EIGENFUNCTION EXPANSIONS ASSOCIATED WITH A NON-SELF-ADJOINT DIFFERENTIAL EQUATION

BERNARD FRIEDMAN AND L. I. MISHOE

1. Introduction. In solving certain characteristic boundary-value problems by the method of separation of variables [2], the problem arose of expanding an arbitrary function $f(x)$ in terms of the eigenfunctions of the equation $(A+\lambda B)u=0$, where A is a second-order and B a first-order differential operator. In this paper we consider a special case of this problem, namely the following:
Expand a function $f(x)$ in terms of the eigenfunctions of the equation

$$u''+q(x)u+\lambda(p(x)u-u')=0\ , \tag{1.1}$$

where $u(0)=u(1)=0$. There has been a long series of investigations concerned with the corresponding self-adjoint problem for the equation $(A-\lambda)u=0$, which often occurs in connection with the boundary-value problems of mathematical physics. However, the problem we are concerned with here does not seem to have been considered previously. F. Browder [1] has considered the eigenfunctions of $A+\lambda B$ where A and B are general partial differential operators, but he has always assumed that B is positive definite. We shall show that the lack of definiteness in B gives rise to peculiar results in the expansion theorem. R. E. Langer [3] has considered the expansion theorem for the following equation, which is similar to (1.1)[1].

$$u''+\{p_{11}\lambda+p_{10}\}u'+\{p_{22}\lambda^2+p_{21}\lambda+p_{20}\}u=0\ .$$

This equation of course reduces to (1.1) if we put

$$p_{10}=p_{22}=0\ ,\qquad p_{11}=-1\ ,\qquad p_{21}=p,\qquad p_{20}=q.$$

However, Langer in his paper made the assumption that the roots of $r^2+p_{11}r+p_{22}=0$ were distinct and nonvanishing. For (1.1), it is clear that $r=0$, $r=+1$, and hence Langer's conditions do not apply. In fact, the results we shall obtain are strikingly different from those of Langer.

Since the operator B is not self-adjoint, we must also consider the adjoint of (1.1), namely

Received February 2, 1955. This research has been made possible through support and sponsorship extended by the Office of Scientific Research, H.Q., Air Research and Development Command, U.S.A.F., Baltimore, Maryland, under Contract No. AF-18(600)-367.

1) A detailed treatment of this expansion problem and related questions has been given by Titchmarsh [4].

$$v'' + q(x)v + \lambda(pv + v') = 0 , \tag{1.2}$$

where $v(0)=v(1)=0$. Let $u_n(x)$ and $v_n(x)$ be the eigenfunctions of (1.1) and (1.2) respectively, corresponding to the eigenvalue λ_n. It is well known that

$$\int_0^1 u_n B^* v_m dx = 0 , \qquad n \neq m .$$

We normalize the solutions so that

$$\int_0^1 u_n B^* v_n dx = 1 .$$

Then we prove the following theorem.

THEOREM. *Let $q(x)$ be continuous and $p(x)$ be such that the second derivative exists and is continuous. If $F(x)$ is of bounded variation in* (0,1) *and if*

$$F(0+) + \exp[-P(0, 1)]F(1-) = 0 , \qquad P(\xi, x) = \int_\xi^x p(t)dt \tag{1.3}$$

then the series

$$\sum_{n=-\infty}^{\infty} a_n u_n(x) , \tag{1.4}$$

where

$$a_n = \int_0^1 F(t)(pv_n + v_n')dt ,$$

converges to $\frac{1}{2}[F(x+0) + F(x-0)]$. *If $F(x)$ does not satisfy the condition* (1.3) *then the series* (1.4) *converges* to

$$g(x) = \frac{1}{2}[F(x+0) + F(x-0) - c \exp[P(0, x)] ,$$

where

$$c = \frac{1}{2}\{F(0+) + F(1-)\exp[-P(0, 1)\} .$$

2. Expansion Theorem. In this section we first derive an orthogonality relationship which will indicate the form of the expansion if it exists; then we derive a contour integral representation for the expansion.

Denote the operator $d^2/dx^2 + q(x)$ by A and its adjoint by A^*; $A = A^*$.

Denote the operator $p-d/dx$ by B and its adjoint, $p+d/dx$, by B^*. We shall now derive the orthogonality relationship

$$(1)\qquad \int_0^1 u_n(x)[p(x)v_m(x)+v_m'(x)]dx=0\,, \qquad \text{for} \quad m\neq n\,.$$

We begin with the differential equations

$$(2)\qquad u_n''+q(x)u_n+\lambda_n(p(x)u_n-u_n')=0\,, \quad \text{or} \quad (A+\lambda_n B)u_n=0\,,$$

$$(3)\qquad v_m''+q(x)v_m+\lambda_m(p(x)v_m+v_m')=0\,, \quad \text{or} \quad (A^*+\lambda_m B^*)v_m=0\,.$$

Multiplying (2) and (3) by $v_m(x)$ and $u_n(x)$ respectively, we obtain

$$(4)\qquad u_n''(x)v_m(x)+q(x)u_n(x)v_m(x)+\lambda_n p(x)u_n(x)v_m(x)-\lambda_n u_n'(x)v_m(x)=0\,,$$

$$(5)\qquad v_m''(x)u_n(x)+q(x)v_m(x)u_n(x)+\lambda_m p(x)v_m(x)u_n(x)+\lambda_m v_m'(x)u_n(x)=0\,.$$

Subtracting (4) from (5) we have

$$(6)\qquad \begin{aligned} u_n''(x)v_m(x)-v_m''(x)u_n(x)+(\lambda_n-\lambda_m)p(x)u_n(x)v_m(x)\\ -\lambda_n u_n'(x)v_m(x)-\lambda_m v_m'(x)u_n(x)=0\,,\end{aligned}$$

which can be written in the form

$$(7)\qquad \begin{aligned}\frac{d}{dx}[v_m(x)u_n'(x)-u_n(x)v_m'(x)]-\lambda_n u_n'(x)v_m(x)-\lambda_m v_m'(x)u_n(x)\\ +(\lambda_n-\lambda_m)p(x)u_n(x)v_m(x)=0\,.\end{aligned}$$

If (7) is integrated over the interval (0, 1), it becomes

$$(8)\qquad \begin{aligned}[v_m(x)u_n'(x)-u_n(x)v_m'(x)]_0^1-\lambda_n\int_0^1 u_n'(x)v_m(x)dx-\lambda_m\int_0^1 v_m'(x)u_n(x)dx\\ +(\lambda_n-\lambda_m)\int_0^1 p(x)u_n(x)v_m(x)dx=0\,.\end{aligned}$$

By an integration by parts it follows that

$$(9)\qquad -\lambda_n\int_0^1 u_n'(x)v_m(x)dx=-\lambda_n\left\{[u_n(x)v_m(x)]_0^1-\int_0^1 u_n(x)v_m'(x)dx\right\}\,.$$

Therefore (8) becomes

$$(10)\qquad \begin{aligned}[v_m(x)u_n'(x)-u_n(x)v_m'(x)]_0^1-\lambda_n[u_n(x)v_m(x)]_0^1+\lambda_n\int_0^1 u_n(x)v_m'(x)dx\\ -\lambda_m\int_0^1 v_m'(x)u_n(x)dx+(\lambda_n-\lambda_m)\int_0^1 p(x)u_n(x)v_m(x)=0\,.\end{aligned}$$

Imposing the boundary conditions $u(1)=u(0)=v(1)=v(0)=0$, we find

$$(11)\qquad (\lambda_n-\lambda_m)\int_0^1 u_n(x)v_m'(x)dx+(\lambda_n-\lambda_m)\int_0^1 p(x)u_n(x)v_m(x)dx=0\ .$$

Combining the two integrals in (11) we obtain

$$(12)\qquad (\lambda_n-\lambda_m)\int_0^1 u_n(x)(p(x)v_m(x)+v_m'(x))dx=0\ ,$$

from which the desired orthogonality relationship (1) follows.

Now assume an eigenfunction expansion exists, and let

$$F(x)=\sum_n a_n u_n(x)\ .$$

Then

$$[p(x)v_m(x)+v_m'(x)]F(x)=\sum_n a_n u_n(x)[p(x)v_m(x)+v_m'(x)]\ .$$

As a consequence of (1) we have

$$(13)\qquad \int_0^1 F(x)[p(x)v_m(x)+v_m'(x)]dx=a_m\int_0^1 u_m(x)[p(x)v_m(x)+v_m'(x)]dx\ .$$

Hence, we obtain

$$(14)\qquad a_m=\frac{\int_0^1 F(x)[p(x)v_m(x)+v_m'(x)]dx}{\int_0^1 u_m(x)[p(x)v_m(x)+v_m'(x)]dx}\ .$$

We derive now a formula for the Wronskian of the differential equation

$$(15)\qquad u''+q(x)u+\lambda(pu-u')=0\ .$$

Let u_1 and u_2 be two fundamental solutions of this equation. We have

$$(16)\qquad u_1''+qu_1+\lambda(pu_1-u_1')=0$$

$$(17)\qquad u_2''+qu_2+\lambda(pu_2-u_2')=0\ .$$

Multiplying (16) by u_2 and (17) by u_1 and then subtracting we obtain

$$u_1''u_2-u_2''u_1-\lambda u_1'u_2+\lambda u_2'u_1=0\ ,$$

which can be written in the form

$$\frac{d}{dx}(u_1u_2'-u_2u_1')=\lambda(u_1u_2'-u_2u_1')\ .$$

Consequently, the Wronskian is given by

$$\omega(x)=u_2'u_1-u_2u_1'=C(\lambda)e^{\lambda x}\ .$$

Let $u_1(x)$ and $u_2(x)$ be solutions of (15), and let $v_1(x)$ and $v_2(x)$ be solutions of the adjoint equation

$$v''+q(x)v+\lambda(pv+v')=0\ . \tag{18}$$

Consider the evaluation by residues of the integral

$$\begin{aligned}\frac{1}{2\pi i}\oint d\lambda\Big[\int_0^x &\frac{F(\xi)(p(\xi)v_1(\xi)+v_1'(\xi))}{C(\lambda)}u_2(x)d\xi \\ &+\int_x^1 \frac{F(\xi)(p(\xi)v_2(\xi)+v_2'(\xi))}{C(\lambda)}u_1(x)d\xi\Big] \\ =\frac{1}{2\pi i}\int_0^x F(\xi)&\Big(\oint\frac{u_2(x)[p(\xi)v_1(\xi)+v_1'(\xi)]}{C(\lambda)}d\lambda\Big)d\xi \\ +\frac{1}{2\pi i}\int_x^1 F(\xi)&\Big(\oint\frac{u_1(x)[p(\xi)v_2(\xi)+v_2'(\xi)]}{C(\lambda)}d\lambda\Big)d\xi\ .\end{aligned} \tag{19}$$

For $\lambda=\lambda_n$ the Wronskian vanishes, and hence the function $C(\lambda)$ has zeros. We may therefore write

$$\begin{aligned}&\frac{1}{2\pi i}\oint\frac{u_2(x)[p(\xi)v_1(\xi)+v_1'(\xi)]}{C(\lambda)}d\lambda \\ &\qquad=\sum u_2(x,\ \lambda_n)\frac{[p(\xi)v_1(\xi,\ \lambda_n)+v_1'(\xi,\ \lambda_n)]}{C'(\lambda_n)}\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\oint u_1(x)\frac{[p(\xi)v_2(\xi)+v_2'(\xi)]}{C(\lambda)}d\lambda \\ &\qquad=\sum u_1(x,\ \lambda_n)\frac{[p(\xi)v_2(\xi,\ \lambda_n)+v_2'(\xi,\ \lambda_n)]}{C'(\lambda_n)}\ ,\end{aligned} \tag{21}$$

where the integrals have been evaluated by means of their residues at the zeros of $C(\lambda)$.

The vanishing of the Wronskian implies that

$$u_1(x)=k(\lambda)u_2(x)\ ,\qquad v_2(\xi)=\frac{v_1(\xi)}{k(\lambda)}\ ,\qquad v_2'(\xi)=\frac{v_1'(\xi)}{k(\lambda)}\ .$$

Using the relation above we rewrite (21) as

$$\begin{aligned}&\sum k(\lambda_n)u_2(x,\ \lambda_n)\int_x^1 F(\xi)\Big\{\frac{p(\xi)v_1(\xi,\ \lambda_n)+v_1'(\xi,\ \lambda_n)}{C'(\lambda_n)k(\lambda_n)}\Big\}d\xi \\ &\qquad=\sum u_2(x,\ \lambda_n)\int_x^1 F(\xi)\Big\{\frac{p(\xi)v_1(\xi,\ \lambda_n)+v_1'(\xi,\ \lambda_n)}{C'(\lambda_n)}\Big\}d\xi\ .\end{aligned} \tag{22}$$

Using (20) and (22) and combining the two integrals we obtain the desired expansion of (19), namely,

$$\sum u_2(x, \lambda_n)\int_0^1 F(\xi)\left\{\frac{p(\xi)v_1(\xi, \lambda_n)+v_1'(\xi, \lambda_n)}{C'(\lambda_n)}\right\}d\xi . \tag{23}$$

Using the relations obtained from the vanishing of the Wronskian we may write (23) as

$$\sum u_1(x, \lambda_n)\int_0^1 F(\xi)\left\{\frac{p(\xi)v_2(\xi, \lambda_n)+v_2'(\xi, \lambda_n)}{C'(\lambda_n)}\right\}d\xi . \tag{24}$$

3. Asymptotic evaluations. In this section asymptotic forms will be derived for the quantities u_1, u_2, $C(\lambda)$, pv_1+v_1', and pv_2+v_2' which appear in (19). These forms will be used in the section following this to show that the value of the integral (19) taken over a large contour in the λ plane is $F(x)$ in the interval $0<x<1$, if $F(x)$ satisfies certain conditions.

In equation (16) we make the substitution $u_1=e^{\lambda x/2}w_1$. Then (16) becomes

$$w_1''+\left(q+\lambda p-\frac{\lambda^2}{4}\right)w_1=0 .$$

Write this as follows:

$$\begin{aligned} w_1''-\left[\frac{1}{4}(\lambda-2p)^2+\frac{p''}{\lambda-2p}+\frac{3p'^2}{(\lambda-2p)^2}\right]w_1 &= -\left[p^2+\frac{p''}{\lambda-2p}+\frac{3p'^2}{(\lambda-2p)^2}+q\right]w_1 \\ &= -g(x)w_1 . \end{aligned} \tag{25}$$

Note that $g(x)$ is bounded as $|\lambda|\to\infty$.

It can now be easily verified that the solution of (25) satisfies the equation

$$w_1=\frac{2\sinh\left\{\frac{\lambda x}{2}-P(0, x)\right\}}{\sqrt{\lambda-2p(x)}\ \sqrt{\lambda-2p(0)}}-\int_0^x\frac{2\sinh\left\{\frac{\lambda}{2}(x-\xi)-P(\xi, x)\right\}g(\xi)w_1(\xi)d\xi}{\sqrt{\lambda-2p(x)}\ \sqrt{\lambda-2p(\xi)}} , \tag{26}$$

$$w_1(0)=0 , \qquad w_1'(0)=1 .$$

In (26) we make the substitution

$$w_1=1/\lambda \exp\left[|\sigma|x/2\right]Z_1(x) ,$$

where σ is the real part of λ. Then we obtain

$$Z_1(x)=\frac{\lambda\left\{\exp\left[-\frac{1}{2}(|\sigma|-\lambda)x-P(0,x)\right]-\exp\left[-\frac{1}{2}(|\sigma|+\lambda)x+P(0,x)\right]\right\}}{\sqrt{\lambda-2p(x)}\sqrt{\lambda-2p(0)}}$$

$$-\frac{\lambda}{\lambda}\int_0^x\frac{\exp\left[-\frac{1}{2}(|\sigma|-\lambda)(x-\xi)-P(\xi,x)\right]-\exp\left[-\frac{1}{2}(|\sigma|+\lambda)(x-\xi)+P(\xi,x)\right]}{\sqrt{\lambda-2p(x)}\sqrt{\lambda-2p(\xi)}}$$

$$\cdot g(\xi)Z_1(\xi)d\xi$$

Now

$$\frac{\lambda}{\sqrt{\lambda-2p(x)}\ \sqrt{\lambda-2p(0)}}\quad\text{and}\quad\frac{\lambda}{\sqrt{\lambda-2p(x)}\ \sqrt{\lambda-2p(\xi)}}$$

are both bounded as $|\lambda|\to\infty$ by some constant C_1, say. Also

$$\left|\exp\left[-\frac{1}{2}(|\sigma|-\lambda)(x-\xi)-P(\xi,x)\right]-\exp\left[-\frac{1}{2}(|\sigma|+\lambda)(x-\xi)+P(\xi,x)\right]\right|$$

and

$$\left|\exp\left[-\frac{1}{2}(|\sigma|-\lambda)x-P(0,x)\right]-\exp\left[-\frac{1}{2}(|\sigma|+\lambda)x+P(0,x)\right]\right|$$

are obviously bounded by some constant C_2, say. Consequently

$$|Z_1(x)|\leqq C_1C_2+\frac{C_1C_2}{|\lambda|}\int_0^x|g(\xi)|\ |Z_1(\xi)|\,d\xi\,.$$

Let μ be the maximum value of $Z_1(x)$ in the interval $0<x<1$. Then

$$\mu\leqq C_1C_2+\frac{C_1C_2}{|\lambda|}\mu\int_0^x|g(\xi)|\,d\xi\,,$$

and hence

$$\mu\leqq\frac{C_1C_2}{1-\dfrac{C_1C_2}{|\lambda|}\displaystyle\int_0^x|g(\xi)|\,d\xi}\,.$$

Therefore μ, and consequently $Z_1(x)$, is bounded as $|\lambda|\to\infty$.

From (26), then,

$$w_1=\frac{2\sinh\left\{\frac{\lambda x}{2}-P(0,x)\right\}}{\sqrt{\lambda-2p(x)}\ \sqrt{\lambda-2p(0)}}$$

$$-\frac{1}{\lambda}\int_0^x \frac{\exp\left[\frac{1}{2}\lambda(x-\xi)+\frac{1}{2}|\sigma|\xi-P(\xi,x)\right]-\exp\left[-\frac{1}{2}\lambda(x-\xi)+\frac{1}{2}|\sigma|\xi+P(\xi,x)\right]}{\sqrt{\lambda-2p(x)}\sqrt{\lambda-2p(\xi)}}$$

$$\cdot g(\xi)Z_1(\xi)d\xi$$

Now, the second term on the right-hand side of this equation is equal to

$$-\exp\left[\frac{1}{2}|\sigma|x\right]$$

$$\cdot\int_0^x \frac{\exp\left[\frac{1}{2}(\lambda-|\sigma|)(x-\xi)-P(\xi,x)\right]-\exp\left[-\frac{1}{2}(\lambda+|\sigma|)(x-\xi)+P(\xi,x)\right]}{\lambda\sqrt{\lambda-2p(x)}\sqrt{\lambda-2d(\xi)}}$$

$$\cdot g(\xi)Z_1(\xi)d\xi$$

$$=O\left(\frac{\exp\left[\frac{1}{2}|\sigma|x\right]}{\lambda^2}\right)$$

Also

$$\left(\sqrt{\lambda-2p(x)}\quad\sqrt{\lambda-2p(0)}\right)^{-1}=\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right),$$

and therefore we have in the interval $0<x<1$

$$w_1(x)=\frac{2\sinh\left\{\frac{\lambda x}{2}-P(0,x)\right\}}{\lambda}+O\left(\frac{\exp\left[\frac{1}{2}|\sigma|x\right]}{\lambda^2}\right).$$

Similarly, substituting $u_2=\exp\left[\frac{\lambda}{2}(x-1)\right]w_2$ into (17) yields the equation

$$w_2(x)=\frac{2\sinh\left\{\frac{\lambda}{2}(x-1)-P(1,x)\right\}}{\sqrt{\lambda-2p(x)}\quad\sqrt{\lambda-2p(0)}}$$

$$-\int_1^x\frac{2\sinh\left\{\frac{\lambda}{2}(x-\xi)-P(\xi,x)\right\}g(\xi)w_2(\xi)d\xi}{\sqrt{\lambda-2p(x)}\quad\sqrt{\lambda-2p(\xi)}}$$

$$w_2(1)=0,\qquad w_2'(1)=1\,.$$

In this case we let

$$w_2=\frac{\exp\left[(1-x)|\sigma|/2\right]}{\lambda}Z_2(x)\,,$$

and by arguments essentially the same as those previously given $Z_2(x)$ can easily be shown to be bounded. Finally, using arguments similar to those given for $w_1(x)$ we obtain

$$w_2(x)=\frac{2\sinh\left\{\frac{\lambda}{2}(x-1)-P(1,x)\right\}}{\lambda}+O\left(\frac{\exp\left[\frac{1}{2}(1-x)|\sigma|\right]}{\lambda^2}\right).$$

Now

$$w_1'(x)=\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(0)}}\cosh\left\{\frac{\lambda x}{2}-P(0,x)\right\}+\frac{2p'(x)\sinh\left\{\frac{\lambda x}{2}-P(0,x)\right\}}{\sqrt{\lambda-2p(0)}\ (\lambda-2p(x))^{3/2}}$$

$$-\int_0^x\left\{\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(\xi)}}\cosh\left\{\frac{\lambda}{2}(x-\xi)-P(\xi,x)\right\}\right.$$

$$\left.+\frac{2p'(x)\sinh\left\{\frac{\lambda}{2}(x-\xi)-P(\xi,x)\right\}}{\sqrt{\lambda-2p(\xi)}\ (\lambda-2p(x))^{3/2}}\right\}w_1(\xi)g(\xi)d\xi .$$

Since we have already shown that $w_1(\xi)=\frac{\exp[|\sigma|\xi/2]}{\lambda}Z_1(\xi)$, where $Z_1(\xi)$ is bounded, it is quite easy to show that

$$w_1'(x)=\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(0)}}\cosh\left\{\frac{\lambda x}{2}-P(0,x)\right\}+O\left(\frac{\exp\left[\frac{1}{2}|\sigma|x\right]}{\lambda}\right).$$

Since

$$\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(0)}}=\frac{\lambda-2p(x)}{\sqrt{\lambda-2p(x)}\ \sqrt{\lambda-2p(0)}}=\frac{\lambda-2p}{\lambda}+O\left(\frac{1}{\lambda}\right),$$

where p is the minimum value of $p(x)$, we have

$$w_1'(x)=\frac{\lambda-2p}{\lambda}\cosh\left\{\frac{\lambda}{2}x-P(0,x)\right\}+O\left(\frac{\exp\left[\frac{1}{2}|\sigma|x\right]}{\lambda}\right).$$

By similar reasoning,

$$w_2'(x)=\frac{\lambda-2p}{\lambda}\cosh\left\{\frac{\lambda}{2}(x-1)-P(1,x)\right\}+O\left(\frac{\exp\left[\frac{1}{2}(1-x)|\sigma|\right]}{\lambda}\right).$$

Using the fact that

$$u_1'(x)=w_1'(x)\exp\left[\frac{\lambda}{2}x\right]+\frac{\lambda}{2}\exp\left[\frac{\lambda}{2}x\right]w_1(x),$$

and

$$u_2'(x)=w_2'(x)\exp\left[\frac{\lambda}{2}(x-1)\right]+\frac{\lambda}{2}\exp\left[\frac{\lambda}{2}(x-1)\right]w_2(x)\,,$$

we obtain for $\mathscr{R}\lambda>0$

$$u_1=\frac{1}{\lambda}\{\exp[\lambda x-P(0,x)]-\exp[P(0,x)]\}+O\left(\frac{e^{\lambda x}}{\lambda^2}\right)$$

$$u_2=\frac{1}{\lambda}\{\exp[\lambda(x-1)-P(1,x)-\exp[P(1,x)]\}+O\left(\frac{1}{\lambda^2}\right)$$

$$u_1'=\frac{1}{\lambda}\{(\lambda-p)\exp[\lambda x-P(0,x)]-p\exp[P(0,x)]\}+O\left(\frac{e^{\lambda x}}{\lambda}\right)$$

$$u_2'=\frac{1}{\lambda}\{(\lambda-p)\exp[\lambda(x-1)-P(1,x)]-p\exp[P(1,x)]\}+O\left(\frac{1}{\lambda}\right).$$

Also, the Wronskian $\omega(x)$ is equal to

$$\begin{aligned}
&u_2'u_1-u_1'u_2\\
&=\frac{1}{\lambda^2}\{(\lambda-p)\exp[\lambda(2x-1)-P(1,x)-P(0,x)]-(\lambda-p)\exp[\lambda(x-1)+P(0,1)]\\
&\quad-p\exp[\lambda x-P(0,1)]+p\exp[P(0,x)+P(1,x)]\\
&\quad-(\lambda-p)\exp[\lambda(2x-1)-P(1,x)-P(0,x)]+(\lambda-p)\exp[\lambda x-P(0,1)]\\
&\quad+p\exp[\lambda(x-1)+P(0,1)]-p\exp[P(0,x)+P(1,x)]\}+O\left(\frac{e^{\lambda x}}{\lambda^2}\right)\\
&\quad=\frac{\lambda-2p}{\lambda^2}\{\exp[\lambda x-P(0,1)]-\exp[\lambda(x-1)+P(0,1)]\}+O\left(\frac{e^{\lambda x}}{\lambda^2}\right)\\
&\quad=\frac{1}{\lambda}\exp[\lambda x-P(0,1)]-\exp[\lambda(x-1)+P(0,1)]+O\left(\frac{e^{\lambda x}}{\lambda^2}\right).
\end{aligned}$$

Since $\omega(x)=C(\lambda)e^{\lambda x}$, we obtain

$$C(\lambda)=\frac{1}{\lambda}\{\exp[-P(0,1)]-\exp[-\lambda+P(0,1)]\}+O\left(\frac{1}{\lambda^2}\right).$$

For $\mathscr{R}\lambda<0$, we obtain

$$u_1=\frac{1}{\lambda}\{\exp[\lambda x-P(0,x)]-\exp[P(0,x)]\}+O\left(\frac{1}{\lambda^2}\right)$$

$$u_2=\frac{1}{\lambda}\{\exp[\lambda(x-1)-P(1,x)]-\exp[P(1,x)]\}+O\left(\frac{e^{\lambda(x-1)}}{\lambda^2}\right)$$

$$u_1'=\frac{1}{\lambda}\{(\lambda-p)\exp[\lambda x-P(0,x)]-p\exp[P(0,x)]\}+O\Big(\frac{1}{\lambda}\Big)$$

$$u_2'=\frac{1}{\lambda}\{(\lambda-p)\exp[\lambda(x-1)-P(1,x)]-p\exp[P(1,x)]\}+O\Big(\frac{e^{\lambda(x-1)}}{\lambda}\Big)$$

For $\mathscr{R}\lambda<0$ we also obtain, by an argument analogous to that given for $\mathscr{R}\lambda>0$,

$$\omega(x)=\frac{1}{\lambda}\{\exp[\lambda x-P(0,1)]-\exp[\lambda(x-1)+P(0,1)]\}+O\Big(\frac{e^{\lambda(x-1)}}{\lambda}\Big),$$

and

$$C(\lambda)=\frac{1}{\lambda}\{\exp[-P(0,1)]-\exp[-\lambda+P(0,1)]\}+O\Big(\frac{e^{-\lambda}}{\lambda^2}\Big).$$

If we make the substitution $u(x)=e^{\lambda x}v(x)$ in equation (15) we obtain our adjoint differential equation (18). Consequently

$$\psi_1(x)\equiv p(x)v_1(x)+v_1'(x)=p(x)e^{-\lambda x}u_1(x)+e^{-\lambda x}u_1'(x)-\lambda e^{-\lambda x}u_1(x)$$
$$=e^{-\lambda x}\{(p(x)-\lambda)u_1(x)+u_1'(x)\}\ .$$

Now

$$u_1(x)=\exp\Big(\frac{\lambda x}{2}\Big)w_1(x)\ .$$

$$u_1'(x)=\frac{\lambda}{2}\exp\Big(\frac{\lambda x}{2}\Big)w_1(x)+\exp\Big(\frac{\lambda x}{2}\Big)w_1'(x)\ ,$$

and therefore,

$$\psi_1(x)=e^{-\frac{\lambda x}{2}}\Big\{\Big(p(x)-\frac{\lambda}{2}\Big)w_1(x)+w_1'(x)\Big\}\ ,$$

$$\Big(p-\frac{\lambda}{2}\Big)w_1+w_1'=\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(0)}}\Big\{\cosh\Big[\frac{\lambda x}{2}-P(0,x)\Big]-\sinh\Big[\frac{\lambda x}{2}-P(0,x)\Big]\Big\}$$
$$+\frac{2p'(x)\sinh\Big[\frac{\lambda x}{2}-P(0,x)\Big]}{\sqrt{\lambda-2p(0)}\ (\lambda-2p(x))^{3/2}}$$
$$-\int_0^x\frac{2p'(\xi)\sinh\Big[\frac{\lambda}{2}(x-\xi)-P(\xi,x)\Big]w_1(\xi)g(\xi)d\xi}{\sqrt{\lambda-2p(\xi)}\ (\lambda-2p(x))^{3/2}}$$
$$-\int_0^x\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(\xi)}}\Big\{\cosh\Big[\frac{\lambda}{2}(x-\lambda)-P(\xi,x)\Big]$$

$$-\sinh\left[\frac{\lambda}{2}(x-\xi)-P(\xi,x)\right]\Big\}\, g(\xi)w_1(\xi)d\xi$$

$$=\frac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(0)}}\exp\left[-\frac{\lambda x}{2}+P(0,x)\right]+\frac{2p'(x)\sinh\left[\frac{\lambda x}{2}-P(0,x)\right]}{\sqrt{\lambda-2p(0)}\,(\lambda-2p(x))^{3/2}}$$

$$-\int_0^x\frac{2p'(\xi)\left\{\exp\left[\frac{\lambda x}{2}-P(\xi,x)+\frac{\xi}{2}(|\sigma|-\lambda)\right]-\exp\left[-\frac{\lambda x}{2}+P(\xi,x)+\frac{\xi}{2}(|\sigma|+\lambda)\right]\right\}}{\sqrt{\lambda-2p(\xi)}\,(\lambda-2p(x))^{3/2}}$$

$$\cdot g(\xi)Z_1(\xi)d\xi$$

$$-\int_0^x\frac{\sqrt{\lambda-2p(x)}}{\lambda\sqrt{\lambda-2p(\xi)}}\exp\left[-\frac{\lambda x}{2}+P(\xi,x)\right]\exp\left[\frac{\xi}{2}(\lambda+|\sigma|)\right]d(\xi)Z_1(\xi)d\xi$$

It is evident that in the above equation, the expressions

$$\frac{2p'(x)\sinh\left\{\frac{\lambda x}{2}-P(0,x)\right\}}{\sqrt{\lambda-2p(0)}\,(\lambda-2p(x))^{3/2}}$$

and

$$\int_0^x\frac{2p'(\xi)\left\{\exp\left[\frac{\lambda}{2}x-P(\xi,x)+\frac{\xi}{2}(|\sigma|-\lambda)\right]-\exp\left[-\frac{\lambda}{2}x+P(\xi,x)+\frac{\xi}{2}(|\sigma|+\lambda)\right]\right\}}{\lambda\sqrt{\lambda-2p(\xi)}\,(\lambda-2p(x))^{3/2}}$$

$$\cdot g(\xi)Z_1(\xi)d\xi$$

are both at least of order $\dfrac{\exp[|\sigma|x/2]}{\lambda^2}$.

For $\mathscr{R}\lambda>0$

$$\int_0^x\frac{\sqrt{\lambda-2p(x)}}{\lambda\sqrt{\lambda-2p(\xi)}}\exp\left[-\frac{\lambda}{2}x+P(\xi,x)\right]\exp\left[\frac{\xi}{2}(\lambda+|\sigma|)\right]g(\xi)Z_1(\xi)d\xi$$

$$=O\left(\frac{\exp\left[-\frac{\lambda x}{2}\right]}{\lambda}\int_0^x e^{\lambda\xi}\,d\xi\right)=O\left(\frac{\exp\left[\frac{\lambda x}{2}\right]}{\lambda^2}\right),$$

while for $\mathscr{R}\lambda<0$ this expression is of order $\dfrac{\exp[-\lambda x/2]}{\lambda}$.

Using the fact that $\dfrac{\sqrt{\lambda-2p(x)}}{\sqrt{\lambda-2p(0)}}=1+O\left(\dfrac{1}{\lambda}\right)$, we conclude that

$$\psi_1(x)=\exp[-\lambda x+P(0,x)]+O\left(\frac{e^{-\lambda x}}{\lambda}\right)+O\left(\frac{1}{\lambda^2}\right),\qquad \mathscr{R}\lambda>0,$$

and

$$\psi_1(x)=\exp[-\lambda x+P(0, x)]+O\left(\frac{e^{-\lambda x}}{\lambda}\right), \qquad \mathscr{R}\lambda<0 .$$

We have also

$$v_2(x)=e^{-\lambda x}u_2(x)=e^{-\lambda x}\exp\left[\frac{\lambda}{2}(x-1)\right]w_2(x)=\exp\left[-\frac{\lambda}{2}(x+1)\right]w_2(x),$$

and, therefore,

$$\psi_2(x)=pv_2+v_2'=\exp\left[-\frac{\lambda}{2}(x+1)\right]\left\{\left(p-\frac{\lambda}{2}\right)w_2(x)+w_2'(x)\right\} .$$

In this case we find by arguments similar to those given for $\psi_1(x)$ that for $\mathscr{R}\lambda>0$

$$\psi_2(x)=\exp[-\lambda x+P(1, x)]+O\left(\frac{e^{-\lambda x}}{\lambda}\right)$$

and for $\mathscr{R}\lambda<0$,

$$\psi_2(x)=\exp[-\lambda x+P(1, x)]+O\left(\frac{e^{-\lambda x}}{\lambda}\right)+O\left(\frac{e^{-\lambda}}{\lambda^2}\right).$$

4. Proof of the expansion theorem. We have already seen that the integral (19), taken over a contour in the λ-plane enclosing the eigenvalues of the system $(A+\lambda B)u=0$, $u(0)=u(1)=0$, is equal to an expansion of the form (23). We shall now show that this integral, taken over a circle whose radius tends to infinity in the λ-plane[2)], tends to $F(x)$, provided $F(x)$ satisfies certain conditions. It is evident that this circle is a contour of the sort described above.

A precise statement of the theorem we shall prove is as follows.

THEOREM. *Let $F(x)$ be a function of bounded variation for $0\leq x\leq 1$ and let $u_n(x)$ be the eigenfunctions of the system $(A+\lambda B)u=0$, $u(0)=u(1)=0$, where A is the operator $d^2/dx^2+q(x)$ and where B is the operator $-d/dx+p(x)$. Furthermore, let $v_n(x)$ be the eigenfunctions of the system adjoint to $(A+\lambda B)u=0$, $u(0)=u(1)=0$; and let $C(\lambda)e^{\lambda x}$ be the Wronskian of the equation $(A+\lambda B)u=0$. If*

2) We require, of course, that our contour does not intersect the eigenvalues of the system. Since the eigenvalues are discrete, it is always possible to choose such a contour. In fact from the form of $C(\lambda)$ we see that the large eigenvalues tend to $\lambda_n=2n\pi i+2P(0, 1)$; consequently if we define the radius of our n^{th} circle, R_n, as $(|\lambda_n|+|\lambda_n+1|)/2$, then for sufficiently large n our contour will not intersect any eigenvalues. In this manner we obtain our desired sequence of circles with radii tending to infinity as $n\to\infty$.

(27) $$F(0+)+\exp[-P(0,1)]F(1-)=0\,,$$

then the series

(28) $$\sum_{-\infty}^{\infty} u_n(x)\int_0^1 F(\xi)\left\{\frac{p(\xi)v_n(\xi)+v_n'(\xi)}{C(\lambda_n)}\right\}d\xi$$

converges to $F(x)$ *at every point where* $F(x)$ *is continuous in* $0<x<1$. *At all other points the series* (28) *converges to* $\frac{1}{2}F(x+0)+\frac{1}{2}F(x-0)$. *If* $F(x)$ *does not satisfy the boundary condition* (27), *then the series* (28) *converges to*

$$\frac{1}{2}F(x+0)+\frac{1}{2}F(x-0)-\frac{1}{2}\cdot\exp[P(0,x)]\{F(0+)+\exp[-P(0,1)]F(1-)\}\,.$$

Using the notation of the previous section we may write the integral (19) in the form

$$\frac{1}{2\pi i}\oint\frac{u_2(x)}{C(\lambda)}\int_0^x F(\xi)\psi_1(\xi)d\xi d\lambda+\frac{1}{2\pi i}\oint\frac{u_1(x)}{C(\lambda)}\int_x^1 F(\xi)\psi_2(\xi)d\xi d\lambda\,.$$

Denote the first integrand by $\gamma(x,\lambda)$, and the second integrand by $\phi(x,\lambda)$. For $\mathscr{R}\lambda>0$ we find from our previously developed forms that

$$\gamma(x,\lambda)=\left\{\frac{\exp[\lambda(x-1)-P(1,x)]-\exp[P(1,x)]+O(1/\lambda)}{\exp[-P(0,1)]-\exp[-\lambda+P(0,1)]+O(1/\lambda)}\right\}\cdot$$
$$\cdot\int_0^x F(\xi)\left\{\exp[-\lambda\xi+P(0,\xi)]+O\left(\frac{e^{-\lambda\xi}}{\lambda}\right)+O\left(\frac{1}{\lambda^2}\right)\right\}d\xi$$

It is quite easily seen that in the term

(29) $$\frac{\exp[\lambda(x-1)-P(1,x)]-\exp[P(1,x)]+O(1/\lambda)}{\exp[-P(0,1)]-\exp[-\lambda+P(0,1)]+O(1/\lambda)}\,,$$

we may immediately pass to the limit as $|\lambda|\to\infty$, since this limit exists. Also, the terms

$$\int_0^x F(\xi)O\left(\frac{e^{-\lambda\xi}}{\lambda}\right)d\xi=O\left(\frac{1}{\lambda^2}\right)$$

and

$$\int_0^x F(\xi)O\left(\frac{1}{\lambda^2}\right)d\xi=O\left(\frac{1}{\lambda^2}\right),$$

both can be seen to tend to zero when integrated over the semi-circle C_1 for which $\mathscr{R}\lambda>0$ and whose radius tends to infinity. Since the limit of (29) as $|\lambda|\to\infty$ is $-\exp[P(0,x)]$, we conclude that

(30) $$\frac{1}{2\pi i}\int_{C_1}\gamma(x,\lambda)d\lambda=\frac{-\exp[P(0,x)]}{2\pi i}\int_{C_1}\int_0^x\exp[-\lambda\xi+P(0,\xi)]F(\xi)d\xi d\lambda\,.$$

Now

$$\int_0^x e^{-\lambda\xi} \exp[P(0,\xi)]F(\xi)d\xi = F(0+)\int_0^x e^{-\lambda\xi}\exp[P(0,\xi)]d\xi - \int_0^x e^{-\lambda\xi} \cdot \exp[P(0,\xi)]\{F(0+)-F(\xi)\}d\xi .$$

Integrating by parts, we have

$$F(0+)\int_0^x e^{-\lambda\xi}\exp[P(0,\xi)]d\xi = F(0+)\Big\{\Big[-\frac{1}{\lambda}\exp[-\lambda+P(0,\xi)]\Big]_0^x + \frac{1}{\lambda}\int_0^x p(\xi)\exp[-\lambda+P(0,\xi)]d\xi\Big\}$$

$$= \frac{F(0+)}{\lambda} + O\Big(\frac{e^{-\lambda x}}{\lambda}\Big) + O\Big(\frac{1}{\lambda^2}\Big) .$$

The terms $O\Big(\frac{e^{-\lambda x}}{\lambda}\Big)$ and $O\Big(\frac{1}{\lambda^2}\Big)$ contribute zero when integrated over the contour, while

$$\int_{C_1} \frac{F(0+)}{\lambda}\, d\lambda = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} iF(0+)d\theta = \pi i F(0+) .$$

Since F is of bounded variation we can write $F(0+)-F(\xi)=h(\xi)-k(\xi)$, where $h(\xi)$ and $k(\xi)$ are positive, steadily increasing, and tend to zero as $\xi\to 0$. We write the integral from 0 to x as the sum of two integrals:

$$\int_0^x \exp[-\lambda\xi+P(0,\xi)][F(0+)-F(\xi)]d\xi = \int_0^x \exp[-\lambda\xi+P(0,\xi)](h(\xi)-k(\xi))d\xi$$

$$= \int_0^{\frac{1}{\sqrt{|\lambda|}}} \exp[-\lambda\xi+P(0,\xi)](h(\xi)-k(\xi))d\xi$$

$$+ \int_{\frac{1}{\sqrt{|\lambda|}}}^x \exp[-\lambda+P(0,\xi)](h(\xi)-k(\xi))d\xi .$$

Note that $0<\frac{1}{\sqrt{|\lambda|}}\leqq x$ as $|\lambda|\to\infty$.

Now

$$\int_{\frac{1}{\sqrt{|\lambda|}}}^x \exp[-\lambda\xi+P(0,\xi)]h(\xi)d\xi = O\Big(\frac{\exp[-|\lambda|x]}{|\lambda|}\Big) + O\Big(\frac{\exp[-\sqrt{|\lambda|}]}{|\lambda|}\Big),$$

and both of these order terms integrated over C_1 tend to zero. By the second mean-value theorem,

$$\int_0^{\frac{1}{\sqrt{|\lambda|}}} \mathscr{R} \exp[-\lambda\xi + P(0, \xi)]h(\xi)d\xi$$

$$= h\left(\frac{1}{\sqrt{|\lambda|}}\right) \mathscr{R} \int_\delta^{\frac{1}{\sqrt{|\lambda|}}} \exp[-\lambda\xi + P(0, \xi)]d\xi = O\left(\frac{h(1/\sqrt{|\lambda|})}{|\lambda|}\right)$$

where $0 \leqq \delta < \dfrac{1}{\sqrt{|\lambda|}}$.

Since $\lim_{\xi \to 0} h(\xi) = 0$, we see that this order term tends to zero when integrated over the contour. Similarly, the corresponding terms with $h(\xi)$ replaced by $k(\xi)$ tend to zero. Using these results together with equation (30) we have

$$\frac{1}{2\pi i}\int_{C_1} \gamma(x, \lambda)d\lambda = -\frac{1}{2}F(0+)\exp[P(0, x)], \qquad \mathscr{R}\lambda > 0 .$$

Consider $\gamma(x, \lambda)$ for $\mathscr{R}\lambda < 0$. We have

$$\gamma(x, \lambda) = \left\{ \frac{\exp[\lambda(x-1) - P(1, x)] - \exp[P(1, x)] + O\left(\frac{e^{\lambda(x-1)}}{\lambda}\right)}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1)] + O\left(\frac{e^{-\lambda}}{\lambda}\right)} \right\}$$

$$\cdot \int_0^x F(\xi)\left\{\exp[-\lambda\xi + P(0, \xi)] + O\left(\frac{e^{-\lambda\xi}}{\lambda}\right)\right\} d\xi .$$

Multiplying the numerator and demoninator of the term in braces by e^{λ}, factoring $e^{\lambda x}$ out of the resulting product and multiplying the integral term by $e^{\lambda x}$, we obtain

$$\gamma(x, \lambda) = \left\{ \frac{\exp[P(1, x)] - \exp[\lambda(1-x) + P(1, x)] + O(1/\lambda)}{\exp[\lambda - P(0, 1)] - \exp[P(0, 1)] + O(1/\lambda)} \right\}$$

$$\cdot \int_0^x F(\xi)\left\{\exp[\lambda(x-\xi) + P(0, \xi)] + O\left(\frac{e^{\lambda(x-\xi)}}{\lambda}\right) d\xi .\right.$$

Again, since $\mathscr{R}\lambda < 0$,

$$\lim_{|\lambda| \to \infty} \frac{\exp[-P(1, x)] - \exp[\lambda(1-x) + P(1, x)] + O(1/\lambda)}{\exp[\lambda - P(0,1)] - \exp[P(0, 1)] + O(1/\lambda)} = -\exp[P(0, x)] ,$$

and

$$\int_0^x F(\xi)O\left(\frac{e^{\lambda(x-\xi)}}{\lambda}\right)d\xi = O\left(\frac{1}{\lambda^2}\right),$$

so that

$$\frac{1}{2\pi i}\int_{C_2}\gamma(x,\lambda)d\lambda=\frac{-\exp[P(0,x)]}{2\pi i}\int_{C_1}d\lambda\int_0^x F(\xi)\exp[\lambda(x-\xi)+P(0,\xi)]d\xi\ ,$$

where C_2 is the semicircle for which $\mathscr{R}\lambda<0$ and whose radius tends to infinity. Write

$$\int_0^x F(\xi)\exp[\lambda(x-\xi)+P(0,\xi)]d\xi=F(x-0)\int_0^x e^{\lambda(x-\xi)}\exp[P(0,\xi)]d\xi$$

$$-\int_0^x e^{\lambda(x-\xi)}\exp[P(0,\xi)](F(x-0)-F(\xi))d\xi\ .$$

Integrating by parts we obtain

$$F(x-0)\int_0^x e^{\lambda(x-\xi)}\exp[P(0,\xi)]d\xi=F(x-0)\left\{\left[\frac{-e^{\lambda(x-\xi)}\exp[P(0,\xi)]}{\lambda}\right]_0^x\right.$$

$$\left.+\frac{1}{\lambda}\int_0^x p(\xi)\exp[\lambda(x-\xi)+P(0,\xi)]d\xi\right\}$$

$$=\frac{-F(x-0)\exp[P(0,x)]}{\lambda}+O\left(\frac{e^{\lambda x}}{\lambda}\right)+O\left(\frac{1}{\lambda^2}\right).$$

The terms $O\left(\frac{e^{\lambda x}}{\lambda}\right)$, and $O\left(\frac{1}{\lambda^2}\right)$ then give zero when integrated over the contour, while

$$\int_{C_1}\frac{-F(x-0)\exp[P(0,x)]}{\lambda}d\lambda=-\pi iF(x-0)\exp[P(0,x)]\ .$$

We again make use of the bounded variation of F to write $F(x-0)-F(\xi)=l(\xi)-m(\xi)$, where $l(\xi)$ and $m(\xi)$ are positive, steadily decreasing and tend to zero as ξ tends to x.

Proceeding as before we write

$$\int_0^x e^{\lambda(x-\xi)}\exp[P(0,\xi](F(x-0)-F(\xi))d\xi=\int_0^x e^{\lambda(x-\xi)}\exp[P(0,\xi)](l(\xi)-m(\xi))d\xi$$

$$=\int_0^{x-\frac{1}{\sqrt{|\lambda|}}} e^{\lambda(x-\xi)}\exp[P(0,\xi)](l(\xi)-m(\xi))d\xi$$

$$+\int_{x-\frac{1}{\sqrt{|\lambda|}}}^x e^{\lambda(x-\xi)}\exp[P(0,\xi)](l(\xi)-m(\xi))d\xi\ .$$

In this case

$$\int_0^{x-\frac{1}{\sqrt{|\lambda|}}} e^{\lambda(x-\xi)}\exp[P(0,\xi)]l(\xi)=O\left(\frac{e^{-|\lambda|^{1/2}}}{|\lambda|}\right)+O\left(\frac{\exp[-|\lambda|x]}{|\lambda|}\right),$$

and both of these order terms tend to zero when integrated over the contour, while by the second mean value theorem, this time applied to a monotonic decreasing function, we obtain

$$\int_{x-\frac{1}{\sqrt{|\lambda|}}}^{x} e^{\lambda(x-\xi)} \exp[P(0, \xi)]l(\xi)d\xi = O\left(\frac{\exp\left[x-\frac{1}{\sqrt{|\lambda|}}\right]}{\lambda}\right).$$

Since $l(\xi)$ tends to zero as ξ tends to x, the above order term also tends to zero upon integration over C_2. Consequently, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_2}\gamma(x,\lambda)d\lambda &= \frac{-\exp[-P(0, x)]}{2\pi i}\int_{C_2}\frac{-F(x-0)\exp[P(0,x)]}{\lambda}d\lambda \\ &= \frac{1}{2}F(x-0) \quad (\mathscr{R}\lambda<0) .\end{aligned}$$

Combining our results for $\mathscr{R}\lambda>0$ and $\mathscr{R}\lambda<0$, we obtain

$$(31)\qquad \frac{1}{2\pi i}\oint\gamma(x, \lambda)d\lambda = -\frac{1}{2}F(0+)\exp[P(0, x)] + \frac{1}{2}(F(x-0) .$$

Consider now $\frac{1}{2\pi i}\oint\phi(x, \lambda)d\lambda$. From our previously developed forms we obtain for $\mathscr{R}\lambda>0$

$$\phi(x, \lambda) = \left\{\frac{\exp[\lambda x - P(0, x)] - \exp[P(0, x)] + O(e^{\lambda\xi}/\lambda)}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1)] + O(1/\lambda)}\right\}$$
$$\cdot\int_x^1 F(\xi)\left\{\exp[-\lambda\xi + P(1, \xi)] + O\left(\frac{e^{\lambda\xi}}{\lambda}\right)\right\} d\xi .$$

Factoring $e^{\lambda x}$ out of the term in brackets and combining it in the integral term we obtain

$$\phi(x, \lambda) = \left\{\frac{\exp[-P(0, x)] - \exp[-\lambda x + P(0, x) + O(1/\lambda)}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1] + O(1/\lambda)}\right\}$$
$$\cdot\int_x^1 F(\xi)\left\{\exp[\lambda(x-\xi) + P(1, \xi)] + O\left(\frac{e^{\lambda(x-\xi)}}{\lambda}\right) d\xi .\right.$$

Since $\mathscr{R}\lambda>0$,

$$\lim_{|\lambda|\to\infty}\left\{\frac{\exp[-P(0, x)] - \exp[-\lambda x + P(0, x)] + O(1/\lambda)}{\exp[-P(0, 1)] - \exp[-\lambda + P(0, 1)] + O(1/\lambda)}\right\} = \exp[P(x, 1)] ,$$

and

$$\int_x^1 F(\xi)O\left(\frac{e^{\lambda(x-\xi)}}{\lambda}\right)d\xi = O\left(\frac{1}{\lambda^2}\right);$$

we conclude that

$$\frac{1}{2\pi i}\int_{C_1}\phi(x,\ \lambda)d\lambda=\frac{\exp[P(x,\ 1)]}{2\pi i}\int_{C_1}\int_x^1 F(\xi)\exp[\lambda(x-\xi)+P(1,\ \xi)]d\xi\, d\lambda\ .$$

Proceeding as before we write

$$\int_x^1 F(\xi)\exp[\lambda(x-\xi)+P(1,\ \xi)]d\xi=F(x+0)\int_x^1 e^{\lambda(x-\xi)}\exp[P(1,\ \xi)]d\xi$$

$$-\int_x^1\exp[\lambda(x-\xi)+P(1,\ \xi)](F(x+0)-F(\xi))d\xi\ ,$$

and again integrating by parts, we obtain

$$F(x+0)\int_x^1 e^{\lambda(x-\xi)}\exp[P(1,\ \xi)]d\xi=F(x+0)\left\{\left[\frac{-1}{\lambda}e^{\lambda(x-\xi)}\exp[P(1,\ \xi)]\right]_x^1\right.$$

$$\left.+\frac{1}{\lambda}\int_x^1 p(\xi)\exp[\lambda(x-\xi)+P(1,\ \xi)]d\xi\right\}$$

$$=\frac{1}{\lambda}F(x+0)\exp[P(1,\ x)]+O\left(\frac{e^{\lambda(x-1)}}{\lambda}\right)+O\left(\frac{1}{\lambda^2}\right).$$

Again by arguments the same as those given twice before, the last two order terms both tend to zero when integrated over the contour. If we write

$$\int_x^1\exp[\lambda(x-\xi)+P(1,\ \xi)](F(x+0)-F(\xi))d\xi$$

$$=\int_x^{x+\frac{1}{\sqrt{|\lambda|}}}\exp[\lambda(x-\xi)+P(1,\ \xi)](F(x+0)-F(\xi))d\xi$$

$$+\int_{x+\frac{1}{\sqrt{|\lambda|}}}^1\exp[\lambda(x-\xi)+P(1,\ \xi)](F(x+0)-F(\xi))d\xi$$

then by arguments essentially the same as those given for the corresponding term in $\gamma(x,\ \lambda)$, the integrals above tend to zero when integrated over the contour. Hence we have

$$\frac{1}{2\pi i}\int_{C_1}\phi(x,\ \lambda)d\lambda=\frac{\exp[P(x,\ 1)]}{2\pi i}\int_{C_1}\frac{F(x+0)\exp[P(1,\ x)]}{\lambda}d\lambda=\frac{1}{2}F(x+0)\ ,$$

$$(\mathscr{R}\lambda>0)\ .$$

For $\mathscr{R}\lambda<0$ we obtain

$$\phi(x,\ \lambda)=\left\{\frac{\exp[\lambda x-P(0,\ x)]-\exp[P(0,\ x)]+O(1/\lambda)}{\exp[-P(0,\ 1)]-\exp[-\lambda+P(0,\ 1)]+O(e^{-\lambda}/\lambda)}\right\}$$

$$\cdot\int_x^1 F(\xi)\left\{\exp[-\lambda\xi+P(1,\ \xi)]+O\left(\frac{e^{-\lambda}}{\lambda}\right)+O\left(\frac{e^{-\lambda}}{\lambda^2}\right)\right\}d\xi\ .$$

Multiplying the denominator of the term in braces and the integral term by e^{λ} we obtain

$$\phi(x, \lambda)=\left\{\frac{\exp[\lambda x-P(0, x)]-\exp[P(0, x)]+O(1/\lambda)}{\exp[\lambda-P(0, x)]-\exp[P(0, x)]+O(1/\lambda)}\right\}$$

$$\cdot\int_x^1 F(\xi)\left\{\exp[\lambda(1-\xi)+P(1, \xi)]+O\left(\frac{e^{\lambda(1-\xi)}}{\lambda}\right)+O\left(\frac{1}{\lambda^2}\right)\right\}d\xi .$$

Since

$$\lim_{|\lambda|\to\infty}\left\{\frac{\exp[\lambda x-P(0, x)]-\exp[\lambda+P(0, x)]+O(1/\lambda)}{\exp[\lambda-P(0, 1)]-\exp[P(0, 1)]+O(1/\lambda)}\right\}=\exp[P(x, 1)] ,$$

$$\int_x^1 F(\xi)O\left(\frac{e^{\lambda(1-\xi)}}{\lambda}\right)d\xi=O\left(\frac{1}{\lambda^2}\right),$$

and

$$\int_x^1 F(\xi)O\left(\frac{1}{\lambda^2}\right)d\xi=O\left(\frac{1}{\lambda^2}\right),$$

we have

$$\frac{1}{2\pi i}\int_{C_2}\phi(x, \lambda)d\lambda=\frac{\exp[-P(x, 1)]}{2\pi i}\int_{C_2}\int_x^1 F(\xi)\exp[\lambda(1-\xi)+P(1, \xi)]d\xi\, d\lambda .$$

Again write

$$\int_x^1 F(\xi)\exp[\lambda(1-\xi)+P(1, \xi)]d\xi=F(1-)\int_x^1 e^{\lambda(1-\xi)}\exp[P(1, \xi)]d\xi$$

$$-\int_x^1 e^{\lambda(1-\xi)}\exp[P(1, \xi)](F(1-)-F(\xi))d\xi .$$

Integrating by parts we obtain

$$F(1-)\int_x^1 e^{\lambda(1-\xi)}\exp[P(1, \xi)]d\xi=F(1-)\left\{\left[\frac{-1}{\lambda}e^{\lambda(1-\xi)}\exp[P(1, \xi)]\right]_x^1\right.$$

$$\left.+\frac{1}{\lambda}\int_x^1 p(\xi)\exp[\lambda(1-\xi)+P(1, \xi)]d\xi\right\}$$

$$=\frac{-F(1-)}{\lambda}+O\left(\frac{e^{\lambda(1-x)}}{\lambda}\right)+O\left(\frac{1}{\lambda^2}\right).$$

As before, the last two order terms tend to zero when integrated over the contour, as well as the integrals

$$\int_x^1 e^{\lambda(1-\xi)}\exp[P(1, \xi)](F(1-)-F(\xi))d\xi$$

$$=\left[\int_x^{1-\frac{1}{\sqrt{|\lambda|}}}+\int_{1-\frac{1}{\sqrt{|\lambda|}}}^1\right]\exp[\lambda(1-\xi)+P(1,\xi)](F(1-)-F(\xi))d\xi .$$

Thus we have

$$\frac{1}{2\pi i}\int_{C_2}\phi(x,\lambda)d\lambda=\frac{-\exp[P(x,1)]}{2\pi i}\int_{C_2}\frac{F(1-)}{\lambda}d\lambda$$

$$=-\frac{1}{2}\exp[-P(x,1)]F(1-), \qquad (\mathscr{R}\lambda<0) .$$

Combining the results for $\mathscr{R}\lambda>0$ and for $\mathscr{R}\lambda<0$, we obtain

(32) $$\frac{1}{2\pi i}\oint\phi(x,\lambda)d\lambda=\frac{1}{2}F(x+0)-\frac{1}{2}\exp[-P(x,1)]F(1-) .$$

From (31) and (32) we obtain[3]

(33) $$\frac{1}{2\pi i}\oint\gamma(x,\lambda)d\lambda+\frac{1}{2\pi i}\oint\phi(x,\lambda)d\lambda=\frac{1}{2}F(x-0)+\frac{1}{2}F(x+0)$$

$$-\frac{1}{2}\exp[P(0,x)]F(0+)-\frac{1}{2}\exp[-P(x,1)]F(1-) .$$

But we have already shown from our residue expansion given in § 2 that (33) is equal to

$$\sum_{-\infty}^{\infty}u_2(x,\lambda_n)\int_0^1F(\xi)\left\{\frac{p(\xi)v_1(\xi,\lambda_n)+v_1'(\xi,\lambda_n)}{C'(\lambda_n)}\right\}d\xi$$

$$=\sum_{-\infty}^{\infty}u_1(x,\lambda_n)\int_0^1F(\xi)\left\{\frac{p(\xi)v_2(\xi,\lambda_n)+v_2'(\xi,\lambda_n)}{C'(\lambda_n)}\right\} .$$

We have, therefore,

(34) $$\sum_{-\infty}^{\infty}u_n(x)\int_0^1F(\xi)\left\{\frac{p(\xi)v_n(\xi)+v_n'(\xi)}{C'(\lambda_n)}\right\}d\xi$$

3) From the nature of the order terms and the fact that no singularities occur on the contour we see at once that we need only consider the cases $\mathscr{R}\lambda>0$ and $\mathscr{R}\lambda<0$. To be precise we imagine our circle in the λ plane to be made up of 6 parts: $-\frac{\pi}{2}\leqq\arg\lambda\leqq-\frac{\pi}{2}+\varepsilon$, $-\frac{\pi}{2}+\varepsilon\leqq\arg\lambda\leqq\frac{\pi}{2}-\varepsilon$, $\frac{\pi}{2}-\varepsilon\leqq\arg\lambda\leqq\frac{\pi}{2}$, $\frac{\pi}{2}\leqq\arg\lambda\leqq\frac{\pi}{2}+\varepsilon$, $\frac{\pi}{2}+\varepsilon\leqq\arg\lambda\leqq-\frac{\pi}{2}-\varepsilon$, and $-\frac{\pi}{2}-\varepsilon\leqq\arg\lambda\leqq-\frac{\pi}{2}$ (where $\varepsilon>0$). We have shown that as $\varepsilon\to0$, the integrals taken over those parts of the contour which tend to C_1 and C_2 tend to our desired result. The integrals over the remaining parts of the contour certainly tend to zero as $\varepsilon\to0$, since the order terms are the same as in the corresponding cases we have considered, and they also tend to zero over the parts of the contour that we have not considered. The contributing terms in each case are of order $B\varepsilon$, where B is a bounded function and hence these terms tend to zero as $\varepsilon\to0$.

$$=\frac{1}{2}F(x-0)+\frac{1}{2}F(x+0)$$

$$-\frac{1}{2}\{\exp[P(0,x)]F(0+)+\exp[-P(x,1)]F(1-)\}$$

$$=\frac{1}{2}F(x-0)+\frac{1}{2}F(x+0)$$

$$-\frac{1}{2}\exp[P(0,x)]\{F(0+)+\exp[-P(0,1)]F(1-)\}\ .$$

Hence, if the boundary condition (27) is satisfied, (34) converges to $\frac{1}{2}F(x-0)+\frac{1}{2}F(x+0)$. If $F(x)$ is continuous, then $F(x-0)=F(x+0)$ and we have

$$\sum_{-\infty}^{\infty} u_n(x)\int_0^1 F(\xi)\left\{\frac{p(\xi)v_n(\xi)+v_n'(\xi)}{C'(\lambda_n)}\right\}d\xi=F(x)\ . \tag{35}$$

Finally, since $C'(\lambda_n)$ is a constant for each value of n, we may define

$$\frac{v_n(\xi)}{C'(\lambda_n)}=V_n(\xi)\ ,$$

and (35) applied to the case $F(x)=u_n(x)$ shows that $\int_0^1 u_n B^* V_n dx=1$. Consequently

$$F(x)=\sum_{-\infty}^{\infty} u_n(x)\int_0^1 F(\xi)(p(\xi)V_n(\xi)+V_n'(\xi))d\xi\ ,$$

which is the form of the expansion indicated in the Introduction (equation (1.4)).

The authors wish to thank Mr. Bertram Levy for his assistance in preparing this paper for publication.

References

1. F. Browder, *On the eigenfunctions and eigenvalues of the general linear elliptic differential operator*, Proc. Nat. Acad. Sci U.S.A., **39** (1953), 433–439.
2. I. Kay, *Diffraction of an arbitrary pulse by a wedge*, Mathematics Research Group, New York University, Research Report No. EM–43.
3. R. E. Langer, *The expansion problem in the theory of ordinary differential systems of the second order*, Trans. Amer. Math. Soc., **31** (1929), 868–906.
4. E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations*, Oxford, Clarendon Press, 1946.

New York University