FAMILIES OF TRANSFORMATIONS IN THE FUNCTION SPACES H^p

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I. Introduction

Let the interior of the unit circle be denoted by Δ; and let the set of functions single-valued and analytic in Δ be denoted by $\mathfrak{A}$.

It is well known that certain subsets of $\mathfrak{A}$ can be made into Banach spaces by the introduction of suitable norms. In particular, if $f \in \mathfrak{A}$, and if, for $1 \leqq p \leqq \infty$,

$$\text{(I.1)} \qquad \mathscr{M}_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}, \qquad p < \infty$$

$$\mathscr{M}_p(f; r) = \sup_{|z|<r} |f(z)|, \qquad p = \infty$$

and if $\sup\limits_{r<1} \mathscr{M}_p(f; r) < \infty$, then f is said to be in the set H^p. Also, H^p is a Banach space with

$$\text{(I.2)} \qquad \|f\|_{H^p} = \sup_{r<1} \mathscr{M}_p(f; r)$$

A proof of these statements, together with a discussion of many properties of the spaces H^p, can be found in [8].

This paper is concerned with certain transformations in the spaces H^p [1].

Let $\omega(z)$ be a function of z which is analytic in Δ and such that $|\omega(z)| < 1$ for $z \in \Delta$. If $f \in \mathfrak{A}$, then so is the function defined by $f[\omega(z)]$. For $f \in \mathfrak{A}$, we define

$$\text{(I.3)} \qquad T_\omega f = g \underset{df}{\Longleftrightarrow} f[\omega(z)] = g(z) \text{ for } z \in \Delta.$$

T_ω is clearly an additive, homogeneous transformation.

It is well known [4] that if $f \in H^p$ and $\omega(0) = 0$, then $T_\omega f \in H^p$ and $\|T_\omega f\| \leqq \|f\|$. In other words, if $\omega(0) = 0$, then $T_\omega \in [H^p]$ (the set of all linear bounded transformations on H^p to H^p), and $\|T_\omega\| \leqq 1$. Our first task is to prove the following.

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[1] In the following, all statements about H^p refer to $1 \leqq p \leqq \infty$ unless further qualified.

THEOREM I.1. *If $\omega \in \mathfrak{A}$ and $|\omega(z)| < 1$ for $z \in \Delta$, and if $|\omega(0)| = \alpha < 1$, then $T_\omega \in [H^p]$ and $\|T_\omega\| \leqq \left(\frac{1+\alpha}{1-\alpha}\right)^{1/p}$ There is at least one such ω for which the equality holds.*

Proof. For $p=\infty$, the theorem is trivial. For $1 \leqq p < \infty$, a simple proof (for which the author is indebted to the referee) is as follows.

For $f \in H^p$, let u be the least harmonic majorant of $|f|^p$ in Δ (see [6]). Then $T_\omega u$ is a harmonic majorant of $|T_\omega f|^p$. Also,

$$\|f\| = \{u(0)\}^{1/p} \text{ and } \|T_\omega f\| \leqq \{(T_\omega u)(0)\}^{1/p} = \{u(\beta)\}^{1/p}$$

where $\beta = \omega(0)$. The Poisson integral for u shows that

$$u(\beta) \leqq u(0)\left(\frac{1+|\beta|}{1-|\beta|}\right)$$

Putting $\alpha = |\beta|$, it follows that

$$\|T_\omega f\| \leqq \|f\| \left(\frac{1+\alpha}{1-\alpha}\right)^{1/p}.$$

To complete the proof, we note that the following statement holds. Define the transformation L_α $(0 \leqq \alpha < 1)$ by

$$L_\alpha f(z) = f\left(\frac{z+\alpha}{1+\alpha z}\right).$$

Then the function

$$f(z) = \left(\frac{z+1}{z-1}\right)^\eta$$

is an eigenfunction of L_α: $L_\alpha f = \lambda f$, belonging to the eigenvalue

$$\lambda = \left(\frac{1+\alpha}{1-\alpha}\right)^\eta,$$

provided $|\Re\eta| < 1/p$. This follows trivially from the fact that $f \in H^p$ provided $|\Re\eta| < 1/p$.

The result stated in Theorem I.1 can be sharpened as follows.

COROLLARY I.1. *For any ω ($\omega \in \mathfrak{A}$, mapping Δ into or onto itself),*

$$\|T_\omega\| \leqq \inf_{\substack{\zeta \in \Delta \\ \eta \in \Delta}} \left\{\left(\frac{1+|\zeta|}{1-|\zeta|}\right)\left(\frac{1+|\eta|}{1-|\eta|}\right)\left(\frac{1+|\Gamma_\omega(\eta, \zeta)|}{1-|\Gamma_\omega(\eta, \zeta)|}\right)\right\}^{1/p} \tag{I.4}$$

where

$$\Gamma_\omega(\eta, \zeta)=\frac{\omega(\eta)+\zeta}{1+\bar{\zeta}\omega(\eta)}$$

Proof. For $\zeta \in \Delta$, define L_ζ by

$$L_\zeta f(z)=f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right)$$

Then

$$T_\omega=L_{-\eta}L_\eta T_\omega L_\zeta L_{-\zeta}$$

where

$$\eta \in \Delta, \ \zeta \in \Delta$$

so that

$$\| T_\omega \| \leqq \| L_{-\eta} \| \ \| L_{-\zeta} \| \ \| L_\eta T_\omega L_\zeta \|$$

Now, $\dfrac{z-\zeta}{1-\bar{\zeta}z}$ takes 0 into $-\zeta$; $\dfrac{z-\eta}{1-\bar{\eta}z}$ takes 0 into $-\eta$;

and $\omega\left(\dfrac{z+\eta}{1+\bar{\eta}z}\right)+\zeta \Big/ 1+\bar{\zeta}\omega\left(\dfrac{z+\eta}{1+\bar{\eta}z}\right)$ takes 0 into $\dfrac{\omega(\eta)+\zeta}{1+\bar{\zeta}\omega(\eta)}$

Applying Theorem I.1, we obtain (I.4).

We are thus assured that a transformation T_ω defined by $T_\omega f(z) = f[\omega(z)]$ is a member of $[H^p]$, $1 \leqq p \leqq \infty$. § II is devoted to a study of semigroups and groups of these transformations. Section III contains a discussion of two examples which illustrate the theorems of § II.

II. Families of Transformations in H^p

A. Definitions and preliminary results. Consider a family of functions $\{\omega(z;\ t)\}$ —also denoted by $\{\omega_t(z)\}$ —where $z \in \Delta$ and t belongs to a set $\mathscr{T}$ of complex numbers. The individual functions will be denoted by $\omega(z;\ t)$ or by $\omega_t(z)$, according to convenience.

Let the set $\mathscr{T}$ satisfy the following conditions.

(CII.1) (i) *If* $t_1,\ t_2 \in \mathscr{T}$, *then* $t_1+t_2 \in \mathscr{T}$.

(ii) $\mathscr{T}$ *contains the origin and some ray originating at the origin.*

(iii) *Every two points in* $\mathscr{T}$ *can be connected by a path*[2] *in* $\mathscr{T}$.

[2] Here a path is defined to mean a finite number of rectifiable Jordan arcs joined together; see [**3**, pp 13, 14].

Let the family $\{\omega(z; t)\}$ satisfy the following conditions:

(CII.2) (i) *For each* $t \in \mathcal{T}$, $\omega_t \in \mathfrak{A}$, *and* ω_t *maps* Δ *into* (*or onto*) *itself.*
(ii) *For* $t_1, t_2 \in \mathcal{T}$ *and* $z \in \Delta$,

$$\omega_{t_2}[\omega_{t_1}(z)] = \omega_{t_1}[\omega_{t_2}(z)] = \omega_{t_1+t_2}(z)$$

(iii) $\omega(z; 0) = z$ *for* $z \in \Delta$.
(iv) *For each* $z \in \Delta$, $\omega(z; t)$ *is differentiable*[3] *with respect to* t *for* $t \in \mathcal{T}$. *Also, if*

$$P(z) = \frac{\partial}{\partial t}\omega(z; t)|_{t=0},$$

then $P \in \mathfrak{A}$.

We can immediately state the following.

LEMMA II.1. *For fixed* $z \in \Delta$,

$$\frac{\partial}{\partial t}[\omega(z; t)] = P[\omega(z; t)] \tag{II.1}$$

Proof. $\omega[\omega(z; t); h] = \omega(z; t+h)$ for $t, h \in \mathcal{T}$

Therefore

$$\frac{\omega(z; t+h) - \omega(z; t)}{h} = \frac{\omega[\omega(z; t); h] - \omega(z; t)}{h}$$

$$= \frac{\omega[\omega(z; t); h] - \omega[\omega(z; t); 0]}{h}$$

Letting $h \to 0$ (in $\mathcal{T}$), we obtain (II.1).

The family of transformations $\{T_{\omega_t}\}$ defined by (I.3) with $\omega = \omega_t$ will henceforth be denoted simply by $\{T_t\}$. This family forms a semi-group (possibly a group) of linear bounded transformations in the spaces H^p. (The boundedness is shown by Theorem I.1.)

We define the generator A of the family $\{T_t\}$ by

$$Af = \lim_{t \to 0} \frac{T_t f - f}{t}, \qquad f \in H^p \tag{II.2}$$

the limit taken in the strong sense in H^p. The domain of A, denoted

[3] Here and in the following, "differentiability with respect to t for $t \in \mathcal{T}$" implies that the difference quotient approaches the *same* limit no matter how t is approached (as long as the approach is made entirely in $\mathcal{T}$).

by $\mathscr{D}(A)$, is defined to be the subset of H^p for which the limit in (II.2) exists as $t \to 0$, $t \in \mathscr{T}$ (the limit to be the same for all modes of approach within $\mathscr{T}$ to 0).

It follows from (II.2) that, for $f \in \mathscr{D}(A)$, and each $z \in \Delta$,

$$Af(z) = \lim_{t\to 0} \frac{T_t f(z) - f(z)}{t} \tag{II·3}$$

This is true since, for fixed $z \in \Delta$, $f(z)$ is a bounded linear functional of f, [7].

Now

$$\begin{aligned} Af(z) &= \lim_{t\to 0} \frac{f[\omega(z;\ t)] - f(z)}{t} \\ &= \lim_{t\to 0} \frac{f[\omega(z;\ t)] - f[\omega(z;\ 0)]}{t} \\ &= \frac{\partial}{\partial t} f[\omega(z;\ t)]|_{t=0} = f'[\omega(z;\ t)] \frac{\partial}{\partial t}\omega(z;\ t)|_{t=0} \end{aligned}$$

or

$$Af(z) = P(z)f'(z) \qquad z \in \Delta,\ f \in \mathscr{D}(A) \tag{II.4}$$

It is thus clear that $\mathscr{D}(A)$ is contained in the subset of H^p consisting of those elements f for which $f'(z)P(z)$ defines an element of H^p.

B. Differentiability properties of the family $\{T_t\}$

THEOREM II.1. *Let f be in H^p, and t_0 be in $\mathscr{T}$; let $g(z) = P(z)f'(z)$ and suppose that*

(i) *There exists a neighborhood $\mathscr{N}_{t_0}$ of t_0 and a positive constant M such that every point t of $\mathscr{N}_{t_0}$ can be connected to t_0 by a polygonal line in $\mathscr{N}_{t_0} \cap \mathscr{T}$ of length $\leq M|t_0 - t|$;*

(ii) *$T_t g \in H^p$ for $t \in \mathscr{N}_{t_0} \cap \mathscr{T}$;*

(iii) *$\| T_t g - T_{t_0} g \| \to 0$ as $t \to t_0$ $(t \in \mathscr{T})$.*

Then, $T_t f$ is strongly differentiable with respect to t at t_0 and

$$\frac{d}{dt} T_t f|_{t=t_0} = T_{t_0} g\ . \tag{II.5}$$

Before giving the proof, the following formal derivation might be of interest

$$\lim_{t\to t_0}\frac{T_t f - T_{t_0} f}{t-t_0} = \lim_{s\to 0} T_{t_0}\left\{\frac{T_s f - f}{s}\right\} \qquad (s=t-t_0)$$

$$= T_{t_0} A f = T_{t_0} g$$

This is however not a rigorous proof, even when $f \in \mathscr{D}(A)$, since $s=t-t_0$ may not be in $\mathscr{T}$ for all $t \in \mathscr{N}_{t_0} \cap \mathscr{T}$.

A rigorous proof is as follows.

Let $f[\omega(z;\ t)]=h(z;\ t)$ and let

$$D(z;\ t;\ t_0) = \frac{h(z;\ t)-h(z;\ t_0)}{t-t_0} - T_{t_0} g(z) \tag{II.6}$$

If $z=re^{i\theta}$, and if $\frac{\partial}{\partial t} h(z;\ t)$ is denoted by $h_t(z;\ t)$, then, from (II.1),

$$D(z;\ t;\ t_0) = \frac{h(re^{i\theta};\ t)-h(re^{i\theta};\ t_0)}{t-t_0} - h_t(re^{i\theta};\ t_0)$$

$$= \frac{1}{t-t_0}\int_{t_0}^{t} [h_t(re^{i\theta};\ \tau) - h_t(re^{i\theta};\ t_0)]d\tau$$

where t is chosen in $\mathscr{N}_{t_0}$ and the integral is taken along a polygonal line in $\mathscr{N}_{t_0} \cap \mathscr{T}$ connecting t and t_0 and of length $\leqq M|t-t_0|$.

First suppose that $1 \leqq p < \infty$. Then

$$\mathscr{M}_p(D;\ r) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |D(re^{i\theta};\ t;\ t_0)|^p\, d\theta\right\}^{1/p} \tag{II.7}$$

$$= \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{t-t_0}\int_{t_0}^{t}[h_t(re^{i\theta};\ \tau)-h_t(re^{i\theta};\ t_0)]d\tau\right|^p d\theta\right\}^{1/p}$$

Let $\tau=\tau(s)$, $0 \leqq s \leqq 1$, $\tau(0)=t_0$, $\tau(1)=t$. Here s is a constant times the arc length. Then [4], [1]

$$\mathscr{M}_p(D;\ r) = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{t-t_0}\int_0^1 [h_t(re^{i\theta};\ \tau)-h_t(re^{i\theta};\ t_0)]\tau'(s)ds\right|^p d\theta\right\}^{1/p}$$

$$\leqq \frac{1}{|t-t_0|}\int_0^1 |\tau'(s)| \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|h_t(re^{i\theta};\ \tau)-h_t(re^{i\theta};\ t_0)\right|^p d\theta\right\}^{1/p}$$

Hence,

$$\|D\| = \left\|\frac{T_t f - T_{t_0} f}{t-t_0} - T_{t_0} g\right\| = \sup_{r<1} \mathscr{M}_p(D;\ r)$$

$$\leq \frac{1}{|t-t_0|}\int_0^1 |\tau'(s)| \left[\sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|h_t(re^{i\theta};\tau)-h_t(re^{i\theta};t_0)\right|^p d\theta\right\}^{1/p}\right] ds$$

$$=\frac{1}{|t-t_0|}\int_0^1 |\tau'(s)|\ \| T_\tau g - T_{t_0} g \|\, ds \leq M \sup_{0\leq s\leq 1} \| T_\tau g - T_{t_0} g\|$$

Now, by (iii), as $t \to t_0$, the quantity $\sup_{0\leq s\leq 1} \| T_\tau g - T_{t_0} g\|$ goes to zero. Thus $\| D \| \to 0$ as $t \to t_0$.

For $p=\infty$, the proof follows similar lines.

COROLLARY II.1–1. *Let f be in H^p, t_0 be in $\mathscr{T}$ and let $g(z)=P(z)f'(z)$. Suppose condition* (i) *of Theorem* II.1 *holds and in addition, suppose that*

(a) $|\omega(z;\ t_0)| < r < 1$ *for* $z \in \Delta$

(b) $\omega(z;\ t)$ *is continuous with respect to t at t_0, uniformly in z for $z \in \Delta$.*

Then, $T_t f$ is differentiable with respect to t at t_0 and (II.5) *holds.*

Proof. By (b), there exists a neighborhood $\mathscr{N}_{t_0}'$ of t_0 such that $|\omega(z;\ t)| < r' < 1$ for $z \in \Delta$, $t \in \mathscr{N}_{t_0}' \cap \mathscr{T}$.

Now, $g(z)$ is analytic in Δ. Therefore for $t \in \mathscr{N}_{t_0}' \cap \mathscr{T}$, $T_t g(z) = g[\omega(z;\ t)]$ is bounded in Δ and therefore $T_t g \in H^p$.

Also, $T_t g(z)$ is continuous with respect to t at t_0, uniformly in z for $z \in \Delta$. Hence $\sup_{z\in\Delta} |T_t g(z) - T_{t_0} g(z)| \to 0$ as $t \to t_0$.

THEOREM II.2. *Suppose*

(i) *Condition* (i) *of Theorem* II.1 *holds for* $t_0=0$;

(ii) $\| T_t f - f\| \to 0$ *as* $t \to 0$ ($t \in \mathscr{T}$) *for every* $f \in H^p$.

Then, $\mathscr{D}(A)$, the domain of the generator A (defined by II.2), *is the set of elements $f \in H^p$ for which $g(z)=f'(z)P(z)$ defines an element g of H^p.*

Proof. Let $\mathscr{G}$ denote the set of elements $f \in H^p$ such that $g(z)=f'(z)P(z)$ defines an element g of H^p. We already know (last paragraph of IIA) that $\mathscr{D}(A) \subset \mathscr{G}$. To show that $\mathscr{G} \subset \mathscr{D}(A)$, one must verify conditions (ii) and (iii) of Theorem II.1 for $f \in \mathscr{G}$, $t_0=0$.

Since $f \in \mathscr{G}$ implies $g \in H^p$, it follows from Theorem I.1 that $T_t g \in H^p$ for all $t \in \mathscr{T}$. Also, condition (iii) of Theorem II.1 is obtained for $t_0=0$ by applying condition (ii) of Theorem II.2 to the function g. Equation (II.5) becomes

$$\text{(II.8)} \qquad Af = g \quad \text{where} \quad g(z) = P(z)f'(z) .$$

THEOREM II.3. *Under conditions* (i) *and* (ii) *of Theorem* II.2, *A is a closed transformation. Also* $\mathscr{D}(A)$ *is dense in* H^p.

Proof. Let f_n be in $\mathscr{D}(A)$; $f_n \to f$ (in the norm of H^p) $Af_n \to g \in H^p$ (in the norm of H^p). Then [7]

$$\left.\begin{array}{r} f_n(z) \to f(z) \\ P(z)f_n'(z) \to g(z) \end{array}\right\} \text{uniformly on compact subsets of } \Delta,$$

that is, $g(z) = P(z)f'(z)$ for $z \in \Delta$.

Therefore, since $g \in H^p$, then, by Theorem II.2, $f \in \mathscr{D}(A)$ and $Af = g$. See [2, Chap. 11] for the fact that $\mathscr{D}(A)$ is dense in H^p.

C. **The family of transformations generated by a given operator of the form** $Af(z) = P(z)f'(z)$. Suppose P is a given function in $\mathfrak{A}$. The following question arises: Is there a set $\mathscr{T}$ in the complex plane and a set of functions $\{\omega_t\}$ satisfying, respectively, conditions CII.1 and CII.2? If so, how, knowing just $P(z)$, can one determine the family $\{\omega_t\}$ and the maximum set $\mathscr{T}$?

To investigate these questions, additional conditions will be imposed on the given function $P(z)$. First,

(CII.3) $1/P(z)$ *is analytic in* Δ *except, possibly, for a single pole.*

Let the function $Q(z)$ be defined by

$$\text{(II.9)} \qquad Q(z) = \int_{z_0}^{z} \frac{d\zeta}{P(\zeta)} \qquad z_0,\ z \in \Delta$$

The path of integration is chosen in Δ so as not to pass through any singularity of $1/P(z)$; also, z_0 is chosen so as not to be a singularity of $1/P(z)$. $Q(z)$ may be a many-valued function.

$Q(z)$ depends on the choice of z_0; however, as will become clear below, it is not worthwhile to express this dependence in the notation. Clearly, all definitions of Q (corresponding to different choices of z_0) differ from each other by additive constants.

The following property of Q is worth noting.

Let z_1 and z_2 be in Δ, and not singularities of $1/P(z)$; let $Q^{(1)}(z_1)$, $Q^{(2)}(z_1)$ be two values of Q at $z = z_1$; and let $Q^{(1)}(z_1) - Q^{(2)}(z_1) = h$. Let $Q^{(1)}(z_2)$ be a value of Q at $z = z_2$. There exists a value of Q at $z = z_2$, which may be denoted by $Q^{(2)}(z_2)$, such that $Q^{(1)}(z_2) - Q^{(2)}(z_2) = h$. This is clear from the definition of Q and from (CII.3).

We shall further assume:

(CII.4) *If z_1 and z_2 are in Δ, are not singularities of $1/P(z)$, and $z_1 \neq z_2$, then $Q(z_1) \neq Q(z_2)$.*

This may, of course, be regarded as a condition on $P(z)$.

Now suppose $P \in \mathfrak{A}$ is given satisfying (CII.3) and (CII.4), and that a set $\mathscr{T}$ and a family $\{\omega_t\}$ exist satisfying (CII.1) and (CII.2). From (II.1) and (CII.2–iii), regarding z as fixed for the moment, one can write

$$\text{(II.10)} \quad \left.\begin{aligned} &\frac{d}{dt}\omega(z;\ t) = P[\omega(z;\ t)] \\ &\omega(z;\ 0) = z \end{aligned}\right\} \begin{aligned} &z \in \Delta \\ &t \in \mathscr{T} \end{aligned}$$

Let z be fixed in Δ and not a singularity of $1/P(z)$. Then, from (II.10), $\omega(z;\ t)$ must satisfy

$$\text{(II.11)} \quad Q[\omega(z;\ t)] = Q(z) + t\,.$$

Now, for fixed $t \in \mathscr{T}$, $\omega(z;\ t)$ must be an analytic function of z in Δ, mapping Δ into itself.

Let I_Q be the image under Q of Δ (excluding the possible singularity of $1/P(z)$. The set I_Q includes *all* values of $Q(z)$ which can be obtained by integrating in (II.9) along paths which are entirely in Δ. If $\omega(z;\ t)$, for fixed $t \in \mathscr{T}$, is defined for all $z \in \Delta$, and such that $|\omega(z;\ t)| < 1$, then (II.11) implies that this t must translate I_Q into a subset of itself: $I_Q + t \subset I_Q$.

Let $\mathscr{T}_Q$ be the set of translations of I_Q into or onto itself. (Clearly $\mathscr{T}_Q$ does not depend on the choice of z_0 in defining Q.) Then $\mathscr{T} \subset \mathscr{T}_Q$.

On the other hand if P being given[4], $\mathscr{T}_Q$ contains a subset $\mathscr{T}^*$ satisfying conditions (CII.1), then a family $\{\omega_t\}$ satisfying (CII.2) exists (with $t \in \mathscr{T}^*$).

Define, for $t \in \mathscr{T}^*$, $z \in \Delta$,

$$\text{(II.12)} \quad \omega(z;\ t) = \begin{cases} Q^{-1}[Q(z)+t], & z \text{ not a singularity of } \dfrac{1}{P(z)} \\ z, & z \text{ a singularity of } \dfrac{1}{P(z)} \end{cases}$$

where Q^{-1} denotes the function inverse to Q.

This definition defines ω uniquely. If $Q(z)$ refers to a particular branch of Q, then ω is uniquely determined (in Δ) because of (CII.4); moreover, by the property of Q mentioned on p. it is seen that the same point ω is defined no matter what branch of Q is used in (II : 12).

[4] $P \in \mathfrak{A}$ and satisfying (CII. 3) and (CII. 4).

It is also clear that $\omega(z; t)$ does not depend on the choice of z_0.

The function $\omega(z; t)$ thus defined is analytic in z for each $t \in \mathscr{T}^*$. This is clear if z is not a singularity of $1/P(z)$. If z_1 is a singularity of $1/P(z)$ in Δ, it is necessary to show that $\omega(z; t)$ is (for fixed t) continuous at $z=z_1$; that is, (from II.12) $\omega_t(z) \to z_1$ as $z \to z_1$.

Since z_1 is a pole of $1/P(z)$, one can say, by the definition of Q, that there exist points $\omega_t(z)$ approaching z_1 as $z \to z_1$, such that (II.12) is satisfied. But, by (CII.4), these points are the only ones in Δ for which (II.12) is satisfied.

The other conditions of (CII.2) are readily verified for the functions $\omega(z; t)$ as defined by (II.12).

The preceding results may be summed up as follows.

THEOREM 11.4. *Let $P(z)$ be in $\mathfrak{A}$, satisfying* (CII.3) *and* (CII.4). *Let $Q(z)$ be defined by* (II.9); *let I_Q be the image of Δ under Q, let $\mathscr{T}_Q$ be the set of translations of I_Q into or onto itself.*

Then, there exists a set $\mathscr{T}$ and a family $\{\omega_t\}$ satisfying (CII.1) *and* (CII.2), *if and only if $\mathscr{T}_Q$ contains a subset $\mathscr{T}^*$ satisfying* (CII.1). *The maximum set $\mathscr{T}$ is the "direct sum" of all subsets of $\mathscr{T}_Q$ which satisfy* (CII.1). *Here "direct sum" is defined as follows: If $\{G_\alpha\}$ is a collection of subsets of the complex plane, each containing the origin, the direcct sum of the sets $\{G^\alpha\}$ is defined to be the set consisting of all elements of the form $t=t_1+\cdots+t_n$ where n is a finite* (*positive*) *integer and where $t_i \in \bigcup_\alpha G_\alpha$.*

The last statement follows from the fact that the direct sum of subsets of $\mathscr{T}_Q$ satisfying (C.II.1) is also a subset of $\mathscr{T}_Q$ which satisfies (C.II.1).

One result of the previous theorem is the following.

THEOREM II.5. *If $P(z) \in \mathfrak{A}$, satisfying* (CII.3) *and* (CII.4), *and if there exists a set $\mathscr{T}$ and a family $\{\omega_t\}$ satisfying* (CII.1) *and* (CII.2), *then $1/P(z)$ can have only a pole of first order in Δ.*

Proof. If $1/P(z)$ had a pole of order higher than the first, then I_Q would have a bounded (and non-null) complement; therefore $\mathscr{T}_Q$ would consist only of the point $t=0$.

Thus, if ζ_0 is the singularity of $1/P(z)$, then $Q(z)$ can be written

$$Q(z)=q_0 \ln (z-\zeta_0)+Q_1(z) \tag{II.13}$$

where $Q_1(z)$ is analytic in Δ.

Theorems II.6 and II.7 refer to families of transformations generated by $P(z)$ satisfying (CII.3) and (CII.4).

THEOREM II.6. *If* $\omega(z_1;\ t)=z_1,\ z_1\in\Delta$, for $t\neq 2\pi ikq_0,\ k=0,\ \pm1,\ \pm2,\ \cdots$, *then* $z_1=\zeta_0$.

Proof. $Q[\omega(z;\ t)]=Q(z)+t$ for $z\neq\zeta_0$.
Therefore $Q[z_1]=Q[z_1]+t$ if $z\neq\zeta_0$.
Therefore $t=2\pi ikq_0,\ k=0,\ \pm1,\cdots$.

THEOREM II.7. If $\omega(z_1;\ t)=\omega(z_2;\ t),\ t\in\mathcal{T}$, *then* $z_1=z_2$.

Proof. Suppose first that $z_1, z_2\neq\zeta_0$. Then $\omega(z_1;t)=\omega(z_2;\ t)$ would imply $Q(z_1)=Q(z_2)$ or, by (CII.4), $z_1=z_2$. On the other hand, if, say, $z_1=\zeta_0$, then $\omega(z_1,\ t)=z_1=\omega(z_2;\ t)$ and so $z_2=z_1$ by Theorem II.6.

Thus, conditions (CII.3) and (CII.4) when imposed on the function $P(z)$ imply that the family $\{\omega_t\}$ is a family of schlicht functions.

It is clear that the functions ω_t as well as the set $\mathcal{T}$ are unaltered if the definition of Q is altered by the addition of an arbitrary constant.

It is also easy to see that multiplying Q (that is, multiplying $1/P$) by a constant $c\neq 0$ yields essentially the same family of transformations: Let $\mathcal{T}$, $\{\omega_t\}$ correspond to $P(z)$ and let $\mathcal{T}'$, $\{\omega'_{t'}\}$ correspond to $\frac{1}{c}P(z)$. (Here the primes do not, of course, imply differentiation.) Then clearly, $\mathcal{T}'=c\mathcal{T}$. Also, for $t'\in\mathcal{T}'$,

$$cQ[\omega'(z;\ t')]=cQ(z)+t'\ ,$$

or

$$Q[\omega'(z;\ t')]=Q(z)+\frac{t'}{c}\ ,$$

so that

$$\omega'(z;t')=\omega\left(z;\frac{t'}{c}\right);\quad t'\in\mathcal{T}',\quad \frac{t'}{c}\in\mathcal{T}. \tag{II.14}$$

In other words, there is a one-to-one correspondence between the transformations corresponding to $P(z)$ and those corresponding to $\frac{1}{c}P(z)$; the correspondence is given by (II.14).

Now consider, for $t\in\mathcal{T}\cap I_Q$, the parameter defined by

$$\beta=Q^{-1}(t)\qquad t\in\mathcal{T}\cap I_Q \tag{II.15}$$

Then $\beta\in\Delta$ and (II.12) becomes, writing $\omega[z;\ t(\beta)]$ simply as $\omega(z;\ \beta)$,

$$\omega(z;\ \beta)=Q^{-1}[Q(z)+Q(\beta)],\qquad z,\ \beta\in\Delta. \tag{II.16}$$

Here β is defined on $Q^{-1}[\mathscr{T} \cap I_Q]$.

It is always possible to define Q in such a way[5] that $\mathscr{T} \subset I_Q$ and therefore $\mathscr{T} \cap I_Q = \mathscr{T}$. In such a case, (II.15) and (II.16) hold for all $t \in \mathscr{T}$. For example, in defining Q by (II.9), it is clear that $Q(z_0)=0$ for $z_0 \in \Delta$. Thus, for Q defined as in (II.9) with $z_0 \in \Delta$, we have $\beta = Q^{-1}(t) = \omega(z_0;\ t)$.

It is, however, often possible and more convenient to define Q such that $\mathscr{T}$ is the closure of I_Q. It is also often possible to extend the definition of Q to the boundary of Δ in such a way that the boundary of Δ goes (under Q) into the boundary of I_Q. (An example of this is given by the family of transformations studied in the next section.) In such cases, (II.15) holds for all $t \in \mathscr{T}$ and, in (II.16), β may be a point on the boundary of Δ.

The law of composition of the transformations $T_{\omega_\beta} = T_\beta$ in terms of the parameter β is

$$\text{(II.17)} \qquad \begin{cases} T_{\beta_1} T_{\beta_2} = T_{\beta_3} \\ \beta_3 = \omega(\beta_1;\ \beta_2) \end{cases}$$

This can be shown as follows.

$$\omega[\omega(z;\ t_1);\ t_2] = \omega(z;\ t_1 + t_2)\,,$$

so

$$\begin{aligned} \omega[\omega(z;\ \beta_1);\ \beta_2] &= \omega[z;\ t = Q(\beta_1) + Q(\beta_2)] \\ &= \omega[z;\ \beta = \omega(\beta_1;\ \beta_2)]\,. \end{aligned}$$

By simply looking at the set I_Q, one is usually able to determine many of the properties of the family $\{T_t\}$. For example, one may determine (a) whether or not such a family exists for the given $P(z)$; (b) what the maximum parameter domain $\mathscr{T}$ is; (c) whether $\{T_t\}$ is a group or a semigroup; (d) which of the functions ω_t transform Δ onto itself and which transform Δ into but not onto itself;

D. Possible applications. The above results provide the basis for obtaining a variety of theorems by rephrasing known results in the theory of transformations in Banach space in terms of transformations in the function spaces H^p of the kind studied above. Three possible categories of results are:

(a) Representations of the transformations T_t in terms of the generator A or the resolvent of A ([2] contains many such formulas).

(b) Application of results in the theory of analytic Banach-space-

[5] The addition of a constant to Q changes I_Q but leaves $\mathscr{T}$ unaltered.

valued functions of a complex variable ([2], [7], [9])

(c) Other theorems concerning properties of semigroups or groups of transformations in Banach space.

III. Two Special Cases

A. The family $\{T_w\}$ defined by $T_w f(z)=f(wz)$, $|w|\leq 1$.

Let

(III.1) $$P(z)=-z$$

and[6]

(III.2) $$Q(z)=\int_1^z \frac{-d\zeta}{\zeta}=-\ln z\,.$$

Then I_Q is the open right half plane: $\Re(z)>0$. $\overline{\mathscr{T}_Q}$ is the closed right half plane: $\Re(z)\geqq 0$. Clearly, $\overline{\mathscr{T}_Q}$ itself satisfies conditions (CII.1) and is therefore the maximum domain $\overline{\mathscr{T}}$ of the parameter t. We have

(III.3) $$\omega(z;\ t)=ze^{-t} \qquad z\in\Delta,\ t\in\overline{\mathscr{T}_Q}$$

or, if we let

(III.4) $$w=e^{-t}$$

then, writing $\omega[z;\ t(w)]$ simply as $\omega(z;\ w)$,

(III.5) $$\omega(z;\ w)=wz \qquad z\in\Delta,\ |w|\leqq 1$$

The corresponding family of transformations $\{T_w\}$ is then given by

(III.6) $$T_w f=g$$

where $$g(z)=f(wz)$$

The generator A is defined for those $f\in H^p$ for which the limit

$$Af=\lim_{w\to 1}\frac{T_w f-f}{1-w} \qquad |w|\leqq 1$$

exists in the H^p norm. Thus,

(III.7) $$Af(z)=-zf'(z) \qquad \text{for } f\in\mathscr{D}(A).$$

For $1\leqq p<\infty$, $\mathscr{D}(A)$ is the set of functions $f\in H^p$ for which $f'(z)$ defines an element of H^p. This follows from Theorem II.2. The crucial point in applying Theorem II.2 is in verifying condition (ii) of

[6] Here $z_0=1$ is not in Δ, but in this case this is immaterial,

that theorem. This amounts to the following. Let h be in H^p $(1 \leqq p < \infty)$, and let $T_w h(z) = h(wz)$ for $|w| \leqq 1$. Then $T_w h \to h$ in the norm of H^p as $w \to 1$ in the closure of Δ. It is not difficult to prove this.

Also, for $1 \leqq p < \infty$, A is a closed operator with domain dense in H^p.

For $p = \infty$, (III.7) still holds, but one cannot verify condition (ii) of Theorem II.2 and it is eassily seen that $\mathscr{D}(A)$ is not dense in H^∞.

B. The family $\{L_\alpha\}$ defined by $L_\alpha f(z) = f\left(\dfrac{z+\alpha}{1+\alpha z}\right)$, $-1 < \alpha < 1$.

Let

$$P(z) = (1 - z^2) \tag{III.8}$$

and[8]

$$Q(z) = \int_0^z \frac{d\zeta}{1 - \zeta^2} = \tanh^{-1} z \tag{III.9}$$

Then I_Q is the strip $|\Im(z)| < \pi/4$. $\mathscr{T}_Q$ is the real axis. Clearly $\mathscr{T}_Q$ satisfies conditions (CII.1) and is therefore the maximum domain $\mathscr{T}$ of the parameter t. We have

$$\omega(z;\ t) = \frac{z + \tanh t}{1 + z \tanh t} \qquad t \in \mathscr{T}_Q,\ z \in \Delta\,. \tag{III.10}$$

If we let

$$\alpha = \tanh t\,, \qquad t \in \mathscr{T}_Q \tag{III.11}$$

then, writing $\omega[z;\, t(\alpha)]$ simply as $\omega(z;\ \alpha)$,

$$\omega(z;\ \alpha) = \frac{z+\alpha}{1+\alpha z}\,, \qquad z \in \Delta,\ -1 < \alpha < 1\,. \tag{III.12}$$

The family of transformations $\{L_\alpha\}$ is given by

$$L_\alpha f = g \tag{III.13}$$

where

$$g(z) = f\left(\frac{z+\alpha}{1+\alpha z}\right)$$

The norm of L_α is

$$\| L_\alpha \|_{H^p} = \left[\frac{1+|\alpha|}{1-|\alpha|}\right]^{1/p} \tag{III.14}$$

[8] The path of integration lying entirely in Δ.

The generator A is defined for those $f \in H^p$ for which the limit

$$Af = \lim_{\alpha \to 0} \frac{L_\alpha f - f}{\alpha}$$

exists in the H^p norm. Hence

(III.15) $$Af(z) = (1-z^2)f'(z) \qquad \text{for } f \in \mathscr{D}(A).$$

For $1 \leqq p < \infty$, $\mathscr{D}(A)$ is the set of functions $f \in P^p$ for which $(1-z^2)f'(z)$ defines an element of H^p; also, A is a closed operator with domain dense in H^p. As with the previous example, these statements do not hold for H^∞.

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