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ON THE HAUSDORFF DIMENSION OF AVERAGE TYPE SUMS OF RADEMACHER FUNCTIONS

Abstract

We prove that for any $a, c \in (0,1)$ and any $b, d \in \mathbb{R}$, the Hausdorff dimension of $\{x \in [0,1] : n^{-a} \sum_{j=1}^{n} r_j(x) \to b$ and $n^{-c} \sum_{j=1}^{n} r_j(x) r_{j+1}(x) \to d\}$, is equal to 1, where $\{r_n(x)\}_{n \geq 1}$, are the Rademacher functions. We give also an extension of this result.

1 Introduction

Let $\{r_n(x)\}_{n\geq 1}, x \in \mathbb{R}$, be the sequence of Rademacher function. Let $a, c \in [0,1]$, $b, d \in \mathbb{R}$, $S_n(x) = \sum\limits_{j=1}^{n} r_j(x)$ and $S'_n(x) = \sum\limits_{j=1}^{n} r_j(x) r_{j+1}(x)$. We consider the sets

$$M(a,b) = \left\{ x \in [0,1] : \lim_{n\to\infty} n^{-a} S_n(x) = b \right\},$$
$$M'(c,d) = \left\{ x \in [0,1] : \lim_{n\to\infty} n^{-c} S'_n(x) = d \right\},$$
$$N(a,b,c,d) = M(a,b) \cap M'(c,d).$$

We will prove the following assertion.

Theorem 1. *For any $a, c \in (0,1)$ and any $b, d \in \mathbb{R}$, we have $\dim N(a,b,c,d) = 1$, where $\dim$ means the Hausdorff dimension.*

This theorem is an extension of some previous results. We point out that A. S. Besicovitch and H. G. Eggleston (see [1]) have computed the Hausdorff dimension of the set $M(1,b)$, $b \in [-1,1]$ and Jun Wu (see [11] Th. 2) has

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proved that for any $a \in (0,1)$ and any $b \in \mathbb{R}$, we have $\dim M(a,b) = 1$. Moreover, A. H. Fan and D. J. Feng (see [7], [8] Th. 1) have computed the Hausdorff dimension of the set $N(1,b,1,d)$, where $b,d \in [-1,1]$. For trigonometric functions, one can find related results in [6]. Finally, for sets defined by digits in dyadic, triadic and other expansions see [2], [5], [3] and [8].

In this work we get lower bounds of the Hausdorff dimension of $N(a,b,c,d)$ by estimating the dimensions of some non-homogeneous Cantor measures. We construct these measures on some subsets of Cantor type of our set. Finally we discuss another proof of the theorem in the case $a,c \in (\frac{1}{2},1)$ and we give an extension of Theorem 1.

2 The Hausdorff Dimension of $\mathbf{N}(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$

Lemma. (i) *Let $\varepsilon_1,\dots,\varepsilon_{2J+1} \in \{\pm 1\}$, $J \in \mathbb{N}$. Then the number of ordered pairs $(\varepsilon_1,\dots,\varepsilon_{2J})$ which satisfies the conditions $\sum\limits_{i=1}^{2J} \varepsilon_i = 0$ and $\sum\limits_{i=1}^{2J} \varepsilon_i\varepsilon_{i+1} = 0$, is equal to $2c_J$, where $c_J = \binom{J}{\frac{J}{2}}\binom{J-1}{\frac{J-2}{2}}$ if J is even and $c_J = \binom{J}{\frac{J-1}{2}}\binom{J-1}{\frac{J-1}{2}}$ if J is odd.*

(ii)
$$\frac{\log c_J}{2J\log 2} \to 1,\ J \to \infty.$$

PROOF OF (i). There exists 2^{2J} ordered pairs of the form $(\varepsilon_1,\dots,\varepsilon_{2J})$. If we have only the condition $\sum\limits_{i=1}^{2J} \varepsilon_i = 0$, then the number of ordered pairs is $\binom{2J}{J}$ and among the numbers $\varepsilon_1,\dots,\varepsilon_{2J}$ the J's are $+1$ and the other J's are -1. Since we desire to have the two conditions simultaneously, we must have $J-1$ or J changes of sign on the pair $(\varepsilon_1,\dots,\varepsilon_{2J})$. We suppose that $\varepsilon_1 = 1$.

(a) If $J = 2n$, then the J numbers with sign + are decomposed into n groups ($\varepsilon_{2J} = -1$) or into $n+1$ groups ($\varepsilon_{2J} = 1$). (In each group we have successively numbers with sign +.) This can be possible with $\binom{2n-1}{n-1} + \binom{2n-1}{n} = \binom{2n}{n}$ ways. The J numbers with sign - are placing in the remainder n positions and this is done with $\binom{2n-1}{n-1}$ ways. Repeating the above process for the case $\varepsilon_1 = -1$ we get that the number of ordered pairs is

$$2\binom{2n}{n}\binom{2n-1}{n-1} = 2\binom{J}{\frac{J}{2}}\binom{J-1}{\frac{J-2}{2}}.$$

(b) If $J = 2n+1$, then the J numbers with sign + are decomposed into n+1 groups. This is possible in $\binom{2n}{n}$ ways. The J numbers with sign - take the remainder n positions ($\varepsilon_{2J} = +1$) or $n+1$ positions ($\varepsilon_{2J} = -1$), and this is done in $\binom{2n}{n} + \binom{2n}{n-1} = \binom{2n+1}{n}$ ways. Repeating the above process for the case $\varepsilon_1 = -1$ we get that the number of ordered pairs is

$$2\binom{2n}{n}\binom{2n+1}{n} = 2\binom{J-1}{\frac{J-1}{2}}\binom{J}{\frac{J-1}{2}}.$$

Proof of (ii). Using Stirling's formula the result follows. $\square$

Proof of Theorem 1. We observe that if $0 < a < a' < 1$, then $M(a,b) \subset M(a',0)$ and if $0 < c < c' < 1$, then $M'(c,d) \subset M'(c',0)$. So it is enough to prove the theorem for $b, d \neq 0$ and which we now assume. For convenience we shall denote by $E_{n,k}$ the interval $[\frac{k}{2^n}, \frac{k+1}{2^n})$, $n \in \mathbb{N}$, $k = 0, 1, \ldots, 2^n - 1$. Let $J \in \mathbb{N}$ and

$$A_1 = \{x \in [0,1] : S_{2J}(x) = S'_{2J}(x) = 0\}. \tag{1}$$

By the lemma we get that A_1 is the union of $2c_J$ intervals of the form $E_{2J+1,k}$. Since $r_j(x) = -r_j(1-x)$, we have that there exist $k_1, k_2 \in \{0, 1, \ldots, 2^{2J+1} - 1\}$, such that $E_{2J+1,k_1} \subset \left[0, \frac{1}{2}\right]$ and $E_{2J+1,k_2} \subset \left[\frac{1}{2}, 1\right]$ and if $x \in E_{2J+1,k_1}$, $y \in E_{2J+1,k_2}$, then

$$\begin{aligned} r_{2J+1}(x)\, r_{2J+1}(y) = -1,\ S_{2J}(x) = S_{2J}(y) = 0, \\ S'_{2J}(x) = S'_{2J}(y) \neq 0 \text{ and } \operatorname{sign} S'_{2J}(x) = \operatorname{sign} b \end{aligned} \tag{2}$$

where $\operatorname{sign} x = 1, 0, -1$ according to $x > 0$, $x = 0$, $x < 0$. Analogous, there exist $k'_1, k'_2 \in \{0, 1, \ldots, 2^{2J+1} - 1\}$ (we assume $J \geq 2$), such that $E_{2J+1,k'_1} \subset \left[0, \frac{1}{2}\right]$, $E_{2J+1,k'_2} \subset \left[\frac{1}{2}, 1\right]$ and if $x \in E_{2J+1,k'_1}$, $y \in E_{2J+1,k'_2}$, then

$$\begin{aligned} r_{2J+1}(x)\, r_{2J+1}(y) = -1,\ S'_{2J}(x) = S'_{2J}(y) = 0, \\ S_{2J}(x) = S_{2J}(y) \neq 0 \text{ and } \operatorname{sign} S_{2J}(x) = \operatorname{sign} d. \end{aligned} \tag{3}$$

Let $A_2 = E_{2J+1,k_1} \cup E_{2J+1,k_2}$, $A_3 = E_{2J+1,k'_1} \cup E_{2J+1,k'_2}$. We construct the following Cantor-type set. The first stage of this construction is to take the intervals $E_{2J+1,k}$ for those k, such that $E_{2J+1,k} \subset A_1 \cup A_2 \cup A_3$ and to remove the others. Each successive step of the construction is essentially the same. Thus, at the second stage, we subdivide each of the remaining intervals $E_{2J+1,k}$ into 4^J equal intervals. We take the intervals $E_{4J+1,\lambda}$, $\lambda \in \{0, 1, \ldots, 2^{4J+1} - 1\}$, for which we have

$$4^J E_{4J+1,\lambda} \bmod 1 \subset A_1 \cup A_2 \cup A_3$$

and remove the others. We proceed by induction. We also remark that by an interval $E_{2J+1,k} \subset A_{i_1}$, $i_1 = 1, 2, 3$, we take at the second stage c_J intervals of the form $E_{4J+1,\lambda}$ with the property $4^J E_{4J+1,\lambda} \bmod 1 \subset A_1$, one with the property $4^J E_{4J+1,\lambda} \bmod 1 \subset A_2$ and one with the property $4^J E_{4J+1,\lambda} \bmod 1 \subset A_3$. Let $A_{i_1 i_2}$ be the union of the intervals $E_{4J+1,\lambda} \subset A_{i_1}$ such that $4^J E_{4J+1,\lambda}$ mod $1 \subset A_{i_2}$. By induction we define the sets $A_{i_1 \ldots i_n, i_{n+1}}$, $i_1, \ldots, i_{n+1} \in \{1,2,3\}$, $n = 1, 2, \ldots$. For these sets we have that

$$A_{i_1 \ldots i_n i_{n+1}} \subset A_{i_1 \ldots i_n},\ 4^{nJ} A_{i_1 \ldots i_n i_{n+1}} \bmod 1 = A_{i_{n+1}}. \tag{4}$$

We consider a sequence of Borel probability measures μ_n, $n \in \mathbb{N}$, such that

$$\mu_n(E_{2Jn+1,k}) = \frac{1}{2} p_{i_1}^{(1)} \ldots p_{i_n}^{(n)},\ E_{2Jn+1,k} \subset A_{i_1 \ldots i_n}, \tag{5}$$

$i_1, \ldots, i_n \in \{1,2,3\},\ n \in \mathbb{N}$ and $c_J p_1^{(n)} + p_2^{(n)} + p_3^{(n)} = 1$. Let

$$S_{n,J}(x) := \sum_{j=2(n-1)J+1}^{2nJ} r_j(x),\ S'_{n,J}(x) := \sum_{j=2(n-1)J+1}^{2nJ} r_j(x) r_{j+1}(x),\ \ n \in \mathbb{N},$$

$$A_{n,i} = \cup_{i_1 \ldots i_{n-1}} A_{i_1 \ldots i_{n-1} i},\ i_1, \ldots, i_{n-1}, i \in \{1,2,3\},\ n = 2, 3, \ldots$$

and for convenience we write $A_{1,i} = A_i,\ i = 1,2,3$. Using the relations (1) – (4) we have that

$$\text{if } x \in A_{n,1}, \text{ then } S_{n,J}(x) = S'_{n,J}(x) = 0, \tag{6}$$

$$\begin{gathered} \text{if } x \in A_{n,2},\ x_0 \in A_2, \text{ then } S_{n,J}(x) = 0, S'_{n,J}(x) = S'_{2J}(x_0) \\ \text{and } \operatorname{sign} S'_{2J}(x_0) = \operatorname{sign} b, \end{gathered} \tag{7}$$

$$\begin{gathered} \text{if } x \in A_{n,3},\ x'_0 \in A_3, \text{ then } S'_{n,J}(x) = 0, S_{n,J}(x) = S_{2J}(x'_0), \\ \text{and } \operatorname{sign} S_{2J}(x'_0) = \operatorname{sign} d. \end{gathered} \tag{8}$$

Let μ be the weak* limit of the sequence of measures $\mu_n,\ n \in \mathbb{N}$. Then, we deduce that

$$\mu(A_{n,1}) = c_J p_1^{(n)}, \mu(A_{n,i}) = p_i^{(n)}, i = 2, 3,\ n \in \mathbb{N}. \tag{9}$$

An easy computation shows that the sequences of functions $S_{n,J}$ and $S'_{n,J},\ n \in \mathbb{N}$, are sequences of independent random variables with respect to μ. We denote by $E(f)$ and by $V(f)$ the expectation and the variance of the function f with respect to μ respectively. From (6) – (9) it follows that

$$\begin{aligned} E(S_{n,J}) &= S_{2J}(x'_0) p_3^{(n)}, & E(S'_{n,J}) &= S'_{2J}(x_0) p_2^{(n)}, \\ V(S_{n,J}) &= S_{2J}^2(x'_0) p_3^{(n)}(1 - p_3^{(n)}), & V(S'_{n,J}) &= S'^2_{2J}(x_0) p_2^{(n)}(1 - p_2^{(n)}). \end{aligned}$$

Let $p_2^{(n)}$, $p_3^{(n)}$ be such that (for sufficient large n)

$$p_2^{(n)} = \frac{(2J)^c|d|}{|S'_{2J}(x_0)|}(n^c - (n-1)^c) \sim \frac{(2J)^c|d|}{|S'_{2J}(x_0)|}\frac{c}{n^{1-c}},$$
$$p_3^{(n)} = \frac{(2J)^a|b|}{|S_{2J}(x'_0)|}(n^a - (n-1)^a) \sim \frac{(2J)^a|b|}{|S_{2J}(x'_0)|}\frac{a}{n^{1-a}}.$$

Using the above relations we take that

$$\sum_{n=1}^{\infty}\frac{V(S_{n,J})}{n^{2a}} < \infty \text{ and } \sum_{n=1}^{\infty}\frac{V(S'_{n,J})}{n^{2c}} < \infty.$$

Hence, by the strong law of large numbers (see [9] p. 364), we conclude that

$$\lim_{n\to\infty}\frac{1}{n^a}\sum_{k=1}^{n} S_{k,J}(x) = (2J)^a b \text{ and } \lim_{n\to\infty}\frac{1}{n^c}\sum_{k=1}^{n} S'_{k,J}(x) = (2J)^c d,\ \mu - a.e.$$

and so $\mu(N(a,b,c,d)) = 1$, as is easy to check. We denote by $E_n(x)$ the interval $E_{n,k}$ which contains x. By (5) we have that

$$\log\mu(E_{2Jn+1}(x)) = \log\mu_n(E_{2Jn+1}(x)) = \log\left[\frac{1}{2}\prod_{k=1}^{n} p_{i_k(x)}^{(k)}\right]$$
$$= -\log 2 + \sum_{k=1}^{n}\left[\chi_{A_{k,1}}(x)\log p_1^{(k)} + \chi_{A_{k,2}}(x)\log p_2^{(k)} + \chi_{A_{k,3}}(x)\log p_3^{(k)}\right],$$

where $i_{k(x)} \in \{1,2,3\}$ and $\chi_{A_{k,i}}(x)$ is the characteristic function of the set $A_{k,i}$. It is a simple matter to see that for $i = 1,2,3$ the sequence of functions

$$\chi_{A_{n,i}}(x)\log p_i^{(n)},\ x \in [0,1],\ n \in \mathbb{N},$$

is a sequence of independent random variables with respect to the measure μ. Since $p_2^{(n)}$, $p_3^{(n)} \to 0$, $n \to +\infty$, the strong law of large numbers and (9) implies that

$$\lim_{n\to\infty}\frac{\log\mu(E_{2Jn+1}(x))}{-(2Jn+1)\log 2} = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left[c_J p_1^{(k)}\log p_1^{(k)} + p_2^{(k)}\log p_2^{(k)} + p_3^{(k)}\log p_3^{(k)}\right]}{-(2Jn+1)\log 2}$$
$$= \frac{\log c_J}{2J\log 2},\ \mu - a.e.$$

and consequently

$$\lim_{n\to\infty} \frac{\log \mu(E_n(x))}{-n \log 2} = \frac{\log c_J}{2J \log 2}, \ \mu - a.e..$$

Since $\mu(N(a,b,c,d)) = 1$, by the well know mass distribution principle, (see [1] p. 141), we obtain that $\dim N(a,b,c,d) \geq \dfrac{\log c_J}{2J \log 2}$. By the lemma we get that $\dim N(a,b,c,d) = 1$, as we desired. □

The theorem implies the following.

Corollary. *For any $c \in (0,1)$ and any $d \in \mathbb{R}$, we have $\dim M'(c,d) = 1$.*

Remark. (i) If $a \in (1/2, 1)$, then we can find $\dim M(a,b)$ using the measure $d\mu = \prod_{n=1}^{\infty} (1 + a_n r_n(x))dx$, where $a_n = (n^a - (n-1)^a)b$, also see [6].

(ii) If $a, c \in (1/2, 1)$, then it is possible to take the result of the theorem using the measure

$$d\mu = \prod_{n=1}^{\infty} [(1 + \frac{1}{2} \operatorname{sign} b \, r_{k_n}(x))(1 + \frac{1}{2} \operatorname{sign} d \, r_{m_n}(x) r_{m_n+1}(x))] \, dx,$$

where the sequences k_n and m_n satisfies the following conditions:

(α_1) $\{k_n, \ n \in \mathbb{N}\} \cap \{m_n, \ n \in \mathbb{N}\} \cap \{m_n + 1, \ n \in \mathbb{N}\} = \emptyset$,

(α_2) $\frac{1}{2} \lim_{n\to\infty} \frac{n}{k_n^a} = |b|, \ \frac{1}{2} \lim_{n\to\infty} \frac{n}{m_n^c} = |d|$.

One can see that $\mu(N(a,b,c,d)) = 1$. Also see [4] about the Hausdorff dimension of μ.

Next we give an extension of our theorem 1 and theorem 3 of [12].

Theorem 2. *Let $(\gamma_n)_{n\geq 0}$ and $(\delta_n)_{n\geq 0}$ be two increasing and unbounded sequences of positive numbers such that $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = \lim_{n\to\infty}(\delta_{n+1} - \delta_n) = 0$. Then for any $b, d \in \mathbb{R}$, the Hausdorff dimension of the set*

$$\Big\{x \in [0,1] : \lim_{n\to\infty} \frac{1}{\gamma_n} S_n(x) = b \ \text{and} \ \lim_{n\to\infty} \frac{1}{\delta_n} S'_n(x) = d\Big\},$$

is equal to 1.

PROOF. The proof is similar to the proof of Theorem 1. For the convenience of the reader we mention that we choose

$$p_2^{(n)} = \frac{|d|}{|S'_{2J}(x_0)|}(\delta_{2Jn} - \delta_{2J(n-1)}) \text{ and } p_3^{(n)} = \frac{|b|}{|S'_{2J}(x_0)|}(\gamma_{2Jn} - \gamma_{2J(n-1)}).$$

In order to apply the strong law of large numbers we make use of the fact that the series $\sum_{n=1}^{\infty} \frac{\gamma_n - \gamma_{n-1}}{\gamma_n^2}$, $\sum_{n=1}^{\infty} \frac{\delta_n - \delta_{n-1}}{\delta_n^2}$ converges, (see [10], Theorem 2.41, p. 57). Finally, we obtain

$$\lim_{n\to\infty} \frac{1}{\gamma_{2Jn}} \sum_{k=1}^{n} S_{k,J}(x) = b \text{ and } \lim_{n\to\infty} \frac{1}{\delta_{2Jn}} \sum_{k=1}^{n} S'_{k,J}(x) = d,\ \mu - a.e..$$

We can now proceed analogously to the proof of Theorem 1. We point out that in [12] (Th. 3), the author estimates the Hausdorff dimension of the set $\left\{x \in [0,1] : \lim\limits_{n\to\infty} \frac{1}{\gamma_n} S_n(x) = b\right\}$.

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