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ON THE PRODUCTS OF UNILATERALLY CONTINUOUS REGULATED FUNCTIONS

Abstract

In this article we describe the products of two unilaterally continuous regulated functions.

Let $\mathbb{R}$ be the set of all reals. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a regulated function ([1, 2, 4]) if for each point $x \in \mathbb{R}$ there are both finite unilateral limits

$$f(x-) = \lim_{t \to x^-} f(t) \text{ and } f(x+) = \lim_{t \to x^+} f(t).$$

In [5] (see also [3]) C. S. Reed calls such functions as jump functions.

Evidently, the sum and the product of two regulated functions is also a regulated function. In [3] I show that each regulated function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two unilaterally continuous regulated functions.

Let (w_n) be an enumeration of all rationals such that $w_n \neq w_m$ for $n \neq m$. In [3] I observed that the function

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x = w_n,\ n \geq 1 \\ 0 & \text{otherwise on} \mathbb{R} \end{cases}$$

is a regulated function, but it is not the product of any finite family of unilaterally continuous regulated functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$.

Moreover, in [3], I prove that each regulated function $f : \mathbb{R} \to (\mathbb{R} \setminus \{0\})$ such that, for each $x \in \mathbb{R}$, the inequality $f(x-) \neq 0 \neq f(x+)$ holds is the product of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$.

Lemma 1. *If a regulated function $f : \mathbb{R} \to \mathbb{R}$ is the product of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$ and $f(x) = 0$, then $f(x-) = 0$ or $f(x+) = 0$.*

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PROOF. Since $0 = f(x) = g(x)h(x)$, we have $g(x) = 0$ or $h(x) = 0$. Assume that $g(x) = 0$. Since g is unilaterally continuous, $g(x+) = g(x) = 0$ or $g(x-) = g(x) = 0$. But $f(x+) = g(x+)h(x+)$ and $f(x-) = g(x-)h(x-)$, so $f(x+) = 0$ or $f(x-) = 0$. If $g(x) \neq 0$, then $h(x) = 0$ and the further reasoning is analogous. This completes the proof. $\square$

The converse theorem is not true. Simply note that the function

$$f(x) = 0 \text{ for } x \neq 0 \text{ and } f(0) = 1$$

is a regulated function which is the product of the two functions

$$g(x) = \begin{cases} 1 & \text{for } x \leq 0 \\ 0 & \text{for } x > 0 \end{cases} \text{ and } h(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0, \end{cases}$$

both of which are unilaterally continuous at each point $x \in \mathbb{R}$. However, $\lim_{x \to 0} f(x) = 0$, but $f(0) = 1 \neq 0$.

Moreover, in the thesis of Lemma 1, the alternative $f(x+) = 0$ or $f(x-) = 0$ cannot be replaced by the conjunction $\lim_{t \to x} f(t) = 0$. For example the function

$$f(x) = 0 \text{ for } x \leq 0 \text{ and } f(x) = 1 \text{ for } x > 0$$

is an unilaterally continuous regulated function and $f(0) = 0$, but it does not have the limit $\lim_{x \to 0} f(x)$.

In [3] (see also [1, 4]) it is proved that the set $D(f)$ of all discontinuity points of a regulated function $f : \mathbb{R} \to \mathbb{R}$ is countable.

From Lemma 1 we obtain the following theorem giving the first necessary condition satisfied by the products of two unilaterally continuous regulated functions.

Theorem 1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the product of two unilaterally continuous regulated functions and $f(x) = 0$, then f is unilaterally continuous at x.*

Now we show the second necessary condition satisfied by all regulated functions which are the products of two unilaterally continuous regulated functions. In the statement of the theorem we are using the following terminology. For a set $A \subset \mathbb{R}$ we denote by $\mathrm{cl}(A)$ the closure of the set A and by $\mathrm{cl}_b(A)$ the set of all points $x \in \mathrm{cl}(A)$ which are bilateral condensation points of A.

Theorem 2. *Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is the product of two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$ and that the closure*

$\mathrm{cl}(f^{-1}(0))$ *of the level set* $f^{-1}(0)$ *is uncountable. Then for each nonempty perfect set* $A \subset \mathrm{cl}(f^{-1}(0))$*, the set*

$$E(f) = \mathrm{cl}_b(A) \cap \{x \in \mathbb{R} : f(x) \neq 0\}$$

is nowhere dense in $\mathrm{cl}_b(A)$.

PROOF. Assume to a contradiction that there are a nonempty perfect set $A \subset \mathrm{cl}(f^{-1}(0))$ and an open interval I such that $A \cap I \neq \emptyset$ and the intersection $I \cap E(f)$ is dense in $B = I \cap \mathrm{cl}_b(A)$. Since f is a regulated function, the set $D(f)$ is countable. Consequently, the set $I \cap \mathrm{cl}_b(A)$ is of the second category in itself and at least one of the sets

$$B_g = \{x \in I \cap \mathrm{cl}_b(A) : g(x) = 0\} \text{ and } B_h = \{x \in I \cap \mathrm{cl}_b(A) : h(x) = 0\}$$

is of the second category in $I \cap \mathrm{cl}_b(A)$. Assume that the set B_g is of the second category in B. There is an open interval $J \subset I$ such that the set $J \cap A \neq \emptyset$ and $J \cap B_g$ is dense in $J \cap A$. There is a point $u \in J \cap E(f)$ which is a bilateral accumulation point of $f^{-1}(0)$. Observe that $\lim_{x \to u} g(x) = 0$ and $0 \neq f(u) = g(u)h(u)$. Consequently, $g(u) \neq 0$ and g is not unilaterally continuous at u, a contradiction with our hypothesis. If the set B_g is of the first category in B then B_h is of the second category in B and we proceed similarly. This finishes the proof. □

Remark 1. *The conjunction of the necessary conditions from Theorems 1 and 2 concerning a regulated function* $f : \mathbb{R} \to \mathbb{R}$ *is not a sufficient condition for the existence of two unilaterally continuous regulated functions* $g, h : \mathbb{R} \to \mathbb{R}$ *with* $f = gh$.

PROOF. Let $C \subset [0, 1]$ be the Cantor ternary set and let (a_n) be an enumeration of all points from C which are unilaterally isolated in C such that $a_n \neq a_m$ for $n \neq m$. The function

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x = a_n, \;\; n \geq 1 \\ 0 & \text{otherwise on } \mathbb{R} \end{cases}$$

is a regulated function continuous at each point $x \in f^{-1}(0)$. It also satisfies the necessary condition from Theorem 2. Assume to a contrary that there are two unilaterally continuous regulated functions $g, h : \mathbb{R} \to \mathbb{R}$ with $f = gh$. Let

$$A = \{x \in C : g(x) = 0\} \text{ and } B = \{x \in C : h(x) = 0\}.$$

Since $H = C \setminus \{a_n : n \geq 1\}$ is of the second category in itself and $H = A \cup B$, at least one of the sets A and B is of the second category in H. Assume that

A is of the second category in H. Then there is an open interval I such that $I \cap C \neq \emptyset$ and $A \cap I$ is dense in $C \cap I$. Fix a positive integer k such that $a_k \in I \cap C$. Assume that a_k is isolated on the left hand in C. Then

$$f(a_k) = g(a_k)h(a_k) = \frac{1}{k} \neq 0$$

and consequently $g(a_k) \neq 0$ and $h(a_k) \neq 0$. Observe that $\lim_{t \to a_k^+} g(t) = 0$. Since $g(a_k) \neq 0$, the function g is not continuous on the right hand at a_k. So the function g is continuous on the left hand at a_k. But f is not continuous on the left hand at a_k, so h is not continuous on the left hand at a_k. Consequently, h is continuous on the right hand at a_k, and there is a positive real r such that

$$[a_k, a_k + r) \subset I \text{ and } |h(t) - h(a_k)| < \frac{|h(a_k)|}{2} \text{ for } t \in [a_k, a_k + r).$$

Let $m > 3$ be a positive integer such that $a_m \in [a_k, a_k + r)$ and a_m is isolated on the left hand in C. Since $\lim_{t \to a_m^+} g(t) = 0$ and $g(a_m) \neq 0$, as above we show that g is continuous on the left hand at a_m and there is a positive real s such that

$$[a_m, a_m + s) \subset (a_k, a_k + r) \text{ and } |g(t) - g(a_m)| < \frac{|g(a_m)|}{2} \text{ for } t \in (a_m, a_m + s).$$

Consequently,

$$f(t) = g(t)h(t) \neq 0 \text{ for } t \in [a_m, a_m + s),$$

a contradiction with the definition of f.

In the case where A is of the first category in H the set B must be of the second category in H and we proceed similarly. This finishes the proof. □

Example 1. Let $C \subset [0, 1]$ be the ternary Cantor set and let $(I_n = (a_n, b_n))$ be an enumeration of all components of the set $\mathbb{R} \setminus C$ such that $I_1 = (-\infty, 0)$, $I_2 = (1, \infty)$ and $I_n \neq I_m$ for $n \neq m$. Let

$$c_n = \frac{a_n + b_n}{2} \text{ for } n > 2.$$

The function

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x \in \{a_n, b_n\},\ n > 2 \\ 0 & \text{for } x = c_n,\ n > 2 \\ \text{interpolatory linear} & \text{on the intervals } [a_n, c_n] \text{ and } [c_n, b_n],\ n > 2 \\ 0 & \text{otherwise on } \mathbb{R} \end{cases}$$

is a unilaterally continuous regulated function which is continuous at each point $x \in f^{-1}(0)$, but for the perfect set $C \subset \mathrm{cl}(f^{-1}(0))$ the set $C \cap \{x : f(x) \neq 0\}$ is dense in C.

So in this example I show that the necessary condition from Theorem 2 cannot replaced by the following condition: *"for each nonempty perfect set $A \subset \mathrm{cl}(f^{-1}(0))$ the set $A \cap \{x : f(x) \neq 0\}$ is nowhere dense in C"*.

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