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SMALL COMBINATORIAL CARDINAL CHARACTERISTICS AND THEOREMS OF EGOROV AND BLUMBERG

Abstract

We will show that the following set theoretical assumption

> $\mathfrak{c} = \omega_2$, the dominating number $\mathfrak{d}$ equals to ω_1, and there exists an ω_1-generated Ramsey ultrafilter on ω

(which is consistent with ZFC) implies that for an arbitrary sequence $f_n\colon \mathbb{R} \to \mathbb{R}$ of uniformly bounded functions there is a set $P \subset \mathbb{R}$ of cardinality continuum and an infinite $W \subset \omega$ such that $\{f_n \restriction P\colon n \in W\}$ is a monotone uniformly convergent sequence of uniformly continuous functions. Moreover, if functions f_n are measurable or have the Baire property then P can be chosen as a perfect set.

We will also show that $\mathrm{cof}(\mathcal{N}) = \omega_1$ implies existence of a magic set and of a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction D$ is discontinuous for every $D \notin \mathcal{N} \cap \mathcal{M}$.

Our set theoretic terminology is standard and follows that of [8]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. We are using symbols $\mathcal{N}$ and $\mathcal{M}$ to denote the ideals of Lebesgue measure zero and meager subsets of $\mathbb{R}$, respectively. For the ideal $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$ its *cofinality* is defined

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by $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{B}| \colon \mathcal{B} \subset \mathcal{I} \text{ generates } \mathcal{I}\}$. A set $L \subset \mathbb{R}$ is a *κ-Luzin set* if $|L| = \kappa$ but $|L \cap N| < \kappa$ for every nowhere dense subset N of $\mathbb{R}$. Recall that Martin's Axiom, MA, implies the existence of a $\mathfrak{c}$-Luzin set. The *dominating number* is defined as

$$\mathfrak{d} = \min\{|T| \colon T \subset \omega^\omega \ \& \ (\forall f \in \omega^\omega)(\exists g \in T)(\forall n < \omega)\ f(n) < g(n)\}.$$

It is well known that $\omega_1 \leq \mathfrak{d} \leq \mathrm{cof}(\mathcal{N})$. (See e.g. [1].) In this paper we use term *Polish space* for a complete separable metric space `without isolated points`.

1 On a Convergence of Subsequences

This section can be viewed as an extension of the discussion around Egorov's theorem presented in [12, Ch. 9]. In 1932 Mazurkiewicz [13] proved the following variant of Egorov's theorem, where a sequence $\langle f_n \rangle_{n<\omega}$ of real-valued functions is *uniformly bounded* provided there exists an $r \in \mathbb{R}$ such that $\mathrm{range}(f_n) \subset [-r, r]$ for every n.

Mazurkiewicz's Theorem *Every uniformly bounded sequence $\langle f_n \rangle_{n<\omega}$ of real-valued continuous functions defined on a Polish space X has a subsequence which is uniformly convergent on some perfect set P.*

Of course Mazurkiewicz' theorem cannot be proved if we do not assume some regularity of the functions f_n even if $X = \mathbb{R}$. But is it at least true that

(∗) for every uniformly bounded sequence $\langle f_n \colon \mathbb{R} \to \mathbb{R} \rangle_{n<\omega}$ the conclusion of Mazurkiewicz' theorem holds for some $P \subset \mathbb{R}$ of cardinality $\mathfrak{c}$?

The consistency of the negative answer follows from the next example, which is essentially due to Sierpiński [16].[1] (See [12, pp. 193-194], where it is proved under the assumption of the existence of ω_1-Luzin set. The same proof works also for our more general statement.)

Example 1. *Assume that there exists a κ-Luzin set. Then for every Polish space X there exists a sequence $\langle f_n \colon X \to \{0,1\} \rangle_{n<\omega}$ with the property that for every $W \in [\omega]^\omega$ the subsequence $\langle f_n \rangle_{n\in W}$ converges pointwise for less than κ-many points $x \in X$.*

In particular, under Martin's Axiom the above sequence exists for $\kappa = \mathfrak{c}$.

Note also that under MA the above example can hold only for $\kappa = \mathfrak{c}$, since MA implies that

[1] Sierpiński constructed this example under the assumption of the Continuum Hypothesis.

> for every set S of cardinality less than $\mathfrak{c}$ every uniformly bounded sequence $\langle f_n \colon S \to \mathbb{R} \rangle_{n<\omega}$ has a pointwise convergent subsequence.

(See [12, p. 195].) Sharper results concerning the above two facts were recently obtained by Fuchino and Plewik [11], in which they relate them to the splitting number $\mathfrak{s}$. (For the definition of $\mathfrak{s}$ see e.g. [1]. For us it is only important that $\omega_1 \leq \mathfrak{s} \leq \mathfrak{d}$.) More precisely, the authors show there that: *For any $X \subset [\mathbb{R}]^{<\mathfrak{s}}$ any sequence $\langle f_n \colon X \to [-\infty, \infty] \rangle_{n<\omega}$ has a subsequence convergent pointwise on X; however for any $X \subset [\mathbb{R}]^{\mathfrak{s}}$ there exists a sequence $\langle f_n \colon X \to [0,1] \rangle_{n<\omega}$ with no pointwise convergent subsequence.*

Our main goal of this section is to prove that $(*)$ is consistent with (so, by the example, also independent from) the usual axioms of set theory ZFC. To state this precisely we need the following terminology and facts.

A maximal non-principal filter $\mathcal{F}$ on ω is said to be *Ramsey* provided for every $B \in \mathcal{F}$ and $h \colon [B]^2 \to \{0,1\}$ there exist $i < 2$ and $A \in \mathcal{F}$ such that $A \subset B$ and $h\left[[A]^2\right] = \{i\}$. We say that a family $\mathcal{W} \subset \mathcal{F}$ *generates* filter $\mathcal{F}$ provided for every $F \in \mathcal{F}$ there exists a $W \in \mathcal{W}$ such that $W \subset F$.

Theorem 2. *Assume that $\mathfrak{d} = \omega_1$ and there exists a Ramsey ultrafilter $\mathcal{F}$ on ω generated by a family $\mathcal{W} \subset \mathcal{F}$ of cardinality ω_1.*

Let X be an arbitrary set and $\langle f_n \colon X \to \mathbb{R} \rangle_{n<\omega}$ be a sequence of functions such that the set $\{f_n(x) \colon n < \omega\}$ is bounded for every $x \in X$. Then there are sequences: $\langle P_\xi \colon \xi < \omega_1 \rangle$ of subsets of X and $\langle W_\xi \in \mathcal{F} \colon \xi < \omega_1 \rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:

> *the sequence $\langle f_n \restriction P_\xi \rangle_{n \in W_\xi}$ is monotone and uniformly convergent.*

The conclusion of Theorem 2 is obvious for sets X with cardinality $\leq \omega_1$, since sets P_ξ can be chosen just as singletons. Thus, we will be interested in the theorem only for the sets X of cardinality greater than ω_1. If X is a Polish space this leads to $\mathfrak{c} = |X| > \omega_1$. Luckily, the assumptions of Theorem 2 are consistent with ZFC+"$\mathfrak{c} = \omega_2$". This holds in the iterated perfect set model. More precisely, the fact that in this model we have $\mathfrak{c} = \omega_2$ and $\mathrm{cof}(\mathcal{N}) = \omega_1$ can be found in [1, p. 339]. The fact that in this model there exists a desired Ramsey ultrafilter has been proved in Baumgartner, Laver [2]. (They proved there that there exists a selective ω_1-generated ultrafilter on ω. But it is well known that an ultrafilter on ω is selective if and only if it is Ramsey.) All these facts follow also from the axiom CPA, which is a subject of a forthcoming monograph [9]. (Some of the results proved here may also be included in [9] as the examples of interesting consequences of CPA.)

In particular, we get the following corollary which, under additional set theoretical assumptions, generalizes Mazurkiewicz' theorem and implies $(*)$.

Corollary 3. *It is consistent with ZFC+"$\mathfrak{c} = \omega_2$" that for every Polish space X and every uniformly bounded sequence $\langle f_n \colon X \to \mathbb{R} \rangle_{n<\omega}$ there exist sequences: $\langle P_\xi \colon \xi < \omega_1 \rangle$ of subsets of X and $\langle W_\xi \in [\omega]^\omega \colon \xi < \omega_1 \rangle$ such that $X = \bigcup_{\xi<\omega_1} P_\xi$ and for every $\xi < \omega_1$:*

the sequence $\langle f_n \restriction P_\xi \rangle_{n \in W_\xi}$ is monotone and uniformly convergent.

In particular, there exists a $\xi < \omega_1$ such that $|P_\xi| = \mathfrak{c}$.

Moreover, if functions f_n are continuous then we can additionally require that all sets P_ξ are closed in X.

Proof. The main part follows immediately from the discussion above and the Pigeon Hole Principle. To see the additional part it is enough to note that for continuous functions sets P_ξ can be replaced by their closures, since for any sequence $\langle f_n \colon P \to \mathbb{R} \rangle_{n<\omega}$ of continuous functions if $\langle f_n \restriction D \rangle_{n<\omega}$ is monotone and uniformly convergent for some dense subset D of P then so is $\langle f_n \rangle_{n<\omega}$. $\square$

PROOF OF THEOREM 2. For every $x \in X$ define $h_x \colon [\omega]^2 \to \{0,1\}$ by putting for every $n < m < \omega$

$$h_x(n,m) = 1 \text{ if and only if } f_n(x) \le f_m(x).$$

Since $\mathcal{F}$ is Ramsey and $\mathcal{W}$ generates $\mathcal{F}$ we can find a $W_x \in \mathcal{W}$ and an $i_x < 2$ such that $h_x[[W_x]^2] = \{i_x\}$. Thus, the sequence $S_x = \langle f_n(x) \rangle_{n \in W_x}$ is monotone. It is increasing when $i_x = 1$ and it is decreasing for $i_x = 0$.

For $W \in \mathcal{W}$ and $i < 2$ let $P^i_W = \{x \in X \colon W_x = W \ \& \ i_x = i\}$. Then $\{P^i_W \colon W \in \mathcal{W} \ \& \ i < 2\}$ is a partition of X and for every $W \in \mathcal{W}$ and $i < 2$ the sequence $\langle f_n \restriction P^i_W \rangle_{n \in W}$ is monotone and pointwise convergent to some function $f \colon P^i_W \to \mathbb{R}$.

To get uniform convergence note that for every $x \in P^i_W$ there exists an $s_x \in \omega^\omega$ such that

$$(\forall k < \omega)\ (\forall n \in W \setminus s_x(k))\ |f_n(x) - f(x)| < 2^{-k}.$$

Since $\mathfrak{d} = \omega_1$, there exists a $T \in [\omega^\omega]^{\omega_1}$ dominating ω^ω. In particular, for every $x \in P^i_W$ there exists a $t_x \in T$ such that $s_x(n) \le t_x(n)$ for all $n < \omega$. For $t \in T$ let

$$P^i_W(t) = \{x \in P^i_W \colon t_x = t\}.$$

Then $\{P^i_W(t) \colon i < 2,\ W \in \mathcal{W},\ t \in T\}$ is the desired covering $\{P_\xi \colon \xi < \omega_1\}$ of X, since every sequence $\langle f_k \restriction P^i_W(t) \rangle_{k \in W}$ is monotone and uniformly convergent. $\square$

2 $\mathrm{cof}(\mathcal{N}) = \omega_1$, Blumberg Theorem, and Magic Set

In this section we will show two consequences of $\mathrm{cof}(\mathcal{N}) = \omega_1$.

In 1922 Blumberg [4] proved that *for every* $f\colon \mathbb{R} \to \mathbb{R}$ *there exists a dense subset* D *of* $\mathbb{R}$ *such that* $f \restriction D$ *is continuous.* This theorem sparked a lot of discussion and generalizations, see e.g. [7, pp. 147–150]. In particular, Shelah [15] showed that there is a model of ZFC in which *for every* $f\colon \mathbb{R} \to \mathbb{R}$ *there is a nowhere meager subset* D *of* $\mathbb{R}$ *such that* $f \restriction D$ *is continuous.* The dual measure result, that is the consistency of a statement *for every* $f\colon \mathbb{R} \to \mathbb{R}$ *there is a subset* D *of* $\mathbb{R}$ *of positive outer Lebesgue measure such that* $f \restriction D$ *is continuous*, has been also recently established by Rosłanowski and Shelah [14]. Below we note that each of these properties contradicts $\mathrm{cof}(\mathcal{N}) = \omega_1$. (We use here the well known inequality $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$. See e.g. [1].)

Theorem 4. *Let* $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}\}$. *If* $\mathrm{cof}(\mathcal{I}) = \omega_1$ *then there exists an* $f\colon \mathbb{R} \to \mathbb{R}$ *such that* $f \restriction D$ *is discontinuous for every* $D \in \P(\mathbb{R}) \setminus \mathcal{I}$.

Proof. We will assume that $\mathcal{I} = \mathcal{N}$, the proof for $\mathcal{I} = \mathcal{M}$ being essentially identical.

Let $\{N_\xi \subset \mathbb{R}^2\colon \xi < \omega_1\}$ be a family cofinal in the ideal of null subsets of $\mathbb{R}^2$ and for each $\xi < \omega_1$ let

$$N_\xi^* = \{x \in \mathbb{R}\colon (N_\xi)_x \notin \mathcal{N}\},$$

where $(N_\xi)_x = \{y \in \mathbb{R}\colon \langle x, y\rangle \in N_\xi\}$. By Fubini's theorem each N_ξ^* is null. For each $x \in N_\xi^* \setminus \bigcup_{\zeta<\xi} N_\zeta^*$ we choose $f(x)$ so that

$$f(x) \notin \bigcup_{\zeta<\xi} (N_\zeta)_x.$$

Then function f is as desired.

Indeed, if $f \restriction D$ is continuous for some $D \subset \mathbb{R}$ then $f \restriction D$ is null in $\mathbb{R}^2$. In particular, there exists a $\xi < \omega_1$ such that $f \restriction D \subset N_\xi$. But this means that $D \subset \bigcup_{\zeta\leq\xi} N_\zeta^*$. □

Note that essentially the same proof works if we assume only that $\mathrm{cof}(\mathcal{I})$ is equal to the additivity number $\mathrm{add}(\mathcal{I})$ of $\mathcal{I}$.

Corollary 5. *Assume* $\mathrm{cof}(\mathcal{N}) = \omega_1$. *Then there exists an* $f\colon \mathbb{R} \to \mathbb{R}$ *such that if* $f \restriction D$ *is continuous then* $D \in \mathcal{N} \cap \mathcal{M}$.

Proof. Let $f_\mathcal{N}$ and $f_\mathcal{M}$ be from Theorem 4 constructed for the ideals $\mathcal{N}$ and $\mathcal{M}$, respectively. Let $G \subset \mathbb{R}$ be a dense G_δ of measure zero and put $f = [f_\mathcal{M} \restriction G] \cup [f_\mathcal{N} \restriction (\mathbb{R} \setminus G)]$. Then this f is as desired. □

Recall that a set $M \subset \mathbb{R}$ is a *magic set* (or *set of range uniqueness*) if for every different nowhere constant functions $f, g \in C(\mathbb{R})$ we have $f[M] \neq g[M]$. It has been proved by Berarducci and Dikranjan [3, thm. 8.5] that a magic set exists under CH. We like to note here that the same is implied by a much weaker assumption that $\mathrm{cof}(\mathcal{M}) = \omega_1$. However, the existence of a magic set is independent of ZFC, as proved by Ciesielski and Shelah in [10].

Proposition 6. *If* $\mathrm{cof}(\mathcal{M}) = \omega_1$ *then there exists a magic set.*

Proof. An uncountable set $L \subset \mathbb{R}$ is a *2-Luzin set* provided for every disjoint subsets $\{x_\xi : \xi < \omega_1\}$ and $\{y_\xi : \xi < \omega_1\}$ of L, where the enumerations are one-to-one, the set of pairs $\{\langle x_\xi, y_\xi\rangle : \xi < \omega_1\}$ is not a meager subset of $\mathbb{R}^2$. In [5, prop. 4.8] it was noticed that every ω_1-dense 2-Luzin set is a magic set. It is also a standard and easy diagonal argument that $\mathrm{cof}(\mathcal{M}) = \omega_1$ implies the existence of a ω_1-dense 2-Luzin set. (The proof presented in [17, prop. 6.0] works also under the assumption $\mathrm{cof}(\mathcal{M}) = \omega_1$.) So, $\mathrm{cof}(\mathcal{M}) = \omega_1$ implies that there is a magic set. □

Recall also that the existence of a magic set for the class D^1 of all differentiable functions can be proved in ZFC. This follows from [6, thm. 3.1], since every function from D^1 belongs to the class (T_2). (Compare also [6, cor. 3.3 and 3.4].)

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