ON THE ORDER OF THE RECIPROCAL SET OF A BASIC SET OF POLYNOMIALS

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1. Introduction. For the general terminology used in this paper the reader is referred to J. M. Whittaker [2], [3]. Let

$$p_n(z) = \sum_i p_{ni} z^i$$

be a basic set, and let

$$z^n = \sum_{i=0}^{D_n} \pi_{ni} \, p_i(z).$$

The order ω and type γ of $\{p_n(z)\}$ are defined as follows. Let $M_i(R)$ be the maximum modulus of $p_i(z)$ in $|z| \le R$. Let

$$\omega_n(R) = \sum_i |\pi_{ni}| \, M_i(R), \tag{1}$$

$$\omega(R) = \limsup_{n \to \infty} \frac{\log \omega_n(R)}{n \log n}, \tag{2}$$

$$\omega = \lim_{R \to \infty} \omega(R); \tag{3}$$

and, for $0 < \omega < \infty$, let

$$\gamma(R) = \limsup_{n \to \infty} \{\omega_n(R)\}^{1/(n\omega)} \, e/(n\omega), \tag{4}$$

$$\gamma = \lim_{R \to \infty} \gamma(R). \tag{5}$$

If

$$P_n(z) = \sum_i \pi_{ni} z^i,$$

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then $\{P_n(z)\}$ is called the reciprocal set of $\{p_n(z)\}$. We shall establish for certain basic sets new formulas expressing upper bounds of the order of the reciprocal set in terms of the data of the original set.

2. **Theorem.** The following theorem holds only if an infinity of $\pi_{nn} \neq 0$; then the whole proof should be carried out for those values of n for which $\pi_{nn} \neq 0$. This is a genuine restriction since there are basic sets such that $\pi_{nn} = 0$ for all n; for example, for $h = 0, 1, 2, \cdots$, let

$$p_{3h}(z) = -\frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+1}(z) = \frac{1}{2} z^{3h} - \frac{1}{2} z^{3h+1} + \frac{1}{2} z^{3h+2},$$

$$p_{3h+2}(z) = \frac{1}{2} z^{3h} + \frac{1}{2} z^{3h+1} - \frac{1}{2} z^{3h+2}.$$

NOTATION. For a fixed n, let $p_{nh'}$ be the set of all nonzero elements p_{nh}, and let

$$\min_{h'} p_{nh'} = p_{n'}.$$

THEOREM 1. *Let $\{p_n(z)\}$ be a basic set of polynomials, such that*

$$\limsup_{n\to\infty} \frac{D_n}{n} = a \qquad (a \geq 1),$$

and of increase less than order ω and type γ, and suppose that

$$\kappa = \liminf_{n\to\infty} \frac{\log|\pi_{nn}|}{n \log n}$$

and

$$k = \liminf_{n\to\infty} \frac{\log|p_{n'}|}{n \log n}.$$

Then its reciprocal set is of order Ω, where

i) *if $k > \omega$, then $\Omega \leq \omega - \kappa$;*

ii) *if* $k \leq \omega$, *then* $\Omega \leq 2\omega - \kappa - k$.

Proof. Let $\gamma_1 > \gamma$; then in view of (4) we have

$$\omega_n(R) \leq \left(\frac{n\,\omega\,\gamma_1}{e}\right)^{n\omega} \tag{6}$$

for values of $n > n_0$ and for sufficiently large values of $R > R_0 > 1$. From (1), we have

$$|\pi_{nn}|\ M_n(R) \leq \omega_n(R).$$

Then

$$|\pi_{nn}|\ |p_{ni}|\ R^i \leq \omega_n(R);$$

that is

$$|p_{ni}| \leq \frac{\omega_n(R)}{|\pi_{nn}|}\ . \tag{7}$$

Also

$$|\pi_{ij}|\ M_j(R) \leq \omega_i(R);$$

then

$$|\pi_{ij}| \leq \frac{\omega_i(R)}{M_j(R)} \leq \frac{\omega_i(R)}{\min\limits_{h'} |p_{ih'}|} = \frac{\omega_i(R)}{|p_{i'}|}\,. \tag{8}$$

From the definition of a reciprocal set, and in view of (1), we get

$$\Omega_n(R) \leq \sum_{i=0}^{D_n} |p_{ni}| \sum_j |\pi_{ij}|\ R^j \leq \frac{\omega_n(R)}{|\pi_{nn}|}\, R^{D_n} \sum_{i=0}^{D_n} \sum_j |\pi_{ij}|$$

by (7); that is, by (8),

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|}\, R^{D_n} \sum_{i=0}^{D_n} N_i\, \frac{\omega_i(R)}{|p_{i'}|}\,.$$

Then

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{\omega_i(R)}{|p_i'|} \right\}$$

$$\leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + \sum_{i=n_0+1}^{D_n} \frac{(i\,\omega\gamma_1)^{i\omega}}{|p_i'|} \right\} \quad \text{by (6)},$$

where $F(R)$ is a function independent of n.

Then for sufficiently large values of $n > n_0$ and $R > R_0$, we get

$$\Omega_n(R) \leq \frac{\omega_n(R)}{|\pi_{nn}|} R^{D_n} \cdot D_n \left\{ F(R) + D_n \left(\frac{n\,\omega\gamma_1}{n^{k_1/\omega}} \right)^{n\omega} \right\} \quad (\text{where } k_1 \geq k).$$

Hence:

i) If $k > \omega$ (this implies $k_1 > \omega$), then $(n\,\omega\gamma_1/n^{k_1/\omega})^{n\omega}$, for values of $n > n_0$, will be a small quantity compared to $F(R)$. Therefore,

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \Omega_n(R)}{n \log n}$$

$$\leq \lim_{R\to\infty} \limsup_{n\to\infty} \left\{ \frac{\log \omega_n(R)}{n\log n} + \frac{D_n \log R}{n \log n} - \frac{\log|\pi_{nn}|}{n\log n} + \frac{\log D_n}{n \log n} + \frac{\log F(R)}{n\log n} \right\},$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa .$$

ii) If $k \leq \omega$, then as k_1 approaches k we find that $F(R)$ will be very small compared to $\{n\,\omega\gamma_1/n^{k_1/\omega}\}^{n\omega}$ for $n > n_0$. Therefore,

$$\lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \omega_n(R)}{n\log n} \leq \lim_{R\to\infty} \limsup_{n\to\infty} \left\{ \frac{\log \omega_n(R)}{n\log n} - \frac{\log|\pi_{nn}|}{n\log n} \right.$$

$$\left. + \frac{D_n \log R + 2\log D_n}{n\log n} + \frac{n\,\omega\left(1 - \frac{k_1}{\omega}\right)\log n}{n \log n} + \frac{n\,\omega \log \omega\gamma_1}{n\log n} \right\}$$

in view of (2) and (3); then

$$\Omega \leq \omega - \kappa + \omega - k = 2\omega - \kappa - k.$$

N. B. *In the case of simple sets, the restriction mentioned above for π_{nn} is satisfied. In this case we have*

$$-\kappa = \limsup_{n\to\infty} \frac{\log |p_{nn}|}{n \log n}.$$

COROLLARY. *If $\{p_n(z)\}$ is a simple set of polynomials,*

$$\left.\begin{array}{ll} \text{i)} & \text{if } k > \omega, \text{ then } \Omega \leq \omega - \kappa \\ \text{ii)} & \text{if } k \leq \omega, \text{ then } \Omega \leq 2\omega - \kappa - k \end{array}\right\} \quad \text{where } \kappa = -\limsup_{n\to\infty} \frac{\log |p_{nn}|}{n \log n}.$$

3. **Examples.** We shall look at four examples.

i) Let $p_n(z) = n^{3n} z^n - n^{2n} z^{n-1} - n^{3n} z^{n+1}$ $\quad$ (n odd),

$p_n(z) = n^{2n} z^n - n^{3n}$ $\quad$ (n even),

$p_0(z) = 1.$

then

$$z^n = n^{-3n} p_n(z) + n^{-n} (n-1)^{-2(n-1)} p_{n-1}(z) + (n+1)^{-2(n+1)} p_{n+1}(z)$$
$$+ \{(n-1)^{(n-1)} n^{-n} + (n+1)^{(n+1)}\} p_0(z) \quad (n \text{ odd}),$$

$$z^n = n^{-2n} p_n(z) + n^n p_0(z) \quad (n \text{ even}).$$

By Theorem (1) of [1], we get $\omega = 1$. Since $\kappa = -3$, $k = 2$, we get, according to case i) of the theorem, $\Omega \leq 1 + 3 = 4$. This is true because $\Omega = 4$ by Corollary (1.1) of [1].

N. B. *This example and the following examples show that the values given in the conclusion of the above theorem are "best possible."*

ii) Let $p_n(z) = n^{2n} z^n - n^{3n/2} z^{2n} - n^{2n}$ $\quad$ (n odd),

$p_n(z) = \left(\frac{n}{2}\right)^{3n/2} z^n - \left(\frac{n}{2}\right)^{2n}$, with $p_0(z) = 1$ $\quad$ (n even), Then

$$z^n = n^{-2n}\, p_n(z) + n^{-7n/2}\, p_{2n}(z) + (1 + n^{n/2})\, p_0(z) \qquad (n \text{ odd}),$$

$$z^n = \left(\frac{n}{2}\right)^{-3n/2} p_n(z) + \left(\frac{n}{2}\right)^{n/2} p_0(z) \qquad (n \text{ even}),$$

Applying theorem (1) of [1], we get $\omega = 1/2$. Now $\kappa = -2$, $k = 3/2$. Then according to case i), of the theorem, we get

$$\Omega \leq \frac{1}{2} + 2.$$

This is true because $\Omega = 5/2$ by Corollary (1.1) of [1].

iii) Let $p_n(z) = n^n z^n - n^{n/2} z^{n-1} - n^{3n/2}$ $\qquad (n \text{ odd}),$

$$p_n(z) = (n+1)^{(n+1)} z^n - (n+1)^{2(n+1)} z^{(n+1)}$$

$$- (n+1)^{5(n+1)/2} \qquad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = \frac{1}{1 - n^{n/2}} \left\{ n^{-n}\, p_n(z) + n^{-3n/2}\, p_{n-1}(z) + (n^{n/2} + n^n)\, p_0(z) \right\} \qquad (n \text{ odd}),$$

$$z^n = \frac{1}{1 - (n+1)^{(n+1)/2}} \left\{ (n+1)^{-(n+1)}\, p_n(z) \right.$$

$$\left. + p_{n+1}(z) + 2(n+1)^{3(n+1)/2}\, p_0(z) \right\} \qquad (n \text{ even}).$$

Applying theorem (1) of [1], we get $\omega = 1$. Now $\kappa = -1$, $k = 1/2$. Then according to case ii) of the theorem, we get

$$\Omega \leq 2 + 1 - \frac{1}{2} = \frac{5}{2}.$$

This is true because $\Omega = 5/2$ by Corollary (1.1) of [1].

iv) Let
$$p_n(z) = \frac{2^{(n-1)}}{2^{(n-1)}\, n^{2n} + (n-1)^{3(n-1)}}\, z^n + \frac{2^{(n-1)}\, n^n}{2^{(n-1)}\, n^{2n} + (n-1)^{3(n-1)}}$$
$$+ \frac{2^{2(n-1)}\,(n-1)^{(n-1)}}{2^{(n-1)}\, n^{2n} + (n-1)^{3(n-1)}}\, z^{n-1} \qquad (n \text{ odd}),$$

$$p_n(z) = \frac{2^{2n}\,(n+1)^{2(n+1)}}{2^n\, n^n\,(n+1)^{2(n+1)} + n^{4n}}\, z^n - \frac{n^n}{2^n\,(n+1)^{2(n+1)} + n^{3n}}\, z^{n+1}$$
$$- \frac{n^n (n+1)^{(n+1)}}{2^n (n+1)^{2(n+1)} + n^{3n}}\, z^{2n+2} \qquad (n \text{ even}),$$

$$p_0(z) = 1.$$

Then

$$z^n = n^{2n}\, p_n(z) - n^{3n}\, p_{2n}(z)$$
$$-(n-1)^{2(n-1)}\, p_{n-1}(z) - n^{5n}\, p_{2n+1}(z) \qquad (n \text{ odd}),$$

$$z^n = \left(\frac{1}{2}\, n\right)^n p_n(z) + \left(\frac{1}{2}\, n\right)^{2n} p_{n+1}(z) \qquad (n \text{ even}).$$

Applying theorem (1) of [1], we get $\omega = 1$. Now $\kappa = 2$, $k = -3$. Then according to case ii) of the theorem, we get

$$\Omega \leq 2 - 2 + 3 = 3.$$

This is true because $\Omega = 3$ by Corollary (1.1) of [1].

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