not proved yet. If $f = \{f_k\}$ is ONS and $f_k \in L^p$ for all k and some $p > 2$, then $s^*(\psi, \sigma) \in L^p$ (see [6]). This result is a consequence of two theorems. According to the first, Banach's theorem (see [3]), every ONS f mentioned above contains an S_p-system. (ONS $\psi = \{\psi_n\}$ is an S_p- system if any polynomial $T = \sum_{n=1}^{N} a_n \psi_n$ satisfies the condition $\|T\|_p \leq C\|a\|$ where $\|*\|_p$ is the L^p- norm and C is an absolute constant.) According to the second theorem, proven by Erdős for the trigonometric system and by Stechkin (by analogous method) for arbitrary systems, all majorants $s^*(\sigma)$ of an S_p- system ψ belong to L^p ([6], p. 322). Note also the following result.

Every uniformly bounded ONS $\{f_k(x)\}_{k=1}^{\infty}$ contains a subsequence $\{f_{k_n}(x)\}_{n=1}^{\infty}$ such that

$$e^{\gamma s^2(x)} \in L \text{ for } \forall\gamma > 0 \tag{1.5}$$

where $s(x) = \sum_{n=1}^{\infty} a_n f_{k_n}(x)$.

This theorem was announced by Gaposhkin in [3]. The proof was communicated by Gaposhkin to Astashkin and published in [7] with his permission.

All the results mentioned above answer only the first question. As for the second question, it is shown in [8] (for even numbers p) that an S_p-system $\{f_{k_n}\}$ can be chosen so that $k_n \le n^{\beta}$, $\beta = \beta(p)$.

In Section 3, we consider the class of uniformly bounded ONS and property (A) of unconditional convergence with exponential estimation of majorants and give an affirmative answer to the second question.

Theorem 3. *Let $\{f_k(x)\}_{k=1}^{\infty}$ be an orthogonal system of uniformly bounded functions, $|f_k| \le M$, on a space X with a finite measure, say, $\mu X = 1$. Then there exist a subsequence $\psi = \{\psi_n(x)\}_{n=1}^{\infty} \equiv \{f_{k_n}(x)\}_{n=1}^{\infty}$ with indices $k_n \le 2^n$ and positive constants C and γ (depending only on M) such that for any permutation σ, any number $\lambda > 0$ and any a, $\|a\| < \infty$ the majorant $s^*(\sigma)$ of the series $\sum_{n=1}^{\infty} a_n \psi_{\sigma(n)}(x)$ satisfies the inequality $\mu(\{s^*(\sigma) > \lambda\}) \le Ce^{-\frac{\gamma\lambda^2}{\|a\|^2}}$.*

Remark. V. F. Gaposhkin communicated to the author that the "weak" version of Theorem 3, without condition $k_n \le 2^n$, can be also obtained by combining the following results. The first, which he established and used for the proof of (1.5), states that every uniformly bounded ONS contains a subsequence for which $\|s\|_p \le C\sqrt{p}\,\|a\|$ for all $p > 2$ and a. The second, an Erdős -type result, states that the inequality $\|s\|_p \le C_p\|a\|$ implies $\|s^*\|_p \le AC_p\|a\|$ where A does not depends on p.

2 Representation of Martingale Differences

Proof of Theorem 1. For given natural K, it follows from the assumptions of Theorem 1 that there exists a partition of Ω into sets Q_n , $1 \le n \le N$, such that each φ_k $(1 \le k \le K)$ is a constant on each of them. Let $c_{k,n}$ denote the value of φ_k on Q_n. Now, following [6] (chapter 8, the proof of Theorem 3), we define random variables $\eta_{k,n}$ $(1 \le k \le K)$ independent on Q_n, which satisfy the conditions:

1 $\int\limits_{Q_n} \eta_{k,n} P(d\omega) = 0$;

2 $\eta_{k,n}$ assumes on Q_n only two values: $-M_k + c_{k,n}$ and $M_k + c_{k,n}$;

3 $\eta_{k,n}(\omega) = 0$ for $\omega \notin Q_n$.

We set

$$\xi_k = \varphi_k - \sum_{n=1}^{N} \eta_{k,n} \quad (k = 1, 2, \dots, K). \tag{2.1}$$

Conditions 2 and 3 yield $|\xi_k| = M_k$. Conditions 1 and 3 lead to $E\xi_k = 0$. Let us introduce the sets

$$Q_{k,n}^{-} = \{\omega \in Q_n \mid \eta_{k,n}(\omega) = -M_k + c_{k,n}\}$$

and

$$Q_{k,n}^{+} = \{\omega \in Q_n \mid \eta_{k,n}(\omega) = M_k + c_{k,n}\}.$$

Since $\eta_{k,n}$ $(1 \le k \le K)$ are independent random variables on Q_n, we have for any $1 \le i \le K$

$$P\left(Q_{1,n}^{-} \cap Q_{2,n}^{-} \cap \dots \cap Q_{i,n}^{-}\right) = \left(P\left(Q_n\right)\right)^{1-i} \prod_{k=1}^{i} P\left(Q_{k,n}^{-}\right) \tag{2.2}$$

The analogous identity holds for the sets $Q_{k,n}^{+}$.

Next, choose one of the values of φ_k and denote it by c_k. Let $T_k \equiv T_k(c_k)$ denote the union of all those Q_n for which $c_{k,n} = c_k$. By definition of the random variables $\eta_{k,n}$, it follows that for any $Q_n \subset T_k$ and $Q_m \subset T_k$

$$\frac{P\left(Q_{k,n}^{-}\right)}{P\left(Q_n\right)} = \frac{P\left(Q_{k,m}^{-}\right)}{P\left(Q_m\right)} \equiv \alpha_k^{-}, \quad \frac{P\left(Q_{k,n}^{+}\right)}{P\left(Q_n\right)} = \frac{P\left(Q_{k,m}^{+}\right)}{P\left(Q_m\right)} \equiv \alpha_k^{+} \tag{2.3}$$

The ratios depend only on c_k.

Let us first prove that ξ_k, $k = 1, 2, \dots, K$ are martingale differences; that is, for any integer j $(2 \le j \le K)$ and any set S of the kind of $\{\xi_1 = \varepsilon_1 M_1,\ \xi_2 = \varepsilon_2 M_2, \dots, \xi_{j-1} = \varepsilon_{j-1} M_{j-1}\}$ where $\varepsilon_k = \pm 1$,

$$\int\limits_{S} \xi_j P(d\omega) = 0. \tag{2.4}$$

We limit ourselves to the case $S = \{\xi_1 = M_1,\ \xi_2 = M_2, \dots, \xi_{j-1} = M_{j-1}\}$. The remaining cases can be treated in the same manner. We set

$$T \equiv T(c_1, c_2, \dots, c_{j-1}) = T_1 \cap T_2 \cap \dots \cap T_{j-1}$$

Since S can be expressed as the union of the pairwise disjoint sets $S\cap T$; namely, $S = \bigcup_{\{c_1,\dots,c_k,\dots,c_{j-1}\}} S\cap T$, it suffices to prove that $\int_{S\cap T} \xi_j P(d\omega) = 0$. Let $D_n = S\cap T\cap Q_n$. This set is either empty or equal to $Q^-_{1,n}\cap Q^-_{2,n}\cap\cdots\cap Q^-_{j-1,n}$. Using Condition 1 and the fact that $\eta_{j,n}$ is independent of $\eta_{i,n}$, $i = 1, 2, \dots, j-1$, we have $\int_{D_n} \eta_{j,n} P(d\omega) = 0$. Denoting the value of φ_j on Q_n by φ_j^n and keeping Condition 3, (2.2), and (2.3) in mind, we obtain for nonempty D_n

$$\begin{aligned}
\int_{D_n} \xi_j P(d\omega) &= \int_{D_n} \varphi_j P(d\omega) = \varphi_j^n \int_{D_n} P(d\omega) \\
&= \varphi_j^n P\left(Q^-_{1,n}\cap Q^-_{2,n}\cap\cdots\cap Q^-_{j-1,n}\right) \\
&= \varphi_j^n \left(P(Q_n)\right)^{2-j} \prod_{k=1}^{j-1} P\left(Q^-_{k,n}\right) \\
&= \varphi_j^n \alpha_1^- \alpha_2^- \dots \alpha_{j-1}^- \left(P(Q_n)\right)^{2-j} \prod_{k=1}^{j-1} P(Q_n) \\
&= \varphi_j^n \alpha_1^- \alpha_2^- \dots \alpha_{j-1}^- \int_{Q_n} P(d\omega) = \alpha_1^- \alpha_2^- \dots \alpha_{j-1}^- \int_{Q_n} \varphi_j P(d\omega).
\end{aligned}$$

Therefore, since T belongs to the algebra F_{j-1} and the random variables $\{\varphi_k\}$ are martingale differences,

$$\begin{aligned}
\int_{S\cap T} \varphi_j P(d\omega) &= \sum_{n:Q_n\subset T} \int_{D_n} \varphi_j P(d\omega) = \sum_{n:Q_n\subset T} \alpha_1^- \alpha_2^- \dots \alpha_{j-1}^- \int_{Q_n} \varphi_j P(d\omega) \\
&= \alpha_1^- \alpha_2^- \dots \alpha_{j-1}^- \int_T \varphi_j P(d\omega) = 0.
\end{aligned}$$

Thus (2.4) is true, and ξ_k $(1 \le k \le K)$ are martingale differences. From this, under the conditions $E\xi_k = 0$ and $|\xi_k| = M_k$, it easily follows by induction that they are independent random variables. Indeed, assume that ξ_k, $1 \le k \le s$ (where $s < K$) are independent; that is, for any set D of the kind of $\{\xi_{k_1} = \varepsilon_1 M_{k_1}, \xi_{k_2} = \varepsilon_2 M_{k_2}, \dots, \xi_{k_r} = \varepsilon_r M_{k_r}\}$ $(1 \le r \le s)$, its probability $P(D) = 2^{-r}$. Taking $\int_D \xi_{s+1} P(d\omega) = 0$ into account, we get

$$P\left(D \cap \{\xi_{s+1} = \varepsilon_{s+1} M_{s+1}\}\right) = \frac{1}{2} P(D) = 2^{-r-1}.$$

This means that ξ_k, $1 \le k \le s+1$ are independent. $\square$

Remark. V. F. Gaposkin communicated to the author a sketch of another proof of the representation. The functions $\{\xi_k\}_1^K$ are a weakly multiplicative system; i.e., $E(\xi_1 \dots \xi_s) = 0$. Any weakly multiplicative system $\{\xi_k\}$ where $\xi_k = \pm M_k$ with probability of $\frac{1}{2}$ is a system of independent random variables. Indeed from the estimates it follows that

$$E \equiv E\left(\xi_1^{l_1} \dots \xi_s^{l_s}\right) = E\xi_1^{l_1} \dots E\xi_s^{l_s} = M_1^{l_1} \dots M_s^{l_s} \text{ if all } l_i \text{ are even;}$$
$$E = 0 \text{ if at least one } l_i \text{ is odd.}$$

These estimates in turn imply $E \exp\left(i\lambda \sum\limits_{k=1}^{m} \xi_k\right) = \prod\limits_{k=1}^{m} E \exp(i\lambda\xi_k)$ for all m, λ. Therefore, $\xi_k, 1 \le k \le K$ are independent.

PROOF OF THEOREM 2. We write $s^*(\varphi, \sigma)$ instead of $s^*(\sigma)$ in order to show that the majorant is related to the series with respect to a system $\varphi = \{\varphi_k\}$. For any natural K, let $s_K^*(\varphi, \sigma) = \sup_{1 \le i \le K} \left| \sum\limits_{k=1}^{i} \varphi_{\sigma(k)} \right|$. Let $\lambda > 0$ be an arbitrary number. One can find a natural number K such that the following inequality holds

$$P(s^*(\varphi, \sigma) > \lambda) \le 2P(s_K^*(\varphi, \sigma) > \lambda) \tag{2.5}$$

Suppose first Ω to be nonatomic and φ_k $(1 \le k \le K)$ to assume a finite number of values. Applying formula (1.1) and letting $\xi = \{\xi_k\}_{k=1}^K$, $\eta_n = \{\eta_{k,n}\}_{k=1}^K$, we obtain

$$s_K^*(\varphi, \sigma) \le s_K^*(\xi, \sigma) + \sum_{n=1}^{N} s_K^*(\eta_n, \sigma).$$

Therefore, since the supports of the systems η_n $(1 \le n \le N)$ are disjoint,

$$P(s_K^*(\varphi, \sigma) > \lambda) \le P\Big(s_K^*(\xi, \sigma) > \frac{\lambda}{2}\Big) + \sum_{n=1}^{N} P\Big(s_K^*(\eta_n, \sigma) > \frac{\lambda}{2}\Big) \tag{2.6}$$

It is known that the exponential estimation of the majorants for the sum of uniformly bounded independent random variables holds. (See, for example, [6], chapter 2, Theorem 5 and inequality (56).) Namely, there are absolute constants C and $\gamma = \gamma(M) > 0$ (not depending on ξ, K, and σ) such that

$$P\Big(s_K^*(\xi, \sigma) > \frac{\lambda}{2}\Big) \le Ce^{-\gamma\lambda^2}. \tag{2.7}$$

(From now on absolute constants C and γ can be different in different inequalities.) Since $|\eta_{k,n}| \leq 2M_k$, $k = 1, 2, \ldots, K$ the following inequality holds for independent on Q_n random variables.

$$P\left(s_K^*\left(\eta_n, \sigma\right) > \frac{\lambda}{2}\right) \leq Ce^{-\gamma\lambda^2} P\left(Q_n\right) \quad (n = 1, 2, \ldots, N) \tag{2.8}$$

(C and γ do not depend on η_n, K, σ) Therefore, we obtain from (2.6)-(2.8)

$$P\left(s_K^*\left(\varphi, \sigma\right) > \lambda\right) \leq Ce^{-\gamma\lambda^2} \tag{2.9}$$

The assertion of the theorem follows from (2.5). □

Now we get rid of the assumptions. The martingale differences φ_k , $k = 1, 2, \ldots, K$ can be approximated by martingale differences $\bar{\varphi}_k$ assuming only a finite number of values so that $\sum\limits_{k=1}^{K} \left|\varphi_k - \bar{\varphi}_k\right| < 1$ a.e. This inequality implies (2.9) if the corresponding inequality holds for $\left\{\bar{\varphi}_k\right\}$. If Ω is not nonatomic, we denote the sets of constancy of $\left\{\bar{\varphi}_k\right\}$ by Q_n, $n = 1, \ldots, N$ and consider a probability space $\{\Omega', F', P'\}$ with continuous measure P', martingale differences $\left\{\bar{\varphi}_k^{'}\right\}$, and sets Q_n', $n = 1, \ldots, N$ such that $P'(Q_n') = P(Q_n)$; $\bar{\varphi}_k^{'}\left(\omega'\right) = \bar{\varphi}_k\left(\omega\right)$ for $\forall\omega' \in Q_n'$, $\forall\omega \in Q_n$ $(1 \leq k \leq K,\ 1 \leq n \leq N)$.

Proof of the Corollary. The exponential estimation (1.3) implies that $s^*\left(\varphi, \sigma\right) < \infty$ a.e. Therefore, the series $\sum\limits_{k=1}^{\infty} \varphi_{\sigma(k)}$ converges a.e. Inequality (1.3) also implies (1.4). This proves the Corollary. □

3 Subsystems of Unconditional Convergence

It suffices to prove Theorem 3 for the number sequences a with $\|a\| \leq 1$. Let us first reduce Theorem 3 to the case of finite orthogonal systems $\{f_k\}_{k=1}^K$ (Proposition 3, below). Let σ_r denote a permutation of the first r natural numbers.

Proposition 1. *Let there exist a sequence $k_n \leq 2^n$, $n = 1, 2, \ldots$ and absolute constants C, $\gamma \equiv \gamma(\dot{M}) > 0$ such that for any integer $K > 0$ the subsequence $\{f_{k_n}\}_{n=1}^L \equiv \{\psi_n\}_{n=1}^L = \psi^L$ $(L \equiv L\left(K\right) = \max\{n|\, k_n \leq K\})$, extracted from the system $\{f_k\}_{k=1}^K$, satisfies*

(A)

$$\textit{for any } \sigma_L \textit{ and } a' = \{a'_n\}_{n=1}^L \textit{ with } \|a'\| \le 1,$$
$$\mu(s_L^*(\psi^L, \sigma_L, a') > \lambda) \le Ce^{-\gamma\lambda^2} \tag{3.1}$$

Then the assertion of Theorem 3 is true (for $\psi = \{\psi_n\} = \{f_{k_n}\}_{n=1}^{\infty}$*).*

Proof. Indeed, for given σ and a, $\|a\| \le 1$, one can choose a number N such that
$$\mu\{s_N^*(\psi, \sigma, a) > \lambda\} \ge \frac{1}{2}\mu\{s^*(\psi, \sigma, a) > \lambda\} \tag{3.2}$$

Let K be a number such that $L > \max_{1\le n\le N}\sigma(n)$. Define σ_L. Let $\sigma_L(n) = \sigma(n)$ for $1 \le n \le N$. For $n = N+1, \ldots, L$ we subsequently set $\sigma_L(n)$ equal to a value in the set $\{1, 2, \ldots, L\}$, which is different from the values determined before. We also set $a_n^{'} = a_n$ $(n = 1, 2, \ldots, N)$, $a_n^{'} = 0$ $(N < n \le L)$ and denote $\{a_n^{'}\}$ by $a^{'}$. Inequality (3.1) holds for $s_L^*\left(\psi^L, \sigma_L, a'\right)$. Since $s_L^*\left(\psi^L, \sigma_L, a'\right) = s_N^*(\psi, \sigma, a)$, inequality (3.2) completes the proof of Theorem 3. □

Proposition 2. *Suppose that for every* K *there exists a subsequence* $\psi^L = \{f_{k_n}\}_{n=1}^L (k_n \equiv k_n(K) \le 2^n,\ n = 1, 2, \ldots, L = L(K),\ k_L \le K,\ 2^{L+1} > K)$ *of the system* $\{f_k\}_{k=1}^K$ *such that condition (A) holds. Then the assertion of Theorem 3 is true.*

Proof. We will show that a sequence $k_n \le 2^n,\ n = 1, 2, \ldots$ can be chosen independent of K, and, therefore, the assumptions of Proposition 1 will be satisfied. We use a diagonal process. Choose one of the numbers (1 or 2) that occurs in the sequence $k_1(K),\ K = 1, 2, \ldots$ infinitely many times and denote it by k_1. Let $T_1 = \{K \mid k_1(K) = k_1\}$. Suppose k_s and T_s have already defined. Then we set k_{s+1} equal to a number that occurs in the subsequence $\{k_{s+1}(K)\}_{K\in T_s}$ infinitely many times and put $T_{s+1} = \{K \in T_s \mid k_{s+1}(K) = k_{s+1}\}$. The sequence k_n, $n = 1, 2, \ldots$ is the desired one because if $K' < K$ and the subsequence $\{f_{k_n}\}_1^{L(K)}$ satisfies the assumption of Proposition 2 , then the subsequence $\{f_{k_n}\}_1^{L(K')}$ also satisfies the assumption. □

Thus it is enough to prove the following.

Proposition 3. *Let* $\{f_k\}_{k=1}^K$ *be an orthogonal system of functions,* $|f_k| \le M$*. Then there exists a subsequence* $\psi = \{f_{k_n}\}_{n=1}^L \equiv \{\psi_n\}_{n=1}^L$ $(\ k_n \le 2^n,\ n = 1, 2, \ldots, L,\ k_L \le K,\ 2^{L+1} > K)$ *such that for any* $a = \{a_n\}_{n=1}^L$, $\|a\| \le 1$, *and an arbitrary permutation* $\sigma = \sigma_L$ $\mu\{s_L^*(\psi, \sigma, a) > \lambda\} \le Ce^{-\gamma\lambda^2}$.

Proof. We may suppose that the functions f_k $(1 \le k \le K)$ assume a finite number of values. Indeed, let $\bar{f}_k$ be a function of this kind sufficiently close to f_k. The Schmidt orthogonalization of $\left\{\bar{f}_k\right\}$ yields an orthogonal system $\{f_k^*\}$

with $\sum_{k=1}^{K} |f_k(x) - f_k^*(x)| < 1$ a.e. This implies the assertion of Proposition 3 if it is true for $\{f_k^*\}$.

Also, it suffices to prove Proposition 3 for $k_n \le 8^n,\ n = 1, 2, \ldots, L$ ($8^{L+1} > K$). Indeed, if Proposition 3 holds for such $\{k_n\}$, we can subsequently extract a subsequence ψ^1 from $\{f_k\}_{k=1}^{K}$, a subsequence ψ^2 from $\{f_k\}_{k=1}^{K} \setminus \psi^1$, and ψ^3 from $\{f_k\}_{k=1}^{K} \setminus (\psi^1 \cup \psi^2)$ so that each of them satisfies the assertion of the proposition with $k_n \le 8^n$. Uniting the functions from ψ^1, ψ^2, and ψ^3 in one sequence and adding (if needed) $f_1, f_2, \ldots, f_6$ to it , we obtain a subsequence ψ that satisfies Proposition 3 with $k_n \le 2^n$. We may assume $M = 1$as well.

We first consider the case $|f_k| = 1\ (1 \le k \le K)$ and define the required subsequence by induction. Suppose that functions $\psi_n \equiv f_{k_n},\ k_n \le 8^n\ (1 \le n \le s,\ 8^{s+1} < K)$ have already been chosen and the following conditions hold.

1 There exist martingale differences $\varphi_n,\ 1 \le n \le s$ such that
$$\mu\left(|\psi_n - \varphi_n| > 2^{-\frac{n}{8}}\right) < 2^{-\frac{n}{4}},\ n = 1, 2, \ldots, s;$$

2 $|\varphi_n| \le 2$;

3 For $n = 1, 2, \ldots, s$, let $Q_1^n, \ldots, Q_r^n\ (r \equiv r(n))$ be the sets of the maximal measure, on which every function $\psi_\nu\ (1 \le \nu \le n)$ is constant. Then the function φ_n is constant on each $Q_\nu^n,\ 1 \le \nu \le r$.

Since every function $\psi_n\ (1 \le n \le s)$ assumes only 2 values, $r = r(s) \le 2^s$. We now choose ψ_{s+1}. Denote the characteristic function of the set $Q_\nu \equiv Q_\nu^s$ by χ_ν and apply Bessel's inequality to it. We have
$$\mu(Q_\nu) \ge \sum_{k=1}^{K} (\chi_\nu, f_k)^2 \ge \sum_{k=8^s+1}^{8^{s+1}} (\chi_\nu, f_k)^2,\ \nu = 1, 2, \ldots, r,$$
whence it follows that at least for one number $k,\ 8^s + 1 \le k \le 8^{s+1}$,
$$(\chi_\nu, f_k)^2 \le \frac{\mu Q_\nu}{4^s} \text{ for all } \nu \le r. \tag{3.3}$$
We denote this number by k_{s+1} and set $\psi_{s+1} = f_{k_{s+1}}$. Let $\bar{\psi}_\nu$ be the average value of the function ψ_{s+1} on the set Q_ν. Let Q' be the union of those Q_ν for which
$$\mu Q_\nu \quad \ge 2^{-\frac{3}{2}s}. \tag{3.4}$$
We define φ_{s+1} by
$$\varphi_{s+1}(x) = \begin{cases} \psi_{s+1}(x) - \quad \bar{\psi}_\nu & \text{for } x \in Q_\nu \subset Q' \\ 0 & \text{for } x \in Q_\nu \not\subseteq Q'. \end{cases} \tag{3.5}$$

It follows from (3.5) that Assumptions 2 and 3 hold for $n = s + 1$ and the average value of $\varphi_{s+1}(x)$ on each Q_ν equals zero. In view of Condition 3, φ_k $(1 \le k \le s+1)$ are martingale differences. Taking (3.3) and (3.4) into account, we have for each set $Q_\nu \subset Q'$

$$\left|\bar{\psi}_\nu\right| = \frac{|(\chi_\nu, \psi_{s+1})|}{\mu Q_\nu} \le \frac{1}{2^s \sqrt{\mu Q_\nu}} \le 2^{-\frac{s}{4}} \le 2^{-\frac{s+1}{8}} \tag{3.6}$$

Therefore, the inequality $|\psi_{s+1}(x) - \varphi_{s+1}(x)| > 2^{-\frac{s+1}{8}}$ can be true only for $x \in X \setminus Q'$. For the set $Q_\nu \not\subseteq Q'$, the inequality opposite to (3.4) holds. Hence

$$\mu(X \setminus Q') < 2^{-\frac{3}{2}s} r \le 2^{-\frac{3}{2}s} 2^s = 2^{-\frac{s}{2}} \le 2^{-\frac{s+1}{4}}$$

Thus, by induction, we can define functions $\psi_n(x)$, $n = 1, 2, \ldots, L$ $(k_n \le 8^n$, $8^{L+1} > K)$ for which Conditions 1-3 holds.

Fix an arbitrary $\lambda > 0$, $\sigma = \sigma_L$ and a sequence $a = \{a_n\}_{n=1}^L$, $\|a\| \le 1$. Denoting the permutation inverse to σ by τ, we have for $l = \min\left(\left[\frac{\lambda}{2}\right]^2, L\right)$

$$\begin{aligned} \sup_{1\le i\le L} \left|\sum_{n=1}^{i} a_n \psi_{\sigma(n)}(x)\right| &\le \sum_{n=1}^{l} \left|a_{\tau(n)}\right| |\psi_n(x)| \\ &\quad + \sup_{1\le i\le L} \left|\sum_{n=1}^{i}{}' a_n \psi_{\sigma(n)}(x)\right| = I_1 + I_2 \end{aligned} \tag{3.7}$$

where the symbol " ' " means that the items with $\sigma(n) \in \{1, 2, \ldots . l\}$ are omitted. Applying Cauchy's inequality to I_1(for $l = 0, I_1 = 0$), we obtain

$$I_1 \le \sqrt{l} \le \frac{\lambda}{2}. \tag{3.8}$$

If $l = L$, then $I_2 = 0$. For $l < L$

$$I_2(x) \le \sup_{1\le i\le L} \left|\sum_{n=1}^{i}{}' a_n \varphi_{\sigma(n)}(x)\right| + \sum_{n=1}^{L}{}' |a_n| \left|\varphi_{\sigma(n)}(x) - \psi_{\sigma(n)}(x)\right|. \tag{3.9}$$

Applying Cauchy's inequality to the second sum and taking Condition 1 into consideration, we convince ourselves that it does not exceed 20 for all x except, perhaps,

$$x \in T_1 \equiv \bigcup_{n=\bar{l}+1}^{L} \left\{x \mid |\psi_n - \varphi_n| > 2^{-\frac{n}{8}}\right\};$$

$$\mu\left(T_1\right) \leq \sum_{n=l+1}^{L} 2^{-\frac{n}{4}} < 10 \cdot 2^{-\frac{l}{4}} \leq Ce^{-\gamma\lambda^2}. \tag{3.10}$$

According to Theorem 2, for $T_2 = \left\{x|\sup_{1\leq i\leq L}\left|\sum\limits_{n=1}^{i} {}^{\backprime}a_n\varphi_{\sigma(n)}(x)\right| > \frac{\lambda}{2} - 20\right\}$ we have

$$\mu\left(T_2\right) \leq Ce^{-\gamma\lambda^2}. \tag{3.11}$$

It follows from (3.7)-(3.11) that

$$\mu\left\{\sup_{1\leq i\leq L}\left|\sum_{n=1}^{i} a_n\psi_{\sigma(n)}(x)\right| > \lambda\right\} \leq \mu\left(T_1\right) + \mu\left(T_2\right) \leq Ce^{-\gamma\lambda^2}$$

Thus the theorem is proved for the special case. The general case can be reduced to the special case. Namely, we represent the functions $f_k,\ k = 1, 2, \ldots, K$ in the form $f_k = \xi_k + \sum\limits_{n=1}^{N} \eta_{k,n},\quad k = 1, 2, \ldots, K$, as it was done in the proof of Theorem 1, with only one difference: now ξ_k, $k = 1, 2, \ldots, K$ are orthogonal functions. The theorem has been proved for them. For the independent functions $\eta_{k,n}\,(k = k_1, k_2, \ldots, k_m)$ estimate (2.8) holds. This leads to (3.1) and completes the proof. □

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