AN ALGEBRAIC APPROACH TO EXTERIOR DIFFERENTIAL SYSTEMS

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This paper concerns E. Cartan's theory of systems of exterior differential forms. We define a purely algebraic model which determines many of the system's properties. By algebraic constructions such concepts as "involutive", "characters" and "prolongations" are defined and the main theorems are given simple algebraic proofs. These methods are applied to characterize systems which reproduce themselves under prolongations. The prolongation theorem of Kuranishi is proved algebraically.

The purpose of this section is to explain the sources of the algebraic structures studied in the rest of the paper. E. Cartan initiated the study of exterior differential systems in [1, 2]. In [2] they were treated in a very algebraic manner which was used in Cartan's theory of infinite groups. Kuranishi in his prolongation theorem [6] again reduces problems to algebraic questions. We wish to isolate and exploit the purely algebraic characteristics of these systems even more fully in this paper.

Cartan studied systems of "Pfaffians", i.e., differential equations defined by a finite number of linearly independent real analytic 1-forms on an open ball in E^n. Let A denote the module of real analytic 1-forms over the ring R of real analytic functions on a fixed open ball in E^n, Euclidean n-space. Consider $\wedge A = \wedge^0 A + \wedge^1 A + \cdots + \wedge^n A$, the exterior algebra generated by A over R. Let S and I be submodules of A. S is the given module of Pfaffians which define for Cartan partial differential equations. I is called the module of independent variables. Let (S) denote the ideal in $\wedge A$ generated by S. We use this ideal because if S is zero on some integral submanifold of E^n, so is every element of (S). Let d denote the exterior derivative.

Choose a linear subspace T of A so that A is the direct sum of I, T, and S. Let $j: \wedge^2 A \to \wedge^2 A/(S) \approx I \wedge I + I \wedge T + T \wedge T$ be the natural epimorphism. Then $jod: S \to I \wedge I + I \wedge T + T \wedge T$. However, in [2, pp. 577–8] Cartan shows that we may always assume T chosen so that $jod: S \to I \wedge T = I \otimes T$, and so we are led to study linear transformations $\delta: S \to I \otimes T$. This was the viewpoint in my recent paper [5]. Observe that $\delta = jod$ is linear over R. For, if $f \in R$, $\sigma \in S$, $d(f\sigma) = df \wedge \sigma + fd\sigma$. Hence $\delta(f\sigma) = jod(f\sigma) = j(df \wedge \sigma) + fj(d\sigma) = fjd(\sigma) = f\delta(\sigma)$, since $df \wedge \sigma \in (S)$.

Historically the use of R instead of the more customary ring of C^{∞}-functions was because the supporting partial differential equations theory exists only in the analytic case. But R also has the property of being an integral domain. If F denotes its quotient field, then we may consider S, I, T, $I \otimes T$ as vector spaces over F, and δ as a linear transformation on these vector spaces.

This will enable us to use the techniques of linear algebra to define Cartan's numerical invariants and his concept of "involutive". These are usually defined pointwise and then their maximum values used in studying the global systems. Cartan wrote of "generic" properties [3], and we return to his concept, defining a generic system by extending the field F in a suitable manner. In §4 these methods are applied to the study of systems which reproduce themselves under prolongations. In §5 the higher order prolongations of a system are identified as certain mappings on tensor products of the original spaces. These results are applied in §6 to given an elementary proof of Kuranishi's prolongation theorem. In this paper it seemed more convenient to study the dual of δ, defined below.

Differential systems and prolongations.

DEFINITION 1.1. A (*differential*) *system* (I, T, d, S) consists in 3 vector spaces I, T, S over a field F and a linear transformation $d: I \otimes T \to S$. The space I is the vector space of *independent variables*.

Denote by i the identity transformation on various spaces and by $j: I \wedge I \to I \otimes I$ the dual of the canonical epimorphism $I^* \otimes I^* \to I^* \wedge I^*$.

DEFINITION 1.2. (I, U, δ, T) is a *prolongation* of (I, T, d, S) if $d(i \otimes \delta)(j \otimes i)(I \wedge I \otimes U) = 0$. That is, the composition

$$(I \wedge I) \otimes U \xrightarrow{j \otimes i} I \otimes (I \otimes U) \xrightarrow{i \otimes \delta} I \otimes T \xrightarrow{d} S$$

is zero.

DEFINITION 1.3. The (*total, normal*) *prolongation* is defined as follows. Let

$$V = \{\phi \in \mathrm{Hom}\,(I,\,T) \mid d(i \otimes \phi)j(I \wedge I) = 0\} \,.$$

There is a natural linear transformation $\delta: I \otimes V \to T$ satisfying, for $\omega \in I$, $\phi \in V$, $\delta(\omega \otimes \phi) = \phi(\omega)$.

In [5] the author showed that the normal prolongation is a prolongation and in fact maximal in a certain sense among all prolongations. In the present paper all prolongations will be normal, so *we shall omit*

the qualification "normal".

2. Involution. E. Cartan characterized algebraically exterior differential systems of equations which could be solved by the Cauchy-Kowalewski system [4]. Systems satisfying these algebraic conditions are called "involutive". We shall define this notion and prove some results about it.

Let (I, T, d, S) be a system. Let $\omega_1, \cdots, \omega_p$ be a basis of I. Let I_h denote the subspace of I spanned by $\omega_1, \cdots, \omega_h$.

DEFINITION 2.1. An *integral element of dimension* h (relative to $\omega_1, \cdots, \omega_p$) is a linear transformation $\phi_h: I_h \to T$ such that $d(i \otimes \phi_h)j(I_h \wedge I_h) = 0$. This means that for all $i \leqq h$, $j \leqq h$,

$$d(\omega_i \otimes \phi_h(\omega_j)) = d(\omega_j \otimes \phi_h(\omega_i)) \ .$$

Let $\mathscr{I}_h$ denote the linear space of integral elements of dimension h. Then $\mathscr{I}_1$ and $\mathscr{I}_p$ are isomorphic to T and V, respectively (see Definition 1.3). There are natural linear transformations $\rho_{h+1}: \mathscr{I}_{h+1} \to \mathscr{I}_h$ obtained by restricting elements in $\mathscr{I}_{h+1}$ to I_h, $h = 1, 2, \cdots, p - 1$.

DEFINITION 2.2. (I, T, d, S) is *h-involutive* (relative to $\omega_1, \cdots, \omega_p$) if ρ_{h+1} is a epimorphism. It is *involutive* (relative to $\omega_1, \cdots, \omega_p$) if it is h-involutive for each $h = 1, 2, \cdots, p - 1$. (I, T, d, S) is *involutive* if it is involutive relative to at least one basis $\omega_1, \cdots, \omega_p$.

Let $\mathscr{P}_h$ be the kernel of ρ_{h-1}, let $\tau_h(\omega_1, \cdots, \omega_p) = \tau_h(\omega) = \dim \mathscr{P}_h$, $\tau_0(\omega) = \dim T$. Then $\dim \mathscr{I}_{h-1} \leqq \tau_h(\omega) + \dim \mathscr{I}_h$, and equality holds if and only if (I, T, d, S) is h-involutive (relative to $\omega_1, \cdots, \omega_p$). Hence one can prove the following:

THEOREM 2.1.

$$\dim V \leqq \tau_0(\omega) + \tau_1(\omega) + \cdots + \tau_{p-1}(\omega) \ ,$$

and equality holds if and only if (I, T, d, S) is involutive (relative to $\omega_1, \cdots, \omega_p$).

Now, let $\omega_1, \cdots, \omega_p$ be any fixed basis. Let $F(x)$ be the field of rational functions over F in variables x_j^i; $i, j = 1, 2, \cdots, p$. Let $I^* = I \otimes_F F(x)$, $T^* = T \otimes_F F(x)$, $S^* = S \otimes_F F(x)$, regarded as vector spaces over $F(x)$, and extend d to be a linear transformation over $F(x)$ on $I^* \otimes_{F(x)} T^*$ into S^*. Let

$$\mu_k = \sum_{i=1}^{p} x_k^i \omega_i \ , \qquad\qquad k = 1, \cdots, p \ .$$

Then $\mu_1, \cdots, \mu_p$ form a basis of I^*. We intend to use $\mu_1, \cdots, \mu_p$ as a "generic" basis of I. Let I_h^* be the subspace spanned by $\omega_1, \cdots, \omega_h$, and define $\mathscr{I}_h^*$, ρ_{h+1}^*, $\mathscr{P}_h^*$, τ_h^* as before.

PROPOSITION 2.1. If $\tau_h(\bar{\omega})$ is defined relative to some basis $\bar{\omega}_1, \cdots, \bar{\omega}_p$ then $\tau_h(\bar{\omega}) \leqq \tau_h^*$, and equality holds for some basis $\bar{\omega}_1, \cdots, \bar{\omega}_p$.

Proof. Let $\pi_1, \cdots, \pi_q$ and $\theta_1, \cdots, \theta_r$ be bases of T and S, respectively. Then $d(\omega_i \otimes \pi_\lambda) = a_{i\lambda}^u \theta_u$. If $\phi \in \mathscr{P}_h$, then $\phi(\mu_i) = 0$ for $i \leqq h$, and $\phi(\mu_{h+1}) = \sum_{\lambda=1}^q c^\lambda \pi_\lambda$, where

$$d(i \otimes \phi)j(\mu_i \wedge \mu_k) = \sum_{\lambda=1}^q c^\lambda d(\mu_i \otimes \pi_\lambda)$$

$$= \sum_{k=1}^p \sum_{\lambda=1}^q c^\lambda x_i^k a_{k\lambda}^u \theta_u = 0 , \qquad i \leqq h .$$

Thus, the c^λ must satisfy the pr linear equations

$$\sum_{k=1}^p \sum_{\lambda=1}^q c^\lambda x_i^k a_{k\lambda}^u = 0 , \qquad \begin{matrix} i = 1, \cdots, h \\ u = 1, \cdots, q . \end{matrix}$$

Conversely, every solution of these equations yields an elements of $\mathscr{P}_h^*$. τ_h^* is the number of linearly independent solutions.

Now suppose $\bar{x}_j^i$ are elements of F such that $\bar{\omega}_i = \sum \bar{x}_i^k \omega_k$. Then the equations for members of $\mathscr{P}_h$ (relative to $\omega_1, \cdots, \omega_p$) are

$$\sum_{k=1}^p \sum_{\lambda=1}^q c^\lambda \bar{x}_i^k a_{k\lambda}^u = 0 , \qquad i \leqq h .$$

Hence $\tau_h(\bar{\omega}) \leqq \tau_h^*$.

We also see from this that it is possible to choose $\bar{x}_i^k \in F$ so that $\tau_h(\bar{\omega}) = \tau_h^*$, all h.

COROLLARY 1. τ_h^* *does not depend on* $\omega_1, \cdots, \omega_p$.

We also observe from the above equations

COROLLARY 2. $\tau_h^* \leqq \tau_{h+1}^*$.

REMARK. This device of introducing $F(x)$ is often of practical value in computing τ_h in examples, since the results are independent of $\omega_1, \cdots, \omega_p$.

Then if τ_h denotes the maximum of $\tau_h(\omega)$ over all bases $\omega_1, \cdots, \omega_p$ we have [3, p. 91].

THEOREM 2.1. $\dim V \leqq \tau_0 + \cdots + \tau_{p-1}$, *and equality holds if and only if* (I, T, d, S) *is involutive.*

To compare our definitions with those of Cartan, see [2, p. 579]. Matrix (14) there is the matrix associated with the equations for the c^λ in Proposition 2.1. The σ_h in Cartan's notation are related to our τ_h by $\tau_h = q - (\sigma_1 + \cdots + \sigma_k)$.

3. Prolongations of involutive systems. We shall now give an elementary proof that the normal prolongation of an involutive system is also involutive [7]. Let (I, T, d, S) be an involutive system relative to a basis $\omega_1, \cdots, \omega_p$ of I. The (normal) prolongation (I, V, δ, T) is defined by $V = \{\phi \in \text{Hom}\,(I, T) \mid d(i \otimes \phi)j(I \wedge I) = 0\}$, $\delta(\omega \otimes \phi) = \phi(\omega)$. Let I_h, $\mathscr{I}_h$, $\mathscr{P}_h$, ρ_{h+1}, $\tau_h(\omega)$ be as before for the system (I, T, d, S). The entities corresponding to these for (I, V, δ, T) will be denoted by I_h, $\mathscr{I}_h'$, $\mathscr{P}_k'$, ρ_{h+1}', $\tau_h'(\omega)$, respectively.

PROPOSITION 3.1. If (I, T, d, S) is involutive (relative to $\omega_1, \cdots, \omega_p$), so is (I, V, δ, T).

Proof. We shall show that ρ_{h-1}' is an epimorphism for each $h = 1, \cdots, p - 1$. Let $\tilde{\phi} \in \mathscr{I}_h'$. We wish to define $\tilde{\phi}$ on ω_{h+1} so that $\delta(i \otimes \tilde{\phi})j(I_{h+1} \wedge I_{h+1}) = 0$, given that $\delta(i \otimes \tilde{\phi})j(I_h \wedge I_h) = 0$. Thus, $\tilde{\phi}(\omega_{h+1})$ must be chosen so that if $i \leq h$, $\delta[\omega_i \otimes \tilde{\phi}(\omega_{h+1})] = \delta[\omega_{h+1} \otimes \tilde{\phi}(\omega_i)]$. By definition this requires that

$$(1) \qquad \tilde{\phi}(\omega_{h+1})(\omega_i) = \tilde{\phi}(\omega_i)(\omega_{h+1}) .$$

Hence, $\tilde{\phi}(\omega_{h-1})$ is defined uniquely on I_h. In fact, if $i, k \leq h$,

$$\begin{aligned}
d[i &\otimes \tilde{\phi}(\omega_{h+1})j](\omega_k \wedge \omega_i)\\
&= d[\omega_k \otimes \tilde{\phi}(\omega_{h+1})(\omega_i)] - d[\omega_i \otimes \tilde{\phi}(\omega_{h+1})(\omega_k)]\\
&= d[\omega_k \otimes \tilde{\phi}(\omega_i)(\omega_{h+1})] - d[\omega_i \otimes \tilde{\phi}(\omega_k)(\omega_{h+1})] \text{ by (1)}\\
&= d[\omega_k \otimes \tilde{\phi}(\omega_i)(\omega_{h+1})] - d[\omega_{h+1} \otimes \tilde{\phi}(\omega_i)(\omega_k)]\\
&\quad + d[\omega_{h+1} \otimes \tilde{\phi}(\omega_i)(\omega_k)] - d[\omega_{h+1} \otimes \tilde{\phi}(\omega_k)(\omega_i)]\\
&\quad + d[\omega_{h+1} \otimes \tilde{\phi}(\omega_k)(\omega_i)] - d[\omega_i \otimes \tilde{\phi}(\omega_k)(\omega_{h+1})]\\
&= d[i \otimes \tilde{\phi}(\omega_i)]j(\omega_k \wedge \omega_{h+1})\\
&\quad + d\{\omega_{h-1} \otimes \delta(i \otimes \tilde{\phi})j(\omega_i \wedge \omega_k)\} - d[i \otimes \tilde{\phi}(\omega_k)]j(\omega_i \wedge \omega_{h+1})\\
&= 0 ,
\end{aligned}$$

since $\tilde{\phi}(\omega_i)$ and $\tilde{\phi}(\omega_k)$ belong to V and $\delta(i \otimes \tilde{\phi})j(I_h \wedge I_h) = 0$. Thus $\tilde{\phi}(\omega_{h+1}) \in \mathscr{I}_h$. Since (I, T, d, S) is involutive, ρ_{h+i} are epimorphisms, so there exists an extension of $\tilde{\phi}(\omega_{h+1})$ to all of I, this extension being in V. In this way $\tilde{\phi}$ is defined on I_{h+1} into V and becomes an element of $\mathscr{I}_{h+1}'$. Thus ρ_{h+1}' is an epimorphism for all h.

PROPOSITION 3.2. If (I, T, d, S) is involutive, $\mathscr{P}_h'$ is isomorphic

to the direct sum

$$\mathscr{P}_h \dot{+} \mathscr{P}_{h+1} \dot{+} \cdots \dot{+} \mathscr{P}_{p-1} \, .$$

Hence $\tau'_h(\omega) = \sum_{j=h}^{p-1} \tau_j(\omega)$.

Proof. Let $\tilde{\phi} \in \mathscr{P}'_h$. Then $\tilde{\phi} \colon I_{h+1} \to V$, $\tilde{\phi}(I_h) = 0$, and $\delta[i \otimes \phi]j(I_{h+1} \wedge I_{h+1}) = 0$. If $i \le h$, $\delta[i \otimes \tilde{\phi}]j(\omega_i \wedge \omega_{h+1}) = \delta[\omega_i \otimes \tilde{\phi}(\omega_{h+1})] = \tilde{\phi}(\omega_{h+1})(\omega_i) = 0$. Thus, $\tilde{\phi}(\omega_{h+1})$ is in V and is zero on I_h. Let $\mathscr{Q}_h = \{\phi \in V \mid \phi(I_h) = 0\}$. Then $\tilde{\phi}(\omega_{h+1}) \in \mathscr{Q}_h$, and conversely every element of $\mathscr{Q}_h$ corresponds to a unique member of $\mathscr{P}'_h$. Hence $\mathscr{P}'_h$ is isomorphic to $\mathscr{Q}_h$.

We have an exact sequence $0 \to \mathscr{Q}_{h+1} \to \mathscr{Q}_h \to \mathscr{P}_h$, and when (I, T, d, S) is involutive the last linear transformation is an epimorphism. Hence

$$\mathscr{Q}_h \approx \mathscr{P}_h \dot{+} \mathscr{Q}_{h+1} \approx \cdots \approx \mathscr{P}_h \dot{+} \mathscr{P}_{h+1} \dot{+} \cdots \dot{+} \mathscr{P}_{p-1} \, .$$

Observe that even when (I, T, d, S) is not involutive one has an exact sequence $0 \to \mathscr{Q}_{h+1} \to \mathscr{Q}_h \to \mathscr{P}_h$, and $\mathscr{Q}_h \approx \mathscr{P}'_h$.

THEOREM 3.1. *If* $\tau'_h(\omega)$, $h = 0, 1, \cdots, p-1$, *are the numbers associated with the normal prolongation* (I, V, δ, T) *of* (I, T, d, S), *then*

$$\tau'_h(\omega) \le \tau_h(\omega) + \tau_{h+1}(\omega) + \cdots + \tau_{p-1}(\omega) \, .$$

If (I, T, d, S) *is involutive, then so is* (I, V, δ, T), *and equality holds in the above relation.*

4. Self-reproducing systems. The methods will now be applied to study certain systems which remain unchanged under prolongations. Let (I, T, d, S) be a given system, (I, V, δ, T) its prolongation, and ω_1 a nonzero element of I. Define $f \colon V \to T$ be $f(\phi) = \phi(\omega_1)$. Let $F = i \otimes f \colon I \otimes V \to I \otimes T$. Define $G \colon T \to S$ by $G(\pi) = d(\omega_1 \otimes \pi)$.

PROPOSITION 4.1. $G\delta = dF$.

Proof. We have $G\delta(\omega \otimes \phi) = d(\omega_1 \otimes \phi(\omega))$ and $dF(\omega \otimes \phi) = d(\omega \otimes \phi(\omega_1))$. Since $\phi \in V$, $0 = d(i \otimes \phi)j(\omega \wedge \omega_1) = d(\omega \otimes \phi(\omega_1)) = d(\omega_1 \otimes \phi(\omega))$.

PROPOSITION 4.2. *If* (I, T, d, S) *is involutive (relative to* $\omega_1, \cdots, \omega_p$), *then* F *is an epimorphism.*

Proof. Given $\omega \otimes \pi \in I \otimes T$, let $\phi(\omega_1) = \pi$, $\phi \in \mathscr{I}_1$. Then ϕ may be extended to I_2, hence to I_3, etc., hence to I. Thus, there is an element $\phi \in V$ such that $\phi(\omega_1) = \pi$. Then $F(\omega \otimes \phi) = \omega \otimes \pi$.

PROPOSITION 4.3.   If (I, T, d, S) is involutive (relative to $\omega_1, \cdots,$ ω_p) and $\tau_1 = 0$, then F is an endomorphism.

Proof.   The exactness of $0 \to \mathscr{P}_h \to \mathscr{J}_{h+1} \to \mathscr{J}_h \to 0$ implies that the restriction linear transformation ρ_{h+1} is an isomorphism for all $h \geq 1$, since Corollary 2 to Proposition 2.1 implies $\mathscr{P}_h = 0$ for all $h \geq 1$. Then given ω_1 and π, there exists a *unique* $\phi \in V$ such that $\phi(\omega_1) = \pi$.

Define $g: T \to V$ by $g(\pi) = \phi$ provided $\phi(\omega_1) = \pi$ is an isomorphism by the above remarks.   Thus, $\dim I \otimes V = \dim I \otimes T = \dim I \otimes V - \dim(\ker F)$.   Hence $\ker F = 0$.

PROPOSITION 4.4.   If (I, T, d, S) is involutive (relative to $\omega_1, \cdots,$ ω_p) and d is an epimorphism, then G is an epimorphism.

Proof. δ is an epimorphism; for, given $\pi \in T$, then as in Proposition 4.2 we can choose $\phi \in V$ so that $\phi(\omega_1) = \pi$.   Hence $\delta(\omega_1 \otimes \phi) = \pi$. F is an epimorphism by Proposition 4.3, so by Proposition 4.1. G must be an epimorphism.

PROPOSITION 4.5.   If (I, T, d, S) is involutive (relative to $\omega_1, \cdots,$ ω_p), d is an epimorphism, and $\dim T = \dim S$, then G is an endomorphism.

Proof.   Immediate consequence of Proposition 4.4.

DEFINITION 4.1.   Two systems (I, T, d, S) and (I', T', d', S') are *equivalent* if there exist isomorphisms $f: T \to T'$ and $G: S \to S'$ such that if $F = i \otimes f: I \otimes T \to I \otimes T'$, then $Gd = d'F$.

We have proved the following

THEOREM 4.1.   *If (I, T, d, S) is involutive (relative to $\omega_1, \cdots, \omega_p$), $\tau_1 = 0$, d is an epimorphism, and $\dim T = \dim S$, then (I, T, d, S) and (I, V, δ, T) are equivalent.   Thus, under these conditions the prolongation reproduces a system equivalent to the original system.*

5.   Higher order prolongations.   In this section we study the process of iterated prolongations and identify the system obtained by prolonging (I, T, d, S) a number of times.   This will be used in § 6 to prove a theorem of Cartan and Kuranishi that every system becomes involutive after a sufficient number of prolongations [6].

If σ is any permutation of $1, 2, \cdots, k$, we denote by p_σ the automorphism on $\otimes^k I$ which satisfies $p_\sigma(\omega_1 \otimes \cdots \otimes \omega_k) = \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(k)}$. A linear transformation $\eta: \otimes^k I \to T$ is called *symmetric* when $\eta = \eta p_\sigma$ for every σ.

DEFINITION 5.1. Let

$$V^k = \{\eta \in \text{Hom}\,(\otimes\,^k I,\, T)\,|\,\eta \text{ is symmetric, and}$$
$$d(i \otimes \eta)(j \otimes i)[(I \wedge I) \otimes {}^{k-1}I] = 0\}$$

if $k > 0$, and $V^0 = T$. If $k > 1$, $\delta^k\colon I \otimes V^k \to V^{k-1}$ is defined such that if $\omega \in I$, $\eta \in V^k$, $\gamma \in \otimes\,^{k-1}I$, then $[\delta^k(\omega \otimes \eta)](\gamma) = \eta(\gamma \otimes \omega)$. Also, $\delta^1 = \delta$ of Definition 1.3. The system $(I,\, V^k,\, \delta^k,\, V^{k-1})$ is called the *kth* (*total, normal*) *prolongation* of $(I,\, T,\, d,\, S)$.

It is a computation to check that δ^k maps into V^{k-1}. The second condition in the definition of V^k means that if $\omega_0,\, \omega_1,\, \cdots,\, \omega_k$ are in I, then

$$d[\omega_0 \otimes \eta(\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_k)] = d[\omega_1 \otimes \eta(\omega_0 \otimes \omega_2 \otimes \cdots \otimes \omega_k)]\ .$$

THEOREM 5.1. *$(I,\, V^k,\, \delta^k,\, V^{k-1})$ is equivalent to the system obtained by successively prolonging $(I,\, T,\, d,\, S)$ k times.*

Proof. For induction, one shows $(I,\, V^{k+1},\, \delta^{k+1},\, V^k)$ is equivalent to the prolongation of $(I,\, V^k,\, \delta^k,\, V^{k-1})$. Let

$$U = \{\phi \in \text{Hom}\,(I,\, V^k)\,|\,\delta^k(i \otimes \phi)j(I \wedge I) = 0\}\ ,$$

let $\mu\colon I \otimes U \to V^k$ be defined by $\mu(\omega \otimes \phi) = \phi(\omega)$. Then it must be shown that $(I,\, U,\, \mu,\, V^k)$ is equivalent to $(I,\, V^{k+1},\, \delta^{k+1},\, V^k)$. In fact, one can define $f\colon U \to V^{k+1}$ as follows: if $\omega_1,\, \cdots,\, \omega_{k+1} \in I$, $\phi \in U$, let $f(\phi)(\omega_1 \otimes \cdots \otimes \omega_{k+1}) = \phi(\omega_1)(\omega_2 \otimes \cdots \otimes \omega_{k+1})$. Then a computation shows that f is well-defined. Also f has an inverse $g\colon V^{k+1} \to U$ constructed similarly. Moreover, $\delta^{k+1}(\omega \otimes f(\phi))(\gamma) = f(\phi)(\omega \otimes \gamma) = \phi(\omega)(\gamma) = \mu(\omega \otimes \phi)$. Hence letting $F = i \otimes f\colon I \otimes U \to I \otimes V^{k+1}$ and $G = \text{identity}\colon V^k \to V^k$, we have $\delta^{k+1}F = G\mu$.

Next we shall identify integral elements of the system $(I,\, V^k,\, \delta^k,\, V^{k-1})$. Let $\omega_1,\, \cdots,\, \omega_p$ be a basis of I. Let I_h be spanned by $\omega_1,\, \cdots,\, \omega_h$.

DEFINITION 5.2. Let, for $1 \leqq k \leqq p$,

$$\mathscr{I}_h^{\,k} = \{\phi \in \text{Hom}\,(\otimes\,^k I \otimes I_h,\, T)\,|\,\phi \text{ is symmetric and}$$
$$d(i \otimes \phi)(j \otimes i)[(I \wedge I) \otimes {}^{k-1}I \otimes I_h] = 0\}\ .$$

("Symmetric" refers to all possible symmetries preserving $\otimes\,^k I \otimes I_h$).
Let $\rho_{h+1}^k\colon \mathscr{I}_{h+1}^{\,k} \to \mathscr{I}_h^{\,k}$ be the restriction linear transformation, with kernel $\mathscr{P}_h^{\,k}$. Let

$$\mathscr{Q}_h^{\,k} = \{\eta \in V^{k+1}\,|\,\eta\,(\otimes\,^k I \otimes I_h) = 0\} \text{ if } h > 0\ ,$$
$$\mathscr{Q}_0^{\,k} = V^{k+1}.$$

Observe that $\mathscr{I}_0{}^k = V^{k+1}$, $\mathscr{Q}_p^k = 0$.

THEOREM 5.2. $\mathscr{I}_h{}^k$ *is isomorphic to the space of integral elements of dimension* h *(relative to* $\omega_1, \cdots, \omega_p$) *on* $(I, V^k, \delta^k, V^{k-1})$. *If we identify these isomorphic spaces,* ρ_{h+1}^k *corresponds to the restriction linear transformation, and* $\mathscr{P}_h{}^k$ *and* $\mathscr{Q}_h{}^{k-1}$ *are isomorphic.*

Proof. The first two assertions follow by a computation. The last isomorphism is defined as follows: $\phi \in \mathscr{P}_h{}^k$ is identified with the element of $\mathscr{Q}_h{}^{k-1}$ whose value on $\gamma \in \bigotimes^k I$ is $\phi(\gamma \otimes \omega_{h+1})$.

PROPOSITION 5.1. *For any element* $\omega_0 \in I$, *there exists a linear transformation* $\lambda^k(\omega_0)$: $\mathscr{Q}_h{}^k \to \mathscr{Q}_h{}^{k-1}$ *which satisfies, for* $\eta \in \mathscr{Q}_h{}^k$, $\gamma \in \bigotimes^k I$,

$$[\lambda^k(\omega_0)(\eta)](\gamma) = \eta(\gamma \otimes \omega_0) .$$

If $\omega_0 = \omega_{h+1}$, *the kernel of* $\lambda^k(\omega_0)$ *is* $\mathscr{Q}_{h+1}^k$.

Proof. A computation shows that $\lambda^k(\omega_0)$ is well-defined. The second assertion follows from the definitions.

THEOREM 5.3. $(I, V^k, \delta^k, V^{k-1})$ *is involutive if and only if* V^{k+1} *is isomorphic to the direct sum*

$$(2) \qquad \mathscr{Q}_0^{k-1} \dotplus \mathscr{Q}_1^{k-1} \dotplus \cdots \dotplus \mathscr{Q}_{p-1}^{k-1} \approx V^{k-1} ,$$

or, if and only if $\lambda^k(\omega_{h+1})$ *is an epimorphism for* $h = 0, 1, \cdots, p - 1$.

Proof. The first condition follows from Theorem 2.1 and the last statement of Theorem 5.2. Next, $\lambda^k(\omega_{h+1})$ is an epimorphism if and only if

$$\mathscr{Q}_h{}^k \approx \mathscr{Q}_h{}^{k-1} \dotplus \mathscr{Q}_{h+1}^k ,$$

and $\mathscr{Q}_0^k = V^{k+1}$. This implies isomorphism (2). If some $\lambda^k(\omega_{h+1})$ were not an epimorphism, the dimensions of the various spaces would preclude any such isomorphism (2).

6. The prolongation theorem. We shall next prove that if (I, T, d, S) is any given system, there exists an integer k_0 such that the kth prolongation $(I, V^k, \delta^k, V^{k-1})$ is involutive for all $k \geq k_0$. In fact, we shall show the existence of one basis $\omega_1, \cdots, \omega_p$, relative to which all $(I, V^k, \delta^k, V^{k-1})$ are involutive for $k \geq$ some k_0.

To show this it is sufficient to show that $\lambda^k(\omega_{h+1})$: $\mathscr{Q}_h{}^k \to \mathscr{Q}_h{}^{k-1}$ is an epimorphism for all $k \geq k_0$, by Theorem 5.3. This will be done inductively, choosing ω_{h+1} after $\omega_1, \cdots, \omega_h$ such that $\lambda^k(\omega_{h+1})$ is an

epimorphism. Choice of ω_{h+1} is accomplished by relating this problem to one in classical ideal theory. Let us temporarily extend $\omega_1, \cdots, \omega_h$ to any basis $\omega_1, \cdots, \omega_p$ of I. Then $\bar{\omega}_{h+1}$ will be chosen later so that using $\bar{\omega}_{h+1}$, $\lambda^k(\bar{\omega}_{h+1})$ will be an epimorphism.

Let $F[x_1, \cdots, x_p]^{(k)}$ be the vector space of polynomials in $x_1, \cdots, x_p$ over F which are homogenous of degree k. Define an epimorphism $\nu: \bigotimes{}^k I \to F[x_1, \cdots, x_p]^{(k)}$ so that $\nu(\omega_{i_1} \otimes \cdots \otimes \omega_{i_k}) = x_{i_1} \cdots x_{i_k}$. If $\eta \in V^k$, η is symmetric, hence $\eta\,(\ker \nu) = 0$. It follows that η induces a unique linear transformation $\bar{\eta}$ on $F[x_1, \cdots, x_p]^{(k)}$ into T.

These $\bar{\eta}$ can be considered linear transformations on $F[x_1, \cdots, x_p]^{(k)} \otimes T^*$ into F, where T^* is the dual of T. If $\dim T = q$ and $\pi_1^*, \cdots, \pi_q^*$ is a basis of T^*, we can identity $F[x_1, \cdots, x_p]^{(k)} \otimes T^*$ with the vector space $F[x_1, \cdots, x_p, t_1, \cdots, t_q]^{(k, 1)} = F[x, t]^{(k, 1)}$ of polynomials in $x_1, \cdots, x_p, t_1, \cdots, t_q$ which are homogenous of degree k in the x_i and degree 1 in the t_λ. Then V^k is identified with a subspace of the dual space of $F[x, t]^{(k, 1)}$. Let

$$A(V^k) = \{X \in F[x, t]^{(k, 1)} \mid \phi(X) = 0 \text{ for every } \phi \in V^k\}\,.$$

Next, consider $\mathscr{Q}_h^{k-1}$, consisting of all members of V^k which are zero on $\bigotimes{}^k I \otimes I_h$. If $(x_1, \cdots, x_h)$ is the ideal in $F[x, t]$ generated by $x_1, \cdots, x_h$; and if $(x_1, \cdots, x_h)^{(k, 1)}$ is its subspace of elements homogenous of degree k in x and 1 in t, let $a[(x_1, \cdots, x_h)^{(k,1)}] = \{\phi \in \mathrm{Hom}\,(F[x, t]^{(k,1)}, F) \mid \phi[(x_1, \cdots, x_h)^{(k, 1)}] = 0\}$ (the dual space to $(x_1, \cdots, x_h)^{(k, 1)}$).

Then by definition

$$\mathscr{Q}_h^{k-1} = V^k \cap a[(x_1, \cdots, x_h)^{(k, 1)}]\,.$$

It follows that

$$A(\mathscr{Q}_h^{k-1}) = A(V^k) + (x_1, \cdots, x_h)^{(k, 1)}\,.$$

PROPOSITION 6.1. $x_i A(\mathscr{Q}_h^{k-1}) \subset A(\mathscr{Q}_h^k)$.

Let $\bar{\omega}$ denote the element of $F[x, t]^{(1, 0)}$ corresponding to $\omega \in I$. Then $\lambda^k(\omega)$ is an epimorphism if and only if the following holds: whenever $\bar{\omega}\gamma \in A(\mathscr{Q}_h^k)$, then $\gamma \in A(\mathscr{Q}_h^{k-1})$.

Proof. $\lambda^k(\omega)$ is the dual of the linear transformation $r_{\bar{\omega}} \colon F[x, t]^{(k,1)} \to F[x, t]^{(k+1, 1)}$ defined by $r_{\bar{\omega}}(\gamma) = \gamma\omega$. Since $\lambda^k(\omega)$ carries $\mathscr{Q}_h^k$ into $\mathscr{Q}_h^{k-1}$, $r_{\bar{x}_i}(A(\mathscr{Q}_h^{k-1})) \subset A(\mathscr{Q}_h^k)$. Hence the first assertion. The second remark also follows from vector space theory.

Now let A_h be the ideal in $F[x, t]$ generated by $A(\mathscr{Q}_h^k)$, $k = 1, 2, \cdots$. By the last Proposition $A_h^{(k, 1)} = A(\mathscr{Q}_h^{k-1})$. If A_0 is the ideal generated by $A(V^k)$, $k = 1, 2, \cdots$, we have $A_h = A_0 + (x_1, \cdots, x_h)$.

PROPOSITION 6.2.   There exists an element $\omega \in F[x, t]^{(1, 0)}$ and an integer k_0 such that if $k \geq k_0$, then $\bar{\omega}\gamma - A(\mathcal{Q}_h^k)$ implies $\gamma \in A(\mathcal{Q}_h^{k-1})$.

Proof.   Since A_h is an ideal in a Noetherian ring it has a decomposition $A_h = q_1 \cap \cdots \cap q_t$ into primary ideals with distinct belonging prime ideals $p_1, \cdots, p_t$ such that

$$(p_i)^{\sigma_i} \subset q_i , \qquad\qquad i = 1, \cdots, t .$$

Let $k_0 > \max (\sigma_1, \cdots, \sigma_t)$.

Now suppose $p_i^{(1, 0)} = F[x, t]^{(1, 0)}$. Then $(p_i^k)^{(k, 1)} = F[x, t]^{(k, 1)}$. Hence if $k \geq k_0 > \sigma_i$, $F[x, t]^{(k, 1)} = (q_i)^{(k, 1)}$, so $\bar{\omega}\gamma \in q_i^{(k, 1)}$ implies $\gamma \in q_i^{(k-1, 1)}$.

Now suppose $p_1^{(1, 0)}, \cdots, p_s^{(1, 0)}$ contain all $F[x, t]^{(1, 0)}$, while $p_{s+1}^{(1, 0)}, \cdots$, $p_t^{(1, 0)}$ do not. Each $p_{s+j}^{(1, 0)}$ is a proper subspace of $F[x, t]^{(1, 0)}$, so we can choose $\bar{\omega} \in F[x, t]^{(1, 0)}$ not in any $p_{s+j}^{(1, 0)}$, $j = 1, \cdots, t - s$. Then if $\bar{\omega}\gamma \in q_{s+i}^{(k, 1)}$, $\bar{\omega} \notin p_{s+j}$, hence $\gamma \in q_{s+i}^{(k-1, 1)}$.

For this $\bar{\omega}$, if $\bar{\omega}\gamma \in A(\mathcal{Q}_h^k) = A_h^{(k+1, 1)} \subset A_h = q_1 \cap \cdots \cap q_t$, then $\gamma \in q_1 \cap \cdots \cap q_t$, provided $k \geq k_0$. Hence $\gamma \in A_h^{(k, 1)} = A(\mathcal{Q}_h^{k-1})$.

COROLLARY.   *There exists $\omega \in I$ and an integer k_0 such that ω is linearly independent of $\omega_1, \cdots, \omega_h$ and $\lambda^k(\omega)$ is an epimorphism for all $k \geq k_0$.*

Proof.   If $F[x, t]^{(k, 1)} \neq A(\mathcal{Q}_h^{k-1})$ for some $k \geq k_0$, let $\gamma \in F[x, t]^{(k, 1)} - A(\mathcal{Q}_h^{k-1})$. If $\bar{\omega}$ chosen in Proposition 6.2 depends on $x_1, \cdots, x_h$, then $\bar{\omega}\gamma \in A(V^{k+1}) + (x_1, \cdots, x_h)^{(k+1, 1)} = A(\mathcal{Q}_h^k)$, while $\gamma \notin A(\mathcal{Q}_h^{k-1})$, a contradiction.   Hence the $\omega \in I$ corresponding to $\bar{\omega}$ cannot depend on $\omega_1, \cdots, \omega_h$.

If $F[x, t]^{(k, 1)} = A(\mathcal{Q}_h^{k-1})$ for all $k \geq k_0$, any $\omega \in I$ will suffice.

THEOREM 6.1.   *It is possible to choose a basis $\omega_1, \cdots, \omega_h$ of I and an integer k_0 such that $\lambda^k(\omega_{h+1})$ is an epimorphism for all h and $k \geq k_0$.*

Proof.   Having chosen $\omega_1, \cdots, \omega_h$, let ω_{h+1} be the ω of the previous Corollary.

THEOREM 6.2.   (*Kuranishi's Prolongation Theorem*) *For any system there exists an integer k_0 such that every kth (normal) prolongation for $k \geq k_0$ involutive.*

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