ON INFINITE GROUPS

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1. Introduction. Several disconnected theorems on infinite groups will be given in this paper. In § 2, a generalization of Poincaré's theorem on the index of the intersection of two subgroups is proved. Other theorems on indices are given. In § 3, the theorem [3, Lemma 1 and Corollary 1] that the layer of elements of infinite order in a group G has order 0 or $o(G)$ is generalized to the case where the order is taken with respect to a subgroup. In § 4, it is shown that the subgroup K of an infinite group G as defined in [3] is overcharacteristic [2]. In § 5, characterizations are obtained for those Abelian groups G, all of whose subgroups H (factor groups G/H) of order equal to $o(G)$ are isomorphic to G (in this connection, compare with [7]). Again the Abelian groups, all of whose order preserving endomorphisms are onto, are found (see [6]).

2. Index theorems. If H is a subgroup of G, let $i(H)$ denote the index of H in G. The cardinal of a set S will be denoted by $o(S)$.

THEOREM 1. *Let H_α be a subgroup of G, $\alpha \in S$. Then*

$$i(\cap H_\alpha) \leq \prod i(H_\alpha).$$

Proof.

$$g_1 g_2^{-1} \in \cap H_\alpha$$

if and only if

$$g_1 g_2^{-1} \in H_\alpha \quad \text{for all} \quad \alpha \in S.$$

Thus each coset of $\cap H_\alpha$ is the intersection of a collection of sets consisting of one coset of H_α for each α, and the conclusion follows.

COROLLARY 1. (Poincaré) *The intersection of a finite number of subgroups of finite index is again of finite index.*

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COROLLARY 2. *If $i(H) = B$, then G has a normal subgroup K such that $i(H) \leq B^B$.*

Proof. Let $N(H)$ denote the normalizer of H, and $Cl(H)$ the conjugate class of H. Then

$$H \subseteq N(H), \quad o(Cl(H)) = i(N(H)) \leq B.$$

Thus if K is the intersection of the conjugates of H, Theorem 1 gives $i(K) \leq B^B$.

REMARKS. For every infinite cardinal A, there is a simple group G of order A (for example, the "alternating" group on A symbols). Thus G has no subgroups of index less than or equal to B if $2^B < A$. In particular, if A is such that $B < A$ implies $2^B < A$, then G has no subgroup of index less than its order A. This is in sharp contrast to the behaviour of Abelian groups, which have 2^A subgroups of index B for $\aleph_0 \leq B \leq A$, $A > \aleph_0$ [4]. It is an unsolved problem as to whether there exists a group G of order A with no subgroups of order A, for $A > \aleph_0$.

Let $\cup$ denote the point set union, and + and Σ direct sums (the lattice union of subgroups will not be used). If T is a nonempty subset of a group G, let

$$i_R(T) = \min o(S) \text{ such that } \cup T x_\alpha = G, \quad \alpha \in S.$$

Define $i_L(T)$ similarly, and let $i(T)$ be the smaller of $i_R(T)$ and $i_L(T)$.

THEOREM 2. *If H_i, $i = 1, \cdots, n$, are subgroups of G such that $i(H_i) \geq A \geq \aleph_0$, then $i(\cup H_i) \geq A$.*

Proof. The theorem is true for $n = 1$. Induction on n. If, contrary to the theorem, $i(\cup H_i) < A$, then, say,

$$G = \bigcup_{\alpha \in S} \left(\bigcup_{i=1}^{n} H_i \right) x_\alpha$$

with $o(S) < A$. Since $i(H_1) \geq A$, there exists an $x \in G$ such that

$$H_1 x \cap (\cup_\alpha H_1 x_\alpha)$$

is empty. Hence

$$H_1 x \subseteq \cup_\alpha (\cup_2^n H_i) x_\alpha .$$

Therefore

$$\bigcup_{i=1}^{n} H_i \subseteq \bigcup_{i=2}^{n} H_i(e \cup (\bigcup_\alpha x_\alpha x^{-1})) = \bigcup_{\beta \in S'} \left(\bigcup_{i=2}^{n} H_i \right) x_\beta,$$

where $o(S') < A$. Hence

$$G = \bigcup_{\alpha \in S} \left(\bigcup_{i=1}^{n} H_i \right) x_\alpha = \bigcup_{\alpha \in S} \bigcup_{\beta \in S'} \bigcup_{i=2}^{n} H_i x_\beta x_\alpha = \bigcup_{\gamma \in S''} \bigcup_{i=2}^{n} H_i x_\gamma, \quad o(S'') < A.$$

This contradicts the induction hypothesis. Hence the theorem is true.

REMARK. For every infinite cardinal A, there is a group G of order A, containing an increasing sequence $\{H_n\}$ of subgroups, each of index A, such that $\bigcup H_n = G$.

Let $1/A = 0$ for $A \geq \aleph_0$.

THEOREM 3. *If H_i is a proper subgroup of G, $(i = 1, \cdots, n)$ and $\sum 1/i(H_i) \leq 1$, then $\bigcup H_i \neq G$.*

Proof. Let $H_1, \cdots, H_r$ have finite index, the others infinite index (if $r = 0$, the theorem follows immediately from Theorem 2). Let

$$D = \bigcap_{1}^{r} H_i.$$

Then D has finite index in G, and it is well known that $(\bigcup_i^r H_i) \cap Dx$ is empty for some $x \in G$. Hence, if $\bigcup_1^n H_i = G$, then $Dx \subseteq \bigcup_{r+1}^n H_i$, whence $\bigcup_{r+1}^n H_i$ has finite "index" in contradiction to Theorem 2. Therefore $\bigcup_1^n H_i \neq G$.

3. Layers. Let T be a subset of G, and let n be a positive integer. Let

$$L(n, T) = \{g \mid g^n \in T, \ g^r \notin T \ \text{ for } \ 0 < r < n\},$$

$$L(\infty, T) = \{g \mid g^n \notin T, \ n = 1, 2, \cdots\}.$$

For $T = e$, the $L(n, T)$ have been called *layers*. The following theorem generalizes [3, Lemma 1].

THEOREM 4. *Let G be an infinite group, H a subgroup, P a set of primes and*

$$S = \left(\bigcup_{p \in P} \bigcup_{\lambda} L(\lambda p, H) \right) \cup L(\infty, H).$$

Then $o(S) = 0$ *or* $o(G)$.

Proof. Deny the theorem. Let $x \in S$. If $x \in L(\lambda p, H)$ then $x^{\lambda} \in L(p, H)$. Hence we may assume that $x \in L(\infty, H)$ or $x \in L(p, H)$, $p \in P$.

Case 1. $o(N(x)) = o(G)$, where $N(x)$ is the normalizer of x. Then $o(N(x) - S) = o(G)$. If $y \in N(x) - S$, then $y^r \in H$ for some r such that $(r, p) = 1$ (if p exists). If $xy \notin S$ then also $(xy)^n \in H$ for some n such that $(n, p) = 1$ (if p exists). Thus

$$(xy)^{rn} = x^{rn} y^{rn} \in H,$$

and $x^{rn} \in H$. But $(rn, p) = 1$ if p exists, and, in any case, we have a contradiction. Hence $xy \in S$ and

$$o(S) \geq o(x(N(x) - S)) = o(N(x) - S) = o(G),$$

a contradiction.

Case 2. $o(N(x)) < o(G)$. Then $o(Cl(x)) = o(G)$.

Case 2.1. $o(H) = o(G)$. Then $o(G)$ right cosets of $N(x)$ intersect H. Thus there are $o(G)$ elements of the form $h^{-1}xh$. But if $(h^{-1}xh)^n \in H$ then $x^n \in H$, whence $n = \lambda p$ and $h^{-1}xh \in S$. Therefore $o(S) = o(G)$, a contradiction.

Case 2.2. $o(H) < o(G)$. We have, since $o(S) < o(G)$,

$$o(G) = o(Cl(x)) = \sum_{\substack{(n, p) = 1 \\ n < \infty}} o(Cl(x) \cap L(n, H)). \tag{1}$$

If $o(G) = \aleph_0$, and $o(x) = \infty$, then since H is finite,

$$Cl(x) \subseteq L(\infty, H) \subseteq S,$$

a contradiction. If $o(G) = \aleph_0$, and $o(x) = m$, then $Cl(x) \cap L(n, H)$ is empty for $n > m$. Hence, by (1), there exists, regardless of the size of $o(G)$, an n such that $(n, p) = 1$ and

$$o(Cl(x) \cap L(n, H)) > o(H)o(S).$$

Let

$$A(n, T) = \{ g \mid g^n \in T \}.$$

Then $A(n, H) \supseteq L(n, H)$, hence

$$o(Cl(x) \cap A(n, H)) = \sum_h o(Cl(x) \cap A(n, h)) > o(H) o(S).$$

Hence there exists an $h_0 \in H$ such that

$$o(Cl(x) \cap A(n, h_0)) > o(S).$$

There is then a $b \in G$ such that $(b^{-1} x b)^n = h_0$, whence

$$x \in Cl(x) \cap A(n, b h_0 b^{-1}).$$

If

$$q \in Cl(x) \cap A(n, b h_0 b^{-1}),$$

then

$$q^n = b h_0 b^{-1} = x^n.$$

Hence if $q^r \in H$, then

$$x^{nr} = q^{nr} \in H$$

and $p \mid nr$, whence $p \mid r$. Thus $q \in S$ in any case. We have

$$o(S) \geq o(Cl(x) \cap A(n, b h_0 b^{-1})) = o(b(Cl(x) \cap A(n, h_0)) b^{-1})$$

$$= o(Cl(x) \cap A(n, h_0)) > o(S).$$

This contradiction shows that the theorem is true.

COROLLARY. *If H is a subgroup of the group G, then $o(L(\infty, H)) = o(G)$ or 0.*

Proof. In Theorem 4, let P be the empty set.

4. An over-characteristic subgroup. Neumann and Neumann [2] have defined a subgroup K of G to be *over-characteristic* in G if and only if (i) K is normal, and (ii) $G/K \simeq G/H$ implies $K \subseteq H$.

Define (see [3]) a subgroup K of an infinite group G as follows. Let $E(x)$ be the set of $g \in G$ such that x is not in the subgroup generated by g, and let K be the set of $x \in G$ such that $o(E(x)) < o(G)$.

THEOREM 5. *If G is infinite, and K is defined as above, then K is an over-characteristic subgroup of G.*

Proof. (i) K is normal since it is fully characteristic [3, Theorem 6].

(ii) Let $G/K \cong G/H$.

Case 1. K is finite. Then [3, Corollary 3 to Theorem 8]

$$K_2 = K(G/K) = e.$$

Hence $K(G/H) = e$. Now

$$o(G/H) = o(G/K) = o(G).$$

If there exists a $k \in K - H$, then

$$o(E(kH)) \leq o(E(k)) < o(G) = o(G/H).$$

Hence $kH \in K(G/H)$. This is a contradiction. Hence $K \subseteq H$, and K is over-characteristic.

Case 2. K is infinite. Then [3, Theorem 5] K is a p^∞ group, and [3, Theorem 8] G/K is finite. If there exists a $k \in K - H$ then

$$k'^{p^n} = k$$

implies $k' \in K - H$, and

$$o(k'H) \geq p^{n+1}.$$

This contradicts the finiteness of G/H. Therefore $K \subseteq H$, and since G/K is finite, $K = H$. Hence K is over-characteristic.

5. Abelian groups with special properties.[1] If G is an Abelian group such that $0 \subset H \subset G$ implies $G \cong H$ for subgroups H, then it is trivial that G is 0 or cyclic of prime or infinite order, and conversely. This naturally leads to the problem of finding those groups which possess the following property:

[1] For the facts used without proof in this section, see [1].

(P_1) G is Abelian, and if H is a subgroup of G such that $o(H) = o(G)$ then $G \cong H$.

THEOREM 6. *G has property (P_1) if and only if* (i) *G is finite Abelian,* (ii) *G is a p^∞ group,* (iii) *G is a direct sum of cyclic groups of order p, p a fixed prime,* (iv) *G is infinite cyclic, or* (v) *G is the direct sum of a non-denumerable number of infinite cyclic groups.*

Proof. If G is of one of the above five types, then it is either trivial or well-known that G has property (P_1).

Conversely, suppose that G is infinite and has property (P_1). Let T be the torsion subgroup of G.

Case 1. $o(T) < o(G)$. Then (see, for example, [3, proof of Theorem 9, Case 1]) there is a free Abelian subgroup H of G such that $o(H) = o(G)$. Hence $G \cong H$. If the rank of G is non-denumerable, we are done. If the rank of G is countable, then G is countable and contains an infinite cyclic subgroup. By (P_1), G is infinite cyclic.

Case 2. $o(T) = o(G)$. Then $G \cong T$, that is, G is periodic. If G_p is a non-zero p-component of G, then $G = G_p + H_p$, hence $G \cong G_p$ or $G \cong H_p$, a contradiction unless $H_p = 0$. Hence G is a p-group. Thus $G = D + R$, where D is a divisible (that is, $nD = D$) and R a reduced (no divisible non-zero subgroups) p-group. Hence $G \cong R$ or $G \cong D$, that is G is reduced or divisible.

Case 2.1. G is a divisible p-group. Then $G = \sum C_\alpha$ where C_α is a p^∞ group. If there is more than one summand, then there is a subgroup

$$H = C^* + \sum C_\alpha,$$

$\alpha \neq \alpha_0$, where C^* is a proper subgroup of C_{α_0}. Hence $o(H) = o(G)$, but H is not divisible, a contradiction. Therefore G is a p^∞ group in this case.

Case 2.2. G is a reduced p-group. Then G has a cyclic direct summand C of order, say, p^n. Zorn's lemma may be applied to sets S of cyclic groups C_α of order p^n such that $\sum C_\alpha$, $C_\alpha \in S$, exists and is pure in G (that is, a servant subgroup of G). There is then a maximal such set S^*, and if $K = \sum C_\alpha$, $C_\alpha \in S^*$, then K is a pure subgroup of bounded order. Hence K is a direct summand, $G = K + A$. It is clear that A has no cyclic direct summands of order p^n. This implies, by property (P_1), that $o(A) < o(G)$, hence $G \cong K$. If, now, $n > 1$, there is a subgroup H of K of order $o(G)$ such that $H \not\cong K$. Therefore $n = 1$.

Theorem 6 has a dual.

(P_2) G is Abelian, and $o(G/H) = o(G)$ implies $G \cong G/H$.

THEOREM 7. *G has property (P_2) if and only if* (i) *G is finite Abelian,* (ii) *G is infinite cyclic,* (iii) *G is a direct sum of cyclic groups of order p,* (iv) *G is a p^∞ group, or* (v) *G is the direct sum of a non-denumerable number of p^∞ groups.*

Proof. If G is of one of the above five types, then it is clear that G has property (P_2).

Conversely suppose that G is infinite and has property (P_2).

Case 1. $o(G/T) = o(G)$. Then, by (P_2) G is torsion-free. Let C be a cyclic subgroup of G. Then $2C$ is cyclic, and $G/2C$ has an element of order 2, hence $o(G/2C) < o(G)$. Therefore $o(G) = \aleph_0$, and $o(G/C)$ is finite, hence G is cyclic.

Case 2. $o(G/T) < o(G)$. Hence $o(T) = o(G)$. Let S be a maximal linearly independent set of elements, B the subgroup generated by S (set $B = 0$ if S is empty). Then $T \cap B = 0$, hence T is isomorphic to a subgroup of G/B, and therefore $o(G/B) = o(G)$. But G/B is periodic, hence G is periodic. It follows, just as in the proof of Theorem 6, that G is either a divisible or a reduced p-group.

Case 2.1. G is a divisible p-group. Then $G = \sum C_\alpha$, where C_α is a p^∞ group. If the number of summands is non-denumerable, we are done. If not, then G is homomorphic to a p^∞ group, and $o(G) = \aleph_0$. Therefore by (P_2), G is a p^∞ group.

Case 2.2. G is a reduced p-group. Then, almost exactly as in Case 2.2 of Theorem 6, it follows that G is the direct sum of cyclic groups of order p.

REMARK. Szélpál [7] has shown that if G is an Abelian group which is isomorphic to all proper quotient groups, then G is a cyclic group of order p or a p^∞ group. Theorem 7 may be considered as a generalization of this theorem.

Szele and Szélpál [6] have shown that if G is an Abelian group such that every non-zero endomorphism is onto, then G is a cyclic group of order p, a p^∞ group, or the rationals. The following theorem may be considered as a generalization.

(P_3) G is Abelian, and if σ is an endomorphism of G such that $o(G\sigma) = o(G)$ then $G\sigma = G$.

THEOREM 8. *G has property (P_3) if and only if* (i) *G is finite Abelian,* (ii) *G is a p^∞ group, or* (iii) *G is the group of rationals.*

Proof. If G is of one of the above three types, then it is clear that (P_3) is satisfied.

Conversely, suppose that G is an infinite group satisfying (P_3).

Case 1. G is torsion-free. Then if $pG \neq G$ for some p, the transformation $g\sigma = pg$ is an isomorphism of G into itself, so that $o(G\sigma) = o(G)$, $G\sigma \neq G$, a contradiction. Hence $pG = G$ for all p, and therefore $G = \sum R_\alpha$, where R_α is is isomorphic to the group of rationals. If there is more than one summand, then there is a projection σ of G onto $\sum R_\alpha$, $\alpha \neq \alpha_0$, a contradiction. Hence G is the group of rationals.

Case 2. G is not torsion-free. Then $G = A + B$ where A is finite (and non-zero) or a p^∞ group. Thus the projection σ of G onto the larger of A and B yields a contradiction unless $B = 0$. But in this case, since G is infinite, $G = A$ is a p^∞ group.

Finally (compare with Szele [5]) consider the following property.

(P_4) G is Abelian, and if σ is an endomorphism of G such that $o(G\sigma) = o(G)$ then σ is an automorphism of G

COROLLARY. *G has property (P_4) if and only if* (i) *G is finite Abelian, or* (ii) *G is the group of rationals.*

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