TRANSFORMATIONS OF SERIES OF E-FUNCTIONS

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1. Introductory. The transformation [1, p. 25, 2, p. 369]

$$(1)\qquad F\begin{pmatrix}\alpha, & \beta, & \gamma, & \delta, & -l;\ 1\\ \alpha-\beta+1, & \alpha-\gamma+1, & \alpha-\delta+1, & \alpha+l+1\end{pmatrix}$$
$$=\frac{(\alpha+1;l)\left(\frac{1}{2}\alpha-\beta+1;l\right)}{\left(\frac{1}{2}\alpha+1;l\right)(\alpha-\beta+1;l)}F\begin{pmatrix}\alpha-\gamma-\delta+1, \frac{1}{2}\alpha, \beta, -l;\ 1\\ \alpha-\gamma+1, \alpha-\delta+1, \beta-\frac{1}{2}\alpha-l\end{pmatrix},$$

where l is a positive integer, is a special case of a formula of Whipple's. It, and other transformations of the same kind, can be employed to obtain transformations of series of E-functions. Two such transformations are:

$$\sum_{n=0}^{l}\frac{(\alpha;n)(\beta;n)(-l;n)}{n!(\alpha-\beta+l;n)(\alpha+l+1;n)}E\left\{\begin{matrix}p;\alpha_r\\ \hline \Delta(m;\rho-n), \Delta(m;\sigma-n), \Delta(m;\alpha+\rho+n)\\ \Delta(m;\alpha+\sigma+n), \rho_1, \rho_2, \cdots, \rho_q\end{matrix}:z\right\}$$
$$(2)\qquad =\frac{(\alpha+1;l)\left(\frac{1}{2}\alpha-\beta+1;l\right)}{\left(\frac{1}{2}\alpha+1;l\right)(\alpha-\beta+1;l)}\sum_{n=0}^{l}\frac{\left(\frac{1}{2}\alpha;n\right)(\beta;n)(-l;n)}{n!\left(\beta-\frac{1}{2}\alpha-l;n\right)}\left(\frac{2}{m}\right)^n$$
$$\times E\left\{\begin{matrix}\Delta(2m;\alpha+\rho+\sigma+n-1), \alpha_1, \alpha_2, \cdots, \alpha_p\\ \hline \Delta(2m;\alpha+\rho+\sigma-1), \Delta(m;\rho), \Delta(m;\sigma),\\ \Delta(m;\alpha+\rho+n), \Delta(m;\alpha+\sigma+n), \rho_1, \rho_2, \cdots, \rho_q\end{matrix}:z\right\};$$

$$\sum_{n=0}^{l}\frac{(\alpha;n)(\beta;n)(-l;n)}{n!(\alpha-\beta+1;n)(\alpha+l+1;n)}E\left\{\begin{matrix}\Delta(m;\gamma+n), \Delta(m;\gamma-\alpha-n), \alpha_1, \cdots, \alpha_p\\ \Delta(m;\sigma+n), \Delta(m;\sigma-\alpha-n), \rho_1, \cdots, \rho_q\end{matrix}:z\right\}$$
$$(3)\qquad =\frac{(\alpha+1;l)\left(\frac{1}{2}\alpha-\beta+1;l\right)}{\left(\frac{1}{2}\alpha+1;l\right)(a-\beta+1;l)}\sum_{n=0}^{l}\frac{(\sigma-\gamma;n)\left(\frac{1}{2}\alpha;n\right)(\beta;n)(-l;n)}{n!\left(\beta-\frac{1}{2}\alpha-l;n\right)(-m^2)^n}$$
$$\times E\left\{\begin{matrix}\Delta(m;\gamma), \Delta(m;\gamma-\alpha-n), \alpha_1, \cdots, \alpha_p\\ \Delta(m;\sigma-\alpha), \Delta(m;\sigma+n), \rho_1, \cdots, \rho_q\end{matrix}:z\right\}.$$

In these formulae m is a positive integer,

$$(4)\qquad (\alpha;0)=1,\ (\alpha;m)=\alpha(\alpha+1)\cdots(\alpha+m-1),$$

and $\Delta(m;\alpha)$ denotes the set of parameters

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$$\frac{\alpha}{m},\ \frac{\alpha+1}{m},\ \cdots,\ \frac{\alpha+m-1}{m}.$$

The proofs of (2) and (3) are given in § 2. The following formulae are required.

If m is a positive integer

$$(5)\qquad \Gamma(mz)=(2\pi)^{[(1/2)-(1/2)m]}m^{[mz-(1/2)]}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right).$$

From this it follows that, if m and n are positive integers,

$$(6)\qquad \Gamma\left(\frac{\alpha+n}{m}\right)\Gamma\left(\frac{\alpha+1+n}{m}\right)\cdots\Gamma\left(\frac{\alpha+m-1+n}{m}\right)$$
$$=\Gamma\left(\frac{\alpha}{m}\right)\Gamma\left(\frac{\alpha+1}{m}\right)\cdots\Gamma\left(\frac{\alpha+m-1}{m}\right)m^{-n}(\alpha;n),$$

and

$$(7)\qquad \Gamma\left(\frac{\alpha-n}{m}\right)\Gamma\left(\frac{\alpha+1-n}{m}\right)\cdots\Gamma\left(\frac{\alpha+m-1-n}{m}\right)$$
$$=\Gamma\left(\frac{\alpha}{m}\right)\Gamma\left(\frac{\alpha+1}{m}\right)\cdots\Gamma\left(\frac{\alpha+m-1}{m}\right)(-m)^n/(1-\alpha;n).$$

The Barnes' integral for the E-function is [2, p. 374]

$$(8)\qquad E(p;\alpha_r;q;\rho_s:z)=\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\Pi\Gamma(\alpha_r-\zeta)}{\Pi\Gamma(\rho_s-\zeta)}z^\zeta d\zeta$$

where $|\operatorname{amp} z|<\pi$ and the integral is taken up the η-axis with loops, if necessary, to ensure that the origin lies to the left of the contour and the points $\alpha_1,\alpha_2,\cdots,\alpha_p$ to the right of the contour. Zero and negative integral values of the parameters are excluded, and the α's must not differ by integral values. When $p<q+1$ the contour is bent to the left at both ends. When $p>q+1$ the formula is valid for $|\operatorname{amp} z|<\frac{1}{2}(p-q+1)\pi$.

2. Proofs of the transformations. Using (8), (6) and (7), the left-hand side of (2), with $p=q=0$, can be written

$$\sum_{n=0}^{l}\frac{(\alpha;n)(\beta;n)(-l;n)}{n!(\alpha-\beta+1;n)(\alpha+l+1;n)}$$
$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)z^\zeta(1-\rho+m\zeta;n)(1-\sigma+m\zeta;n)d\zeta}{\prod_{\rho,\sigma}\left[\prod_{u=0}^{m-1}\left\{\Gamma\left(\frac{\rho+u}{m}-\zeta\right)\Gamma\left(\frac{\alpha+\rho+u}{m}-\zeta\right)\right\}(\alpha+\rho-m\zeta;n)\right]}$$
$$=\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)z^\zeta}{\prod_{\rho,\sigma}\left[\prod_{u=0}^{m-1}\left\{\Gamma\left(\frac{\rho+u}{m}-\zeta\right)\Gamma\left(\frac{\alpha+\rho+u}{m}-\zeta\right)\right\}\right]}Id\zeta\ ,$$

where

$$I=F\begin{pmatrix}\alpha, & \beta, & 1-\rho+m\zeta, & 1-\sigma+m\zeta, & -l; & 1\\ \alpha-\beta+1, & \alpha+\rho-m\zeta, & \alpha+\sigma-m\zeta, & \alpha+l+1 & \end{pmatrix},$$

From (1) it follows that this can be put in the form

$$\frac{(\alpha+1;l)\left(\frac{1}{2}\alpha-\beta+1;l\right)}{\left(\frac{1}{2}\alpha+1;l\right)(\alpha-\beta+1;1)}$$
$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)z^{\zeta}}{\prod_{\rho,\sigma}\left[\prod_{u=0}^{m-1}\left\{\Gamma\left(\frac{\rho+u}{m}-\zeta\right)\Gamma\left(\frac{\alpha+\rho+u}{m}-\zeta\right)\right\}\right]}J\,d\zeta\,,$$

where

$$J=F\begin{pmatrix}\alpha+\rho+\sigma-1-2m\zeta, \frac{1}{2}\alpha, \beta, -l; 1\\ \alpha+\rho-m\zeta, \alpha+\sigma-m\zeta, \beta-\frac{1}{2}\alpha-l\end{pmatrix}.$$

Now the integral is equal to

$$\sum_{n=0}^{l}\frac{\left(\frac{1}{2}\alpha;n\right)(\beta;n)(-l;n)}{n!\left(\beta-\frac{1}{2}\alpha-l;n\right)2\pi i}$$
$$\times\int\frac{\Gamma(\zeta)z^{\zeta}\Gamma(\alpha+\rho+\sigma-1+n-2m\zeta)}{\prod_{\rho,\sigma}\left[\sum_{u=0}^{m-1}\left\{\Gamma\left(\frac{\rho+u}{m}-\zeta\right)\Gamma\left(\frac{\alpha+\rho+u}{m}-\zeta\right)\right.\right.}\Bigg/$$
$$\Bigg/\frac{\Gamma(\alpha+\rho-m\zeta)\Gamma(\alpha+\sigma-m\zeta)d\zeta}{\left.\left.\Gamma(\alpha+\rho+n-m\zeta)\right\}\right]\Gamma(\alpha+\rho+\sigma-1-2m\zeta)},$$

and, on applying (5) to the gamma functions whose arguments contain $-2m\zeta$ or $-m\zeta$, the right-hand side of (2) with $p=q=0$ is obtained. Formula (2) can then be derived by generalising.

Formula (3) can be proved in the same way. It should be noted that

$$(\alpha-\gamma+1+m\zeta;n)=(-1)^n(\gamma-\alpha-n-m\zeta;n)\,.$$

The restrictions on amp z and on the parameters can be removed by analytical continuation, provided that the functions exist.

References

1. W. N. Bailey, *Generalized Hypergeometric Series,* Cambridge 1935.
2. T. M. MacRobert, *Functions of a Complex Variable,* London 1954.