Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 267–272

Yuan-Chuan Li,* Department of Applied Mathematics, National Chung-Hsing University, Taichung, 402 Taiwan. email: `ycli@amath.nchu.edu.tw`

A NOTE ON AN IDENTITY OF THE GAMMA FUNCTION AND STIRLING'S FORMULA

Abstract

Short and elementary proofs of the well-known Stirling formula for the discrete Gamma function $\Gamma(n)$ have been given by several authors. In this note, a well-known identity and Stirling's formula for the continuous Gamma function $\Gamma(x)$ are deduced in a different and short way from a simple and elementary proposition.

It is well known that the Gamma function, $\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}\,dt,\ x > 0$, satisfies the identity

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2}) \tag{1}$$

and Stirling's formula

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}\sqrt{2\pi}} = 1. \tag{2}$$

In 2000, Romik [8] gives a very short proof of the Stirling's formula for $\Gamma(n)$. Other different proofs of (2) can be found in [1, pp. 20–24], [6], [4, pp. 216–218], and [9, pg. 194]. See also [3], [5], and [7] for various proofs of the case $x = n \in \mathbb{N}$ of (2).

The purpose of this note is to deduce (1) and (2) in a different way from the following elementary and simple proposition, which also holds for vector-valued functions.

Key Words: convex function, Gamma function, Stirling's formula
Mathematical Reviews subject classification: Primary 38B15; Secondary 54A41
Received by the editors May 17, 2006
Communicated by: B. S. Thomson
*Research supported in part by the National Science Council of Taiwan

Recall that a function $f:(a,b)\to\mathbb{R}$ is said to be *convex*, where (a,b) is an interval of $\mathbb{R}$, if it satisfies

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \text{ for all } x,y\in(a,b) \text{ and } 0\le\lambda\le 1.$$

It is well-known that convex functions have the following properties:
(C1) Every convex function is continuous [2, Thm. 6.2.5],
(C2) If $f:(a,b)\to\mathbb{R}$ is continuous and midpoint convex; i.e.,

$$f(\frac{1}{2}x+\frac{1}{2}y)\le\frac{1}{2}f(x)+\frac{1}{2}f(y) \text{ for all } x,y\in(a,b),$$

then f is convex [9, pg. 101].
(C3) If $f:(a,b)\to\mathbb{R}$ is differentiable, then f is convex if and only if f' is non-decreasing on (a,b) (see [2, Thm. 6.2.3]). In particular, if $f''(x)>0$ on (a,b), then f is convex on (a,b).

Proposition 1. *Let $f:(0,\infty)\to\mathbb{R}$ and $\Delta f(x)\equiv f(x+1)-f(x)$, $x>0$.*

(a) *$\lim\limits_{x\to\infty} f(x)$ exists if and only if $\sum\limits_{n=1}^{\infty}\Delta f(n)$ converges and f satisfies*

(3) $$\lim_{n\to\infty}[f(n+1+x)-f(n+1)-x\Delta f(n)]=0 \text{ uniformly on } 0\le x\le 1.$$

(b) *If f is convex and $\lim\limits_{n\to\infty}\Delta^2 f(n)=0$, then (3) holds.*

PROOF. (a) The necessity is obvious. For the sufficiency, suppose that $\sum\limits_{n=1}^{\infty}\Delta f(n)$ converges, and f satisfies (3). Then $\Delta f(n)\to 0$ and

$$f(n+1)=f(1)+\sum_{k=1}^{n}\Delta f(k)\to f(1)+\sum_{n=1}^{\infty}\Delta f(n) \text{ as } n\to\infty.$$

From these facts and (3), we easily deduce that

$$\lim_{x\to\infty} f(x)=f(1)+\sum_{n=1}^{\infty}\Delta f(n).$$

(b) Since f is convex, we have for every $n=1,2,\dots$ and $0\le x\le 1$

$$f(n+1)=f(\frac{x}{x+1}n+\frac{1}{x+1}(n+1+x))\le\frac{x}{x+1}f(n)+\frac{1}{x+1}f(n+1+x)$$

and

$$f(n+1+x)=f((1-x)(n+1)+x(n+2))\le(1-x)f(n+1)+xf(n+2).$$

From these two inequalities, we obtain

$$x\Delta f(n) = x[f(n+1)-f(n)] \le f(n+1+x)-f(n+1) \le x[f(n+2)-f(n+1)] = x\Delta f(n+1),$$

and hence

$$0 \le f(n+1+x) - f(n+1) - x\Delta f(n) \le x[\Delta f(n+1) - \Delta f(n)] = x\Delta^2 f(n).$$

Now (3) follows from the assumption $\lim_{n\to\infty} \Delta^2 f(n) = 0$. □

Corollary 2. (cf. [9, pg. 194]) $\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2})$ *for all* $x > 0$.

Proof. Let $h(x) := \frac{2^{x-1}}{\sqrt{\pi}}\Gamma(\frac{x}{2})\Gamma(\frac{x+1}{2})$, $x > 0$. Then $h(1) = 1 = \Gamma(1)$. Since $\Gamma(x)$ is continuous on $(0,\infty)$, so is the function $\ln\Gamma(x)$. Using the Cauchy-Schwarz inequality, we obtain from the definition of Gamma function that

$$\ln\Gamma(\frac{1}{2}x + \frac{1}{2}y) \le \ln\Big[\Gamma(x)^{1/2}\Gamma(y)^{1/2}\Big] = \frac{1}{2}\ln\Gamma(x) + \frac{1}{2}\ln\Gamma(y)$$

for all $x, y > 0$; i.e., $\ln\Gamma(x)$ is midpoint convex on $(0,\infty)$. It follows from (C2) that $\ln\Gamma(x)$ is convex on $(0,\infty)$. Hence, the function

$$\ln h(x) = (x-1)\ln 2 - \frac{1}{2}\ln\pi + \ln\Gamma(\frac{x}{2}) + \ln\Gamma(\frac{x+1}{2})$$

is also convex, and we have for every $x > 0$

$$\begin{aligned}\Delta \ln h(x) &= \ln 2 + \ln\Gamma(\tfrac{x+1}{2}) - \ln\Gamma(\tfrac{x}{2}) + \ln\Gamma(\tfrac{x+2}{2}) - \ln\Gamma(\tfrac{x+1}{2})\\ &= \ln 2 + \ln\tfrac{x}{2} = \ln x = \Delta\ln\Gamma(x),\end{aligned}$$

so that $\Delta^2\ln h(x) = \Delta^2\ln\Gamma(x) = \Delta\ln x = \ln\frac{x+1}{x} \to 0$ as $x\to\infty$.

By Proposition 1(b), both $\ln h(x)$ and $\ln\Gamma(x)$ satisfy (3) with $\Delta\ln h(x) = \Delta\ln\Gamma(x) = \ln x$. Thus, the function $f(x) := \ln h(x) - \ln\Gamma(x)$ satisfies (3) with $\Delta f(x) = 0$ for all $x > 0$, and $f(1) = \ln h(1) - \ln\Gamma(1) = 0$. It follows from Proposition 1(a) that $c := \lim_{x\to\infty} f(x)$ exists. Therefore, for every $x > 0$,

$$\ln h(x) - \ln\Gamma(x) = f(x) = f(x+1) = \cdots = f(n+x) \to c \text{ as } n\to\infty.$$

Since $f(1) = 0$, this proves $c = 0$ and so $h(x) \equiv \Gamma(x)$. □

Corollary 3. (Stirling's formula) $\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}\sqrt{2\pi}} = 1$.

PROOF. Since $\Gamma(x+1) = x\Gamma(x)$, we have

$$\begin{aligned}\ln \frac{\Gamma(x+1)}{x^{x+\frac{1}{2}}e^{-x}\sqrt{2\pi}} &= \ln\Gamma(x+1) - (x+\tfrac{1}{2})\ln x + x - \tfrac{1}{2}\ln(2\pi)\\ &= \ln\Gamma(x) - \phi(x) - \tfrac{1}{2}\ln(2\pi),\end{aligned}$$

where $\phi(x) := (x - \frac{1}{2})\ln(x) - x$. Hence, it suffices to show $\lim\limits_{x\to\infty}[\ln\Gamma(x) - \phi(x) - \frac{1}{2}\ln(2\pi)] = 0$. Since $\phi'' > 0$ on $(0,\infty)$, ϕ is convex on $(0,\infty)$. Also, $\Delta\phi(x) = \ln x + r(x)$, where $r(x) = (x+\frac{1}{2})\ln(1+\frac{1}{x}) - 1$ for $x > 0$. Thus, we have

$$\begin{aligned}x^2 r(x) &= x^2\Big[(x+\tfrac{1}{2})\sum_{n=0}^{\infty}\tfrac{(-1)^n}{n+1}x^{-n-1} - 1\Big]\\ &= \sum_{n=1}^{\infty}\Big[\tfrac{(-1)^{n+1}}{n+2} - \tfrac{(-1)^{n+1}}{2(n+1)}\Big]x^{-n+1} \to \tfrac{1}{12} \text{ as } x\to\infty.\end{aligned}$$

If follows that $\Delta^2\phi(x) = \Delta\ln x + \Delta r(x) \to 0$ as $x\to\infty$. Hence, ϕ satisfies (3) by Proposition 1(b). Thus, the function $f(x) := \ln\Gamma(x) - \phi(x)$ satisfies (3) with

$$\Delta f(x) = \Delta\ln\Gamma(x) - \Delta\phi(x) = \ln x - \Delta\phi(x) = -r(x).$$

Since $\sum\limits_{n=1}^{\infty}\Delta f(n) = -\sum\limits_{n=1}^{\infty} r(n)$ converges by limiting comparison test with the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$, it follows from Proposition 1(a) that $c := \lim\limits_{x\to\infty}[\ln\Gamma(x) - \phi(x)]$ exists. Since

$$\begin{aligned}&\phi(x+1) - \phi(\tfrac{x+1}{2}) - \phi(\tfrac{x+2}{2}) - x\ln(2)\\ &= \tfrac{1}{2}(1+\ln(2)) + \tfrac{x+1}{2}\ln(\tfrac{x+1}{x+2}) \to \tfrac{1}{2}\ln(2) \text{ as } x\to\infty,\end{aligned}$$

using Corollary 2, we have

$$\begin{aligned}c &= \lim_{x\to\infty}\big[\ln\Gamma(x+1) - \phi(x+1)\big]\\ &= \lim_{x\to\infty}\Big[\big(x\ln(2) + \ln\Gamma(\tfrac{x+1}{2}) + \ln\Gamma(\tfrac{x+2}{2}) - \tfrac{1}{2}\ln(\pi)\big)\\ &\quad -\big(\phi(\tfrac{x+1}{2}) + \phi(\tfrac{x+2}{2}) + x\ln(2) + \tfrac{1}{2}\ln(2)\big)\Big]\\ &= c + c - \tfrac{1}{2}\ln(2\pi).\end{aligned}$$

This shows that $c = \frac{1}{2}\ln(2\pi)$, and hence $\lim\limits_{x\to\infty}[\ln\Gamma(x) - \phi(x) - \frac{1}{2}\ln(2\pi)] = 0$. The proof is complete. $\square$

References

[1] E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, Inc., New York, 1964.

[2] G. de Barra, *Measure Theory and Integration,* Ellis Horwood Series in Mathematics and its Applications, Halsted Press, New York, 1981.

[3] C. R. Blyth and P. K. Pathak, *A note on easy proofs of Stirling's theorem*, Amer. Math. Monthly, **93** (1986), 376–379.

[4] R. C. Buck, *Advanced Calculus*, 2nd ed., McGraw-Hill, New York, 1965.

[5] P. Diaconis and D. Freeman, *An elementary proof of Stirling's formula*, Amer. Math. Monthly, **93** (1986), 123–125.

[6] W. Feller, *A direct proof of Stirling's formula*, Amer. Math. Monthly, **74** (1967), 1223-1225 and **75** (1968), 518.

[7] J. M. Patin, *A very short proof of Stirling's formula*, Amer. Math. Monthly, **96** (1989), 41–42.

[8] D. Romik, *Stirling approximation for* $n!$*: the ultimate short proof?*, Amer. Math. Monthly, **107** (2000), 556–557.

[9] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, Inc., New York, 1976.