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QUALITATIVE BEHAVIOR OF SOLUTIONS OF A THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

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This paper investigates the behavior of nonoscillatory solutions and the existence of oscillatory solutions of the differential equation

$$y''' + p(t)y' + q(t)y^r = 0$$

where $p(t)$ **and** $q(t)$ **are continuous and real valued on a half axis** $[a, \infty)$ **and** r **is the quotient of odd positive integers. The two cases** $p(t), q(t) \leqq 0$ **and** $p(t), q(t) \geqq 0$ **are discussed.**

One theorem improves an oscillation criterion of Waltman [16]. **Other results supplement those obtained by Lazar** [10].

1. In this paper, real valued solutions of

$$y''' + p(t)y' + q(t)y^r = 0 \tag{1.1}$$

are investigated where $p(t)$ and $q(t)$ are continuous real valued functions defined on some interval $[a, \infty)$ with $a > 0$. Furthermore $q(t)$ is not eventually (i.e., for sufficiently large t) identically zero. r is assumed to be the quotient of odd integers. This insures that solutions with real initial conditions are real and also that the negative of a solution of (1.1) is also a solution of (1.1).

Motivation for the study of this equation comes from two directions. The equation

$$y''' + p(t)y' + q(t)y = 0$$

has been studied extensively. Some recent papers are those of Gregus [3], Hanan [5], Lazer [10], and Svec [15]. On the other hand, the equation

$$y^{(n)} + q(t)y^r = 0, \qquad n \geqq 2$$

has been investigated by Licko and Svec [11] for $r \neq 1$, by Kiguradze [9] for $r < 1$, and by Mikusinski [12] for $r = 1$. Equation (1.1) has been studied recently by Waltman [16].

A solution of (1.1) is said to be continuable if it exists on $[a_1, \infty)$ for some $a_1 \geqq a$. A nontrivial solution of (1.1) is called oscillatory if it is continuable and has zeros for arbitrarily large t. A nontrivial solution of (1.1) is called nonoscillatory if it is continuable and not oscillatory.

Two cases, $p(t) \leqq 0, q(t) \leqq 0$ and $p(t) \geqq 0, q(t) \geqq 0$ are discussed in this paper. Most of the theorems deal with the behavior of non-oscillatory solutions. However, in each of the two cases considered, these results lead to criteria for the existence of oscillatory solutions. For the case $p(t) \geqq 0$ and $q(t) \geqq 0$, an oscillation criterion of Waltman [16] is generalized.

Unless otherwise stated all results are new for the linear as well as the nonlinear case. The existence of nonoscillatory solutions is not discussed in this paper. For the linear case Lazer [10] has shown that they exist under general conditions. It can be easily verified that most theorems in this paper are precise in some sense. Examples are given only where they seem to be particularly illustrative.

Since this paper discusses the behavior of continuable solutions, the following two theorems are of interest. The first one shows that all solutions of (1.1) are continuable if $r \leqq 1$. The second shows that under certain conditions the noncontinuable solutions of (1.1) have an infinite number of zeros on a finite interval.

THEOREM 1.1. *If $r \leqq 1$ and $[t_0, b]$ is an arbitrary compact interval such that $a \leqq t_0$, then any solution of* (1.1) *which exists at t_0 can be continued on $[t_0, b]$.*

Proof. Let $|p(t)| + 1 \leqq M$ and $|q(t)| \leqq M$ on $[t_0, b]$. Write (1.1) in vector form

$$\bar{y}' = f(t, \bar{y}) \qquad \bar{y} = (y_1, y_2, y_3) \tag{1.4}$$

where $f_1(t, \bar{y}) = y_2, f_2(t, \bar{y}) = y_3$, and $f_3(t, \bar{y}) = -q(t)y_1^r - p(t)y_2$. Then to a solution y of (1.1) corresponds a solution $\bar{y} = (y_1, y_2, y_3)$ where $y = y_1, y' = y_2$, and $y'' = y_3$.

Define $U(t, u) = M(u + 1)$. Then $\|f(t, \bar{y})\| \leqq U(t, \|\bar{y}\|)$. The theorem now follows from a theorem of Wintner (Hartman [6, p. 29]).

THEOREM 1.2. *Suppose that $p(t) \geqq 0, q(t) \geqq 0$, and $p'(t) \leqq 0$ and continuous. Then any solution of* (1.1) *which is not continuable has an infinite number of zeros on a finite interval.*

Proof. Suppose that the solution $y(t)$ exists and has only a finite number of zeros on $[t_0, b)$ where $b < \infty$. Then there is a $t_1 > t_0$ such that $y(t) \neq 0$ on $[t_1, b)$. Suppose that $y(t) > 0$ on $[t_1, b)$. Then $y'''(t) + p(t)y'(t) \leqq 0$ on $[t_1, b)$. By integrating twice the last inequality from t_1 to $t, t_1 < t$, it is seen that $y(t)$ is bounded on $[t_1, b)$. Now, by integrating (1.1) twice between t_1 and $t, t_1 < t$, it is seen that both $y'(t)$ and $y''(t)$ are bounded on $[t_1, b)$. Thus

$$\lim_{t \to b^-} [(y(t))^2 + (y'(t))^2 + (y''(t))^2] < \infty$$

and $y(t)$ may be extended beyond b (see [2, p. 61]). This proves the theorem.

2. The case $p(t) \leqq 0$ and $q(t) \leqq 0$ is considered in this section. The first lemma is a generalization of a result of Lazer [10, p. 448].

LEMMA 2.1. *Let $p(t) \leqq 0$ on $[a, \infty)$. Suppose that on the same interval $q(t) < 0$ if $0 < r < 1$ and $q(t) \leqq 0$ if $r \geqq 1$. If $y(t)$ is a nonoscillatory solution of* (1.1), *then there is a number $c \geqq a$ such that either $y(t)y'(t) > 0$ for $t \geqq c$ or $y(t)y'(t) \leqq 0$ for $t \geqq c$.*

Proof. If $r \geqq 1$ then solutions of (1.1) are unique. Therefore the argument given by Lazer [10, p. 448] for the linear case proves the lemma.

If $0 < r < 1$ proceed as follows. Suppose that $y(t) > 0$ for $t \geqq t_0$ where $a \leqq t_0$. It is asserted that the zeros of $y'(t)$ are isolated in $[t_0, \infty)$. To show this let $t \geqq t_0$ be an accumulation point for zeros of $y'(t)$. Then $y'(t_1) = 0$ by continuity of $y'(t)$. By Rolle's Theorem, t_1 is an accumulation point for the zeros of $y''(t)$. Hence $y''(t_1) = 0$ by continuity of $y''(t)$. Similarly $y'''(t_1) = 0$. But since $q(t_1) \neq 0$ and $y(t_1) > 0$ this contradicts the fact that $y(t)$ is a solution of (1.1).

If $y'(t)$ has at most one zero in (t_0, ∞) the lemma is clear. If $y'(t)$ has two or more zeros in (t_0, ∞) proceed as follows. Consider two consecutive zeros, say t_2 and t_3, of $y'(t)$ satisfying $t_0 < t_2 < t_3$. Multiplying (1.1) by $y'(t)$ and integrating by parts between t_2 and t_3 yields

$$-\int_{t_2}^{t_3}(y''(s))^2 ds + \int_{t_2}^{t_3} p(s)(y'(s))ds + \int_{t_2}^{t_3} q(s)(y(s))^r y'(s)ds = 0 .$$

Since the first two terms are nonpositive, $q(t) < 0$, and $y(t) > 0$, it follows that $y'(t) < 0$ for t in (t_2, t_3). Since the zeros of $y'(t)$ are isolated, this argument can be repeated to show that $y'(t) \leqq 0$ for $t \geqq t_2$. This proves the lemma.

REMARK. On the basis of the preceding lemma it will be assumed throughout the rest of this section that

$$\text{(H)} \qquad \begin{array}{ll} p(t) \leqq 0 & t \text{ in } [a, \infty) \\ q(t) < 0, 0 < r < 1 & t \text{ in } [a, \infty) \\ q(t) \leqq 0, r \geqq 1 & t \text{ in } [a, \infty) . \end{array}$$

LEMMA 2.2. *Let $f(t)$ be a real valued function defined in $[t_0, \infty)$ for some real number $t_0 \geqq 0$. Suppose that $f(t) > 0$ and that $f'(t)$, $f''(t)$ exist for $t \geqq t_0$. Suppose also that if $f'(t) \geqq 0$ eventually, then*

$\lim_{t\to\infty} f(t) = A < \infty$. *Then*

$$\liminf_{t\to\infty} | t^\alpha f''(t) - \alpha t^{\alpha-1} f'(t) | = 0$$

for any $\alpha \leqq 2$.

Proof. First suppose that $f'(t)$ has both positive and negative values for arbitrarily large t. Then the function $G(t)$ defined by $G(t) = t^\alpha f''(t) - \alpha t^{\alpha-1} f'(t)$ has both positive and negative values for arbitrarily large t. Therefore, by the intermediate value theorem, $G(t) = 0$ for arbitrarily large t. Thus the lemma is proved in this case.

Now suppose that $f'(t)$ is eventually either nonpositive or nonnegative. It is asserted that

$$\liminf_{t\to\infty} | t^{\alpha-1} f'(t) | = 0 .$$

To show this, the mean value theorem is used to write

$$\frac{t_c^{\alpha-1}}{c}(f(2c) - f(c)) = t_c^{\alpha-1} f'(t_c)$$

where $t_0 \leqq c$ and $1 < c < t_c < 2c$. Therefore

$$| t_c^{\alpha-1} f'(t_c) | \leqq 2 | f(2c) - f(c) | .$$

The right side tends to zero as c becomes infinite since $\lim_{t\to\infty} f(t)$ exists and is finite.

It follows that either $\lim_{t\to\infty} t^{\alpha-1} f'(t) = 0$ or $\lim_{t\to\infty} t^{\alpha-1} f'(t)$ does not exist. In the first case it is claimed that $\liminf_{t\to\infty} | t^\alpha f''(t) | = 0$. To show this the mean value theorem is again used to write

$$\frac{t_d^\alpha}{d}(f'(2d) - f'(d)) = t_d^\alpha f''(t_d) .$$

where $t_0 \leqq d$ and $1 < d < t_d < 2d$. Therefore

$$| t_d^\alpha f''(t_d) | \leqq | 2(2d)^{\alpha-1} f'(2d) - 2^\alpha d^{\alpha-1} f'(d) | .$$

Again, the right side tends to 0 as $d \to \infty$ since $\lim_{t\to\infty} t^{\alpha-1} f'(t) = 0$ by assumption. It follows that $\liminf_{t\to\infty} | t^\alpha f''(t) - \alpha t^{\alpha-1} f'(t) | = 0$.

Now suppose that $\lim_{t\to\infty} t^{\alpha-1} f'(t)$ does not exist. Since

$$\liminf_{t\to\infty} | t^{\alpha-1} f'(t) | = 0 ,$$

there is a sequence (t_n), $\lim_{n\to\infty} t_n = \infty$, such that $\lim_{n\to\infty} t_n^{\alpha-1} f'(t_n) = 0$ and $| t_n^{\alpha-1} f''(t_n) + (\alpha - 1) t^{\alpha-2} f'(t_n) = (t^{\alpha-1} f'(t))' |_{t_n} = 0$ for $n = 1, 2, 3, \cdots$. Since $\lim_{n\to\infty} - (2\alpha - 1) t_n^{\alpha-1} f'(t_n) = 0$ it follows that

$$\lim_{n\to\infty}\,[t_n f''(t_n) - \alpha t_n^{\alpha-1} f'(t_n)] = 0\ .$$

Therefore the case where $\lim_{t\to\infty} t^{\alpha-1}f'(t)$ does not exist is disposed of and the lemma is proved.

THEOREM 2.3. *Suppose that (H) is satisfied,* $-\infty < -M \leqq p(t)t^\alpha$ *in* $[a, \infty)$ *and* $\int_a^\infty s^\alpha q(s)ds = -\infty$ *for some* $\alpha \leqq 2$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $y(t)y'(t) \leqq 0$, *then* $\lim_{t\to\infty} y(t) = 0$.

Proof. Suppose $y(t) > 0$ and hence $y'(t) \leqq 0$ for $t \geqq t_0$. Suppose $\lim_{t\to\infty} y(t) = A > 0$. Multiply (1.1) by t^α and integrate between t_0 and t, $t > t_0$ to obtain

$$\begin{aligned} s^\alpha y''(s)\Big|_{t_0}^{t} - s^{\alpha-1}y'(s)\Big|_{t_0}^{t} + \alpha(\alpha-1)\int_{t_0}^{t} s^{\alpha-2}y'(s)ds \\ - M\int_{t_0}^{t} y'(s)ds + \int_{t_0}^{t} s^\alpha q(s)(y(s))^r ds \geqq 0\ . \end{aligned} \tag{2.1}$$

Note that the terms $\alpha(\alpha-1)\int_{t_0}^{t} s^{\alpha-2}y'(s)ds$ and $-M\int_{t_0}^{t} y'(s)ds$ are both bounded as $t \to \infty$ since $y(t)$ has finite limit and $\alpha \leqq 2$. Therefore (2.1) can be written

$$t^\alpha y''(t) - \alpha t^{\alpha-1}y'(t) \geqq K - \int_{t_0}^{t} s^\alpha q(s)(y(s))^r ds \tag{2.2}$$

where K is a finite constant. Since $\lim_{t\to\infty} y(t) = A > 0$, the right hand side of (2.2) tends to ∞ as $t \to \infty$. However, by Lemma 2.2

$$\liminf_{t\to\infty}\, |\, t^\alpha y''(t) - \alpha t^{\alpha-1}y'(t)\,| = 0\ .$$

This contradiction proves the theorem.

THEOREM 2.4. *Suppose that* (H) *holds and that* $\int_a^\infty sp(s)ds > -\infty$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1), *then* $y'(t)y(t) > 0$ *eventually.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) and assume that $y(t) > 0$ for $t \geqq t_0$. The assertion is then that $y'(t) > 0$ eventually. Suppose not. Then by Lemma 2.1, there is a $t_1 \geqq t_0$ such that $y'(t) \leqq 0$ for $t \geqq t_1$.

Pick $t_2 \geqq t_1$ such that $\int_{t_2}^\infty sp(s)ds \geqq -1$. Multiply (1.1) by t and integrate by parts between t_2 and t, $t_2 < t$, to obtain

$$\begin{aligned} sy''(s)\Big|_{t_2}^{t} - y'(s)\Big|_{t_2}^{t} + y'(t)\int_{t_2}^{t} sp(s)ds \\ - \int_{t_2}^{t} y''(s)\int_{t_2}^{s} up(u)duds = -\int_{t}^{t} sq(s)(y(s))^r ds\ . \end{aligned} \tag{2.3}$$

Since $-y'(t) \geqq y'(t)\int_{t_2}^{t} sp(s)ds \geqq 0$, (2.3) becomes

$$ty''(t) - 2y'(t) + y'(t_2) - \int_{t_2}^{t} y''(s)\int_{t_2}^{s} up(u)duds \geqq t_2y''(t_2) - \int_{t_2}^{t} sq(s)(y(s))^r ds . \tag{2.4}$$

Note that $y''(t) \leqq 0$ eventually is impossible with $y'(t) \leqq 0$ and $y(t) > 0$. Suppose that $y''(t) \geqq 0$ for $t \geqq t_2$ (change t_2 if necessary). Then

$$-\int_{t_2}^{t} y''(s)\int_{t_2}^{s} up(u)duds \leqq \int_{t_2}^{t} y''(s)ds = y'(t) - y'(t_2) .$$

Therefore (2.4) becomes

$$ty''(t) - y'(t) \geqq t_2y''(t_2) - \int_{t_2}^{t} sq(s)(y(s))^r ds . \tag{2.5}$$

By lemma 2.2 $\liminf_{t\to\infty} ty''(t) - y'(t) = 0$. But this contradicts the fact that the right hand side of (2.5) is positive and increasing. Thus the theorem is proved for the case $y''(t) \geqq 0$.

Suppose now that $y''(t)$ has positive and negative values for arbitrarily large t. Then there is a sequence of points (t_n), $n \geqq 3$, $t_2 < t_3$, $\lim_{t\to\infty} t_n = \infty$, with the following properties:

(i) $t_i < t_{i+1}$, $i = 3, 4, 5, \cdots$,

(ii) $y''(t_i) = 0$, $i = 3, 4, 5, \cdots$,

(iii) $\lim_{i\to\infty} y'(t_i) = 0$.

The existence of such a sequence (t_n) is clear since $y'(t) \leqq 0$ and $\limsup_{t\to\infty} y'(t) = 0$.

Now, let $L = \int_{t_3}^{\infty} up(u)ds$. $L > -1$ by the choice of $t_3 > t_2$. Thus

$$\begin{aligned} -\int_{t_3}^{t} y''(s)\int_{t_3}^{s} up(u)duds &= \int_{t_3}^{t} y''(s)\left(\int_{s}^{\infty} up(u)du - L\right)ds \\ &= \int_{t_3}^{t} y''(s)\int_{s}^{\infty} up(u)duds - L\int_{t_3}^{t} y''(s)ds \\ &\leqq \int_{t_3}^{t} y''(s)\int_{s}^{\infty} up(u)duds - y'(t_3) . \end{aligned}$$

Substituting this into (2.4) (replacing t_2 by t_3) gives

$$ty''(t) - 2y'(t) + \int_{t_3}^{t} y''(s)\int_{s}^{\infty} up(u)duds \geqq -\int_{t_3}^{t} sq(s)(y(s))^r ds . \tag{2.6}$$

Let $Q(s) = \int_{s}^{\infty} up(u)ds$. Then

$$\begin{aligned}\int_{t_3}^{t} y''(s)Q(s)ds &= y''(t)\int_{t_3}^{t} Q(s)ds - \int_{t}^{t} y'''(s)\int_{t_3}^{t} Q(u)duds \\ &= y''(t)\int_{t_3}^{t} Q(s)ds + \int_{t_3}^{t} p(s)y'(s)\int_{t_3}^{s} Q(u)duds \\ &+ \int_{t_3}^{t} q(s)(y(s))^r\int_{t_3}^{s} Q(u)duds \\ &\leqq y''(t)\int_{t_3}^{t} Q(s)ds + \int_{t_3}^{t} q(s)(y(s))^r\int_{t_3}^{s} Q(u)duds \\ &\leqq y''(t)\int_{t_3}^{t} Q(s)ds - \int_{t_3}^{t} (s-t_3)q(s)(y(s))^r ds\ ,\end{aligned}$$

where the last inequality depends on the fact that $|Q(u)| \leqq 1$. Substituting this into (2.6) yields

$$\begin{gathered} ty''(t) - 2y'(t) + y''(t)\int_{t_3}^{t} Q(s)ds \\ -\int_{t_3}^{t}(s-t_3)q(s)(y(s))^r ds \geqq -\int_{t_3}^{t} sq(s)(y(s))^r ds\ .\end{gathered}$$

Combining the last two terms gives

$$ty''(t) - 2y'(t) + y''(t)\int_{t_3}^{t} Q(s)ds \geqq -t_3\int_{t_3}^{t} q(s)(y(s))^r ds\ . \tag{2.7}$$

Replacing t by t_i in (2.7) where (t_i) is the sequence defined above yields

$$-2y'(t_i) \geqq -t_3\int_{t_3}^{t_i} q(s)(y(s))^r ds\ . \tag{2.8}$$

The right hand side of (2.8) is positive and increasing in t_i while the left hand side of (2.8) converges to zero as $i \to \infty$. This contradiction proves the theorem.

THEREM 2.5. *Suppose* (H) *holds and that* $-2/t^2 \leqq p(t) \leqq 0$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1), *then* $y(t)y'(t) > 0$ *eventually.*

Proof. Suppose $y(t) > 0$ for $t \geqq t_0$. It is to be shown that $y'(t) > 0$ eventually. Suppose to the contrary (by Lemma 2.1) that $y'(t) \leqq 0$ eventually, say for $t \geqq t_0$. Because of the assumption on $p(t)$, (1.1) can be written

$$y'''(t) - (2/t^2)y'(t) + q(t)(y(t))^r \geqq 0 \tag{2.9}$$

for $t \geqq t_0$. Since $y''(t) < 0$ eventually is impossible ($y'(t) \leqq 0$ and $y(t) > 0$), pick $t_1 > t_0$ such that $y''(t_1) \geqq 0$. Now multiply (2.9) by t^2 and integrate by parts between t_1 and t, $t > t_1$, to obtain

$$(2.10) \qquad t^2y''(t) - 2ty'(t) \geqq t_1^2y''(t_1) - 2t_1y'(t_1) - \int_{t_1}^{t} s^2q(s)(y(s))^r ds .$$

The right hand side of (2.10) is positive and increasing for large t. However by Lemma 2.2, with $\alpha = 2$, it follows that the lim inf of the left side of (2.10) is zero. This contradiction proves the theorem.

It is noteworthy that in the last two theorems no restriction was placed on the magnitude of $q(t)$. The following example shows the sharpness of the last result.

EXAMPLE. The equation $y''' - (K/t^2)y' + q(t)y^r = 0, t > 1$ where

$$q(t) = -\frac{K-2}{t^3(\log t)^{2-r}} + \frac{6}{t^3(\log t)^{3-r}} + \frac{6}{t^3(\log t)^{4-r}}$$

has the solution $y(t) = (\log t)^{-1}$. If $K > 2$ then $q(t) < 0$ eventually.

THEOREM 2.6. *Suppose that* (H) *holds and that* $\int_a^\infty s^2q(s)ds = -\infty$ *and that* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $y'(t)y(t) > 0$ *eventually. Then* $|y(t)| \to \infty$ *as* $t \to \infty$.

Proof. It may be assumed that $y(t) > 0$ and $y'(t) > 0$ for $t \geqq t_0$. Multiply (1.1) by t^2 and integrate from t_0 to t obtaining

$$(2.11) \qquad t^2y''(t) - 2ty'(t) + 2y(t) \geqq -\int_{t_0}^{t} s^2q(s)(y(s))^r ds + K$$

where K is a constant. Since $y'''(t) \geqq 0$ for $t \geqq t_0$, it follows that $y''(t)$ is eventually of one sign. If $y''(t) > 0$ eventually, the proof is complete. If $y''(t) < 0$ eventually, then, since the right side of (2.11) tends to ∞ as $t \to \infty$ and all nonconstant terms on the left side of (2.11) except $2y(t)$ are negative, $y(t) \to \infty$ as $t \to \infty$. This proves the theorem.

THEOREM 2.7. *Let* $p(t), q(t) \leqq 0, r > 1$ *and* $\int_a^\infty s^2q(s)ds = -\infty$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $y'(t)y(t) > 0$ *eventually, then* $y''(t)y'(t) \geqq 0$ *eventually and* $\lim_{t\to\infty}|y''(t)| = \lim_{t\to\infty}|y'(t)| = \lim_{t\to\infty}|y(t)| = \infty$.

Proof. It may be supposed that $y(t) > 0$ for $t \geqq t_0$. Thus $y'(t) > 0$ eventually, say for $t \geqq t_0$. This implies that $y'''(t) \geqq 0$ for $t \geqq t_0$ which shows that $y''(t)$ is eventually of one sign. It is asserted that $y''(t) \geqq 0$ eventually. Suppose to the contrary that $y''(t) \leqq 0$ eventually, say for $t \geqq t_0$.

Multiply (1.1) by $t^2/(y(t))^r$ and integrate between t_0 and t obtaining

$$(2.12)\qquad \int_{t_0}^{t}\frac{s^2y'''(s)ds}{(y(s))^r}=\int_{t_0}^{t}\frac{s^2p(s)y'(s)ds}{(y(s))^r}-\int_{t_0}^{t}s^2q(s)ds\ .$$

Expand the first integral by parts, obtaining

$$\int_{t_0}^{t}\frac{s^2y'''(s)ds}{(y(s))^r}=\frac{t^2y''(t)}{(y(t))^r}+r\int_{t_0}^{t}\frac{s^2y''(s)y'(s)ds}{(y(s))^{r+1}}-\frac{2ty'(s)}{(y(t))^r}$$
$$-2r\int_{t_0}^{t}\frac{s(y'(s))^2ds}{(y(s))^{r+1}}-\frac{2(r-1)}{(y(t))^{r-1}}+K$$

where K is a constant. All of the nonconstant terms on the left side of (2.12) are negative while the right side tends to ∞. This contradiction shows that $y''(t)\geqq 0$ eventually.

Clearly $\lim_{t\to\infty}y'(t)=\lim_{t\to\infty}y(t)=\infty$. To show that $\lim_{t\to\infty}y''(t)^2=\infty$ proceed as follows. Since $y'''(t)\geqq 0$ eventually, there is a $t_1\geqq t_0$ and an $A>0$ such that $y''(t)\geqq 2A$ for $t\geqq t_1$. Thus $y(t)\geqq A(t-t_1)$. Now integrate (1.1) between t_1 and t, replacing $y(s)$ by $A(s-t_1)^2$ to obtain

$$(2.13)\qquad y''(t)-y''(t_0)\geqq-\int_{t_1}^{t}p(s)y'(s)ds-\int_{t_1}^{t}A^rq(s)(s-t_1)^{2r}ds\ .$$

Since

$$-A^r\int_{t_1}^{t}q(s)(s-t_1)^{2r}ds\geqq-A^r\int_{t_1}^{t_1+1}q(s)(s-t_1)^{2r}ds$$
$$-A^r\int_{t_1+1}^{t}q(s)(s-t_1)^2ds$$

for $t\geqq t_1+1$, and the divergence of $\int_{t_1+1}^{\infty}q(s)(s-t_1)^2ds$ is equivalent to the divergence of $\int_{t_1+1}^{\infty}q(s)s^2ds$, the right hand side, and therefore the left hand side, of (2.13) tends to ∞. This proves the theorem.

THEOREM 2.8. *Let $p(t)\leqq 0, q(t)\leqq 0$, and suppose that*

$$\int_{t_0}^{\infty}q(s)u(s)ds=-\infty\,,\ t_0>\max\{1,a\}$$

where $u(t)$ is one of the functions

$$t^{2-\alpha},\ t^2(\log t)^{-1-\alpha},\ t^2(\log t)^{-1}(\log(\log t))^{-1-\alpha}$$

for some $0<\alpha<1$.

If $y(t)$ is a nonoscillatory solution of (1.1) with $r=1$ such that $y'(t)y(t)>0$ eventually, then $y''(t)y'(t)\geqq 0$ eventually and

$$\lim_{t\to\infty}|y''(t)|=\lim_{t\to\infty}|y'(t)|=\lim_{t\to\infty}|y(t)|=\infty\ .$$

REMARK. This sequence of functions was used by Hille [7] in different circumstances.

Proof. As before it may be supposed that $y(t) > 0, y'(t) > 0$, and $y'''(t) \geqq 0$ for $t \geqq t_0$. Therefore $y''(t)$ is eventually of one sign. Assume that $y''(t) \leqq 0$ eventually, say for $t \geqq t_0$. Therefore $\lim_{t\to\infty} y'(t) = B$ exists and $0 \leqq B < \infty$. Suppose that $B > 0$. Then $y'(t) \geqq B > 0$ for $t \geqq t_0$, which implies that $y(t) \geqq B(t - t_0)$ for $t \geqq t_0$. Now multiply (1.1) by $u(t)/t$ to obtain

$$\begin{aligned} \frac{y'''(t)u(t)}{t} &= -\frac{p(t)y'(t)u(t)}{t} - \frac{Bq(t)(t-t_0)u(t)}{t} \\ &\geqq -\frac{Bq(t)u(t)}{2} \end{aligned} \tag{2.14}$$

for $t \geqq 2t_0$. Integrating (2.14) between $2t_0$ and t gives

$$\int_{2t_0}^{t} \frac{y'''(s)u(s)ds}{s} \geqq -\frac{B}{2}\int_{2t_0}^{t} q(s)u(s)ds\ . \tag{2.15}$$

Evaluating the left hand side of (2.15) gives

$$\begin{aligned} \int_{2t_0}^{t} \frac{y'''(s)u(s)ds}{s} &= \frac{y''(t)u(t)}{t} + \int_{2t_0}^{t} \frac{y''(s)u(s)ds}{s^2} - \frac{y'(t)u'(t)}{t} \\ &\quad - \int_{2t_0}^{t} \frac{y'(s)u'(s)ds}{s^2} + \int_{2t_0}^{t} \frac{y'(s)u''(s)ds}{s} + K \end{aligned}$$

where K is a constant. Since $y'(t) \geqq 0$ and $y''(t) \leqq 0$, $\int_{t_0}^{\infty} y'(s)s^{-1}u''(s)ds$ is finite. Therefore the left hand side of (2.15) consists of bounded or negative terms while the right hand side of (2.15) tends to ∞. This contradiction shows that $\lim_{t\to\infty} y'(t) = 0$.

It follows by L'hospital's rule that $\lim_{t\to\infty} (y(t)/t) = 0$. Thus there is a $t_1 \geqq t_0$ such that $y(t) \leqq t$ for $t \geqq t_1$.

Now, multiply (1.1) by $u(t)/y(t)$ and integrate from t_1 to t, obtaining

$$\int_{t_1}^{t} \frac{y'''(s)u(s)ds}{y(s)} \geqq -\int_{t_1}^{t} q(s)u(s)ds\ . \tag{2.16}$$

The right hand side of (2.16) tends to ∞. Evaluating the left hand side gives

$$\begin{aligned} \int_{t_1}^{t} \frac{y'''(s)u(s)ds}{y(s)} &= K + \frac{y''(t)u(t)}{y(t)} + \int_{t_1}^{t} \frac{(y'(s))^2u'(s)ds}{(y(s))^2} - \frac{y'(t)u'(t)}{y(t)} \\ &\quad - \int_{t_1}^{t} \frac{(y'(s))^2u'(s)ds}{(y(s))^2} + (\log y(t))u''(t) - \int_{t_1}^{t} (\log y(s))u'''(s)ds\ , \end{aligned} \tag{2.17}$$

where K is a constant.

Since $y(t) \leqq t$ for $t \geqq t_1$, the last two terms in (2.17) are bounded. Since the other terms on the right hand side of (2.17) are constant or are negative, a contradiction to (2.16) is obtained. This shows that $y''(t) \geqq 0$ eventually.

It follows immediately that $\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y(t) = \infty$. Since $u'''(t) \leqq 0$, $\lim_{t\to\infty} u''(t) = D$ exists and $0 \leqq D < \infty$. Consider the function $f(t) = kt^2$ where $k = (D + 1)/2$. Then $(u(t)/f(t)) < 1$ eventually by L'hospital's rule so that $f(t) > u(t)$ eventually. Therefore $\int^\infty s^2 q(s)ds = -\infty$ since $\int^\infty f(s)q(s)ds = -\infty$. Since $y'''(t) \geqq 0$, $y''(t) \geqq 2A$ for $t \geqq t_2$ for some $A > 0$ and some $t_2 \geqq t_1$. Thus $y(t) = A(t - t_2)^2$ for $t \geqq t_2$. Now integrate (1.1) between t_2 and t to get

$$y''(t) - y''(t_2) \geqq -A\int_{t_2}^t q(s)(s - t_2)^2 ds .$$

since $-A\int_{t_2}^\infty (s - t_2)^2 q(s)ds = \infty$, $y''(t) \to \infty$. This proves the theorem.

COROLLARY 2.9. *Suppose that the hypotheses of Theorem* 2.8 *and the hypotheses of either Theorem* 2.4 *or Theorem* 2.5 *hold. Suppose also that* $\int_a^\infty s^4(p'(s) - 2q(s))ds = \infty$. *Then* (1.1), *with* $r = 1$, *has oscillatory solutions.*

Proof. This corollary follows immediately from Theorem's 2.4, 2.5, 2.8 and a theorem of Lazer [10, p. 449].

For the sake of completeness a theorem is stated which considers the case $0 < r < 1$. This theorem can be proved in a similar manner as a theorem of Licko and Svec [11]. However an easier proof can be given by proceeding as in Theorem 2.7 and using Lemma 3.1. The details are omitted here. See also Kiguradze [9, p. 101].

THEOREM 2.10. *Let* $p(t) \leqq 0, q(t) < 0$, *and* $\int^\infty s^{2r} q(s)ds = -\infty$. *Suppose* $y(t)$ *is a nonoscillatory solution of* (1.1) *with* $0 < r < 1$ *and such that* $y(t)y'(t) > 0$. *Then* $y'(t)y''(t) > 0$ *eventually and*

$$\lim_{t\to\infty} |y'''(t)| = \lim_{t\to\infty} |y'(t)| = \lim_{t\to\infty} |y(t)| = \infty .$$

3. In this section the case $p(t) \geqq 0$ and $q(t) \geqq 0$ is considered. The first lemma is an easy adaptation of a result of Kiguradze [8, 649 (Soviet Math.)] and will not be proved here.

LEMMA 3.1. *Let* $f(t)$ *be a continuous nonnegative function defined on* $[t_0, \infty)$ *for some* $t_0 \geqq 0$. *If* $f^{(n)}(t) \leqq 0, n \geqq 2$, *and* $f^{(n-k)}(t) \geqq 0, k = 1, \cdots, n - 1$, *on* $[t_0, \infty)$ *then there are constants* $A_k > 0, k = 1, \cdots,$

$n-1$ such that

$$\frac{f(t)}{f^{(n-k)}(t)} \geqq A_k t$$

for sufficiently large t.

THEOREM 3.2. *Let $p(t) \geqq 0$ and $q(t) \geqq 0$. Suppose also that*

(i) $\int_a^\infty s^{2r} q(s) ds = \infty$ *if* $0 < r < 1$;

(ii) $\int_a^\infty u(s)q(s)ds = \infty$ *if* $r = 1$ *where $u(t)$ is one of the functions* $t^{2-\alpha}$, $t^2 (\log t)^{-1-\alpha}$, $t^2 (\log t)^{-1} \log (\log t)^{-1-\alpha}, \cdots$ *for some* $\alpha > 0$;

(iii) $\int_a^\infty s^{1+r} q(s) ds = \infty$ *if* $1 < r$.

If $y(t)$ is a nonoscillatory solution of (1.1), *then $|y(t)|$ is not eventually nondecreasing.*

Proof. Suppose to the contrary that $|y(t)|$ is eventually nondecreasing. It may be supposed that $y(t) > 0$ and thus that $y'(t) \geqq 0$ for large t, i.e., $t \geqq t_0$. Therefore, by (1.1), $y'''(t) \leqq 0$ for $t \geqq t_0$. This implies that $y''(t) > 0$ eventually, in fact, for $t \geqq t_0$. The three cases are now considered separately.

Case (i). First of all, choose $t_1 \geqq t_0$ and $A > 0$ such that $y(t)/y''(t) \geqq At^2$ for $t \geqq t_1$ (use Lemma 3.1). Now divide (1.1) by $(y''(t))^r$ and integrate between t_1 and t, $t > t_1$, to obtain

$$\frac{(y''(t_1))^{1-r} - (y''(t))^{1-r}}{1-r} \geqq A^r \int_{t_1}^t s^{2r} q(s) ds \,. \tag{3.1}$$

Since $r < 1$ and $y''(t)$ is positive and decreasing, the right hand side of (3.1) is bounded, contradicting the integral condition (i).

Case (ii). As before, choose $t_1 \geqq t_0$ and $A > 0$ such that $y(t)/y''(t) \geqq At^2$ for $t \geqq t_1$. Now multiply (1.1) by $u(t)/y(t)$ and integrate between t_1 and t, $t \geqq t_1$, to obtain

$$\int_{t_1}^t \frac{y'''(s)u(s)ds}{y(s)} \leqq -\int_{t_1}^t q(s)u(s)ds \,. \tag{3.2}$$

But

$$\int_{t_1}^t \frac{y'''(s)u(s)ds}{y(s)} = \frac{y''(t)u(t)}{y(t)} - \frac{y''(t_1)u(t_1)}{y(t_1)} + \int_{t_1}^t \frac{y''(s)u(s)y'(s)ds}{(y(s))^2} - \int_{t_1}^t \frac{y''(s)u'(s)ds}{y(s)} \,.$$

Since $y''(t)/y(t) \leqq A^{-1}t^{-2}$ for $t \geqq t_1$, $\int_{t_1}^\infty \frac{y''(s)u'(s)ds}{y(s)}$ is finite. Therefore all terms on the left hand side of (3.2) are bounded, constant, or positive while the right hand side of (3.2) tends to $-\infty$. This proves the theorem for case (ii).

Case (iii). By Lemma 3.1 there is a $t_2 \geqq t_0$ such that $y(t)/y'(t) \geqq Bt$ for some $B > 0$ and for $t \geqq t_2$. Multiply (1.1) by $t/(y'(t))^r$ and integrate from t_2 to t to obtain

$$\int_{t_2}^{t} \frac{sy'''(s)ds}{(y'(s))^r} \leqq -\int_{t2}^{t} sp(s)(y'(s))^{1-r}ds - B^r\int_{t_2}^{t} s^{1+r}q(s)ds \,. \tag{3.3}$$

Integrating by parts the left hand member of (3.3) yields

$$\int_{t_1}^{t} \frac{sy'''(s)ds}{(y'(s))^r} = \frac{ty''(t)}{(y'(t))^r} - \frac{t_2y''(t_2)}{(y'(t_2))^r}$$
$$+ r\int_{t_2}^{t} (y''(s))^2 s(y'(s))^{-1-r}ds + \frac{(y'(t))^{1-r}}{(r-1)} - \frac{(y'(t_2))^{1-r}}{(r-1)} \,.$$

By the integral condition (iii) the right side of (3.3) tend to $-\infty$ as $t \to \infty$ while all terms on the left side are either positive or constant. This contradiction proves the theorem for this case.

REMARK. It is easy to construct examples showing the sharpness of this theorem in all three cases.

The next lemma is an easy generalization of a lemma proved by Lazer [10, p. 454] for the linear case.

LEMMA 3.3. *Suppose that* $p(t) \geqq 0, q(t) \geqq 0$, *and* $p'(t) \leqq 0$ *in* $[a, \infty)$. *It* $y(t)$ *is a nonoscillatory, eventually positive solution of* (1.1) *such that* $F(y(c)) \geqq 0$ *for some* $c \in [a, \infty)$ *where*

$$F(y(t)) = (y'(t))^2 - 2y(t)y''(t) - p(t)(y(t))^2 \,,$$

then there is a $d \geqq c$ *such that* $y(t) > 0, y'(t) > 0, y''(t) > 0$ *and* $y'''(t) \leqq 0$ *for* $t \geqq d$.

REMARK. In the linear case of Lemma 3.3 the condition $p'(t) \leqq 0$ may be replaced by the weaker condition $2q(t) - p'(t) \geqq 0$.

COROLLARY 3.4. *Suppose that the hypotheses of Theorem* 3.2 *and Lemma* 3.3 *are fulfilled. Let* $y(t)$ *be a continuable solution of* (1.1) *defined at some* $t_0 \geqq a$. *Then* $y(t)$ *is nonoscillatory if and only if* $F(y(t)) < 0$ *for all* $t \in [t_0, \infty)$.

Proof. If $F(y(t)) < 0$ for $t \geqq t_0$, it is clear that $y(t)$ can't have any zeros for $t \geqq t_0$. Hence $y(t)$ is nonoscillatory.

Now suppose that $F(y(t_1)) \geqq 0$ for some $t_1 \geqq t_0$. By Theorem 3.2

and Lemma 3.3 $y(t)$ is not nonoscillatory and eventually positive. Suppose that $y(t)$ is nonoscillatory and eventually negative. Then $-y(t)$ is nonoscillatory and eventually positive. But $F(-y(t_1)) = F(y(t_1)) \geqq 0$. Thus a contradiction to Theorem 3.2 Lemma 3.3 and is again obtained. Therefore $y(t)$ is oscillatory.

REMARK. Corollary 3.4 is an oscillation criterion for (1.3). It improves a theorem of Waltman [16] since $y(t_2) = 0$ implies that $F(y(t_2)) \geqq 0$.

The remainder of this section investigates further the behavior of nonoscillatory solutions of (1.3). The following lemma is due to Nehari [14, p. 431].

LEMMA 3.5. *If* $u'' + p(t)u = 0$ *has no oscillatory solutions in* $[a, \infty)$ *and* $v(t)$ *is any function of class* C^1 *on* $[b, c]$ *such that* $v(b) = 0$ *and* $v(t) \not\equiv 0$ *on* (b, c), *then*

$$\int_b^c (v'(s))^2 ds > \int_b^c p(s)(v(s))^2 ds$$

where $a \leqq b < c$.

THEOREM 3.6. *Let* $p(t) \geqq 0$, $q(t) > 0$, *and suppose that* $u'' + p(t)u = 0$ *is nonoscillatory. If* $y(t)$ *is a nonoscillatory solution of* (1.1) *then there is a* $d > a$ *such that either* $y(t)y'(t) \geqq 0$ *for* $t \geqq d$ *or* $y(t)y'(t) < 0$ *for* $t \geqq d$.

REMARK. Two known sufficient conditions that $u'' + p(t)u = 0$ be nonoscillatory are that $\int^\infty sp(s)ds < \infty$ or that $\limsup_{t\to\infty} t^2 p(t) \leqq \dfrac{1}{4}$ ([2, p. 103] and [6, p. 362]).

Proof. Suppose that $y(t) > 0$ for $t \geqq t_0$. Suppose that t_1 and t_2, $t_0 \leqq t_1 < t_2$, are consecutive zeros of $y'(t)$. The proof of the isolation of zeros of $y'(t)$ given in the first part of Lemma 2.1 did not depend on the sign of $p(t)$ and $q(t)$ and hence applies here. Now multiply (1.1) by $y'(t)$ and integrate by parts between t_1 and t_2 to obtain

$$\int_{t_1}^{t_2} p(s)(y'(s))^2 ds + \int_{t_1}^{t_2} q(s)(y(s))^r (y'(s)) ds = \int_{t_1}^{t_2} (y''(s))^2 ds$$

since $y'(t_1) = y'(t_2) = 0$. By Nehari's Lemma

$$\int_{t_1}^{t_2} p(s)(y'(s))^2 ds < \int_{t_1}^{t_2} (y''(s))^2 ds \,.$$

Therefore $\int_{t_1}^{t_2} q(s)(y(s))^r y'(s) ds > 0$ and since t_1 and t_2 are consecutive zeros of $y'(t)$, this implies that $y'(t) > 0$ on (t_1, t_2). Therefore $y'(t) > 0$ between any two successive zeros greater than t_0. Since the zeros of $y'(t)$ are isolated, this shows that $y(t)$ is monotone for $t \geqq t_0$.

EXAMPLE. It will be shown that the condition $u'' + p(t)u = 0$ is nonoscillatory is not sharp. In fact, all nonoscillatory solutions of a

$$y''' + \frac{2}{t^2}y' + q(t)y^r = 0 \tag{3.4}$$

are monotone. To see this, let $y(t) > 0$ be a nonoscillatory solution of (3.4) and suppose that $y'(t)$ has positive and negative values for arbitrarily large t. Pick $t_2 \geqq t_0$ such that $y'(t_2) = 0$ and $y''(t_2) < 0$. Now multiply (3.4) by t^2 and integrate by parts between t_2 and $t, t > t_2$ to obtain

$$t^2y''(t) - 2ty'(t) \leqq -\int_{t_2}^{t} s^2q(s)(y(s))^r ds\ . \tag{3.5}$$

The right side of (3.5) is negative and decreasing while the left side of (3.5) equals zero for arbitrarily large values of t (see Lemma 2.2). This contradiction shows that $y'(t)$ is eventually either nonpositive or nonnegative and hence that $y(t)$ is monotone. On the other hand, by Kneser's criterion (Hartman [6, p. 362]), $u'' + (2/t^2)\, u = 0$ has only oscillatory solutions.

EXAMPLE. The equation

$$y''' + y' + \left(\frac{(1/t^2 + 6/t^4)}{(1 + \sin t + 1/t)^r}\right)y^r = 0$$

has the nonoscillatory, nonmonotone solution $y(t) = 1 + \sin t + 1/t$. A precise condition on $p(t)$ needed to prove Theorem 3.6 remains to be determined.

THEOREM 3.7. *Let* $p(t) \geqq 0, q(t) \geqq 0, \int_a^\infty s^2q(s)ds = \infty$, *and* $t^2p(t) \leqq M < \infty$. *If* $y(t)$ *is a nonoscillatory solutions of* (1.1) *such that* $y(t)y'(t) \leqq 0$ *eventually, then* $\lim_{t\to\infty} y(t) = 0$.

Proof. Suppose that $y(t) > 0$ and hence that $y'(t) \leqq 0$ for $t \geqq t_0 > a_1$. Multiply (1.1) by t^2 and integrate by parts from t_0 to t to obtain

$$t^2y''(t) - 2ty'(t) + (2 + M)y(t) + K + \int_{t_0}^{t} s^2q(s)(y(s))^r ds \leqq 0 \tag{3.6}$$

where all constants have been combined to give K. If $\lim_{t\to\infty} y(t) > 0$, then it follows from (3.6) that $y''(t) < 0$ eventually. But since $y'(t) \leqq 0$ eventually, this contradicts $y(t) > 0$. Thus $\lim_{t\to\infty} y(t) = 0$.

THEOREM 3.8. *Let* $p(t) \geqq 0, q(t) \geqq 0, p'(t) \leqq 0$, *and* $\int_a^\infty q(s)ds = \infty$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1), *then* $\liminf_{t\to\infty} |y(t)| = 0$.

Proof. Suppose that $y(t) > 0$ for $t \geqq t_0 \geqq a$. Multiply (1.1) by $y(t)$ and integrate form t_0 to t obtaining

$$y(t)y''(t) - \frac{(y'(t))^2}{2} + \frac{p(t)(y(t))^2}{2} - \int_{t_0}^t \frac{p'(s)(y(s))^2 ds}{2} + \int_{t_0}^t q(s)(y(s))^{r+1} ds + K = 0 \tag{3.7}$$

where K is a constant. Suppose that $\liminf_{t\to\infty} y(t) = A > 0$. By Theorem (3.2) $y(t)$ can't be eventually monotone nondecreasing. Therefore either $y'(t)$ has arbitrarily large zeros or $y'(t) < 0$ eventually.

Suppose first that there is a sequence $\{t_n\} \to \infty, t_i < t_{i+1}, i = 1, 2, 3, \cdots$ such that $y'(t_i) = 0, i = 1, 2, 3, \cdots$. Replacing t by t_n in (3.7) shows that $y''(t_n)$ is eventually negative. This contradiction shows that the first case is impossible.

Now suppose that $y'(t) < 0$ for $t \geqq t_1 \geqq t_0$. Suppose that

$$\lim_{t\to\infty} y(t) = A > 0 .$$

Since $\limsup_{t\to\infty} y'(t) = 0$, pick a sequence $\{u_n\} \to \infty, u_i < u_{i+1}, i = 1, 2, 3, \cdots$ such that $y'(u_n) \to 0$ as $n \to \infty$ and $y'(u_i) < y'(u_{i+1}), i = 1, 2, 3, \cdots$. Let $u_j^* = \sup\{t \in [u_j, u_{j+1}]: y'(t) = y'(u_j)\}$ for each $j = 1, 2, 3, \cdots$. It follows that $u_j \leqq u_j^* < u_{j+1}$ for $j = 1, 2, 3, \cdots$ and that $y'(u_j^*) = y'(u_j)$. By the mean value theorem there is a v_j in the interval (u_j^*, u_{j+1}) such that

$$y''(v_j) = \frac{y'(u_{j+1}) - y'(u_j^*)}{u_{j+1} - u_j^*} > 0 .$$

By construction $y'(v_j) > y'(u_j^*) = y'(u_j)$. This is done in each $[u_j, u_{j+1}]$ to obtain a sequence $\{v_j\} \to \infty$ with the property that

$$\limsup_{j\to\infty} (2y''(v_j)y(v_j) - (y'(v_j))^2) \geqq 0 .$$

However, since $p'(t) \leqq 0, \int^\infty q(s)ds = \infty$ and $\liminf_{t\to\infty} y(t) > 0$, it follows from (3.7) that

$$y''(t)y(t) - \frac{(y'(t))^2}{2} \longrightarrow -\infty$$

as $t \to \infty$. This contradiction eliminates the second case and proves the theorem.

REMARK. It is clear from the proof that in the linear case ($r = 1$), the conditions $p'(t) \leqq 0$ and $\int_t^\infty q(s)ds = \infty$ can be replaced by the condition $\int_a^\infty (2q(s) - p'(s))ds = \infty$.

The following two lemmas can be proved in exactly the same manner as they were proved by Lazer [10, p. 462, 463] for special cases.

LEMMA 3.9. *Suppose that the hypotheses of Theorem 3.2 are fulfilled and that $p'(t) \leqq 0$. If $y(t)$ is a nonoscillatory solution of (1.1), then $(y'(t))^2 \leqq Ky(t)$ eventually for some constant K. If $r = 1$, then the condition $p'(t) \leqq 0$ can be replaced by $q(t) - p'(t) \geqq 0$.*

LEMMA 3.10. *Suppose the function $f(t)$ is nonnegative, continuous, and differentiable in $[a, \infty)$. Suppose also that $\int_a^\infty (f(s))^\alpha ds < \infty$ for some real $\alpha > 0$ and that $\limsup_{t\to\infty} f(t) = M > 0$. Suppose $0 < d \leqq M/2$. Then the set $\{y'(t): t \in [a, \infty)$ and $d \leqq y(t) \leqq 2d\}$ is unbounded on (b, ∞) for any $b > 0$.*

THEOREM 3.11. *Let the hypotheses of Theorem 3.2 be satisfied if $r \leqq 1$ and suppose that $\int_a^\infty s^2 q(s)ds = \infty$ if $r > 1$. In addition suppose that*

(i) $M/t^\alpha \leqq q(t)$ for some $\alpha \geqq 0$, $M > 0$, and

(ii) $p'(t) \leqq 0$ and $(p(t)t^\alpha)' + (t^\alpha)''' \leqq 0$.

If $y(t)$ is a nonoscillatory solution of (1.1), then

$$\lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y(t) = 0 .$$

Proof. Suppose that $y(t) > 0$ for $t \geqq t_0$. Multiplying (1.1) by t^α and using (i) of the hypothesis gives

$$t^\alpha y'''(t) + t^\alpha p(t)y'(t) + M(y(t))^r \leqq 0 .$$

Now integrate this inequality from t_0 to t to obtain

$$H(y(t)) - H(y(t_0)) - \int_{t_0}^t [(s^\alpha)''' + (s^\alpha p(s))']y(s)ds + \int_{t_0}^t M(y(s))^r \leqq 0$$

where

$$H(y(t)) = t^\alpha y''(t) - \alpha t^{\alpha-1}y'(t) + \alpha(\alpha - 1)t^{\alpha-2}y(t) + t^\alpha p(t)y(t) .$$

Therefore

$$\int_{t_0}^t M(y(s))^r \leqq H(y(t_0)) - H(y(t)) \tag{3.8}$$

and $H(y(t))$ is nonincreasing in t.

It will be shown first that $\lim_{t\to\infty} y(t) = 0$. Two cases are considered. Suppose first that $0 \leqq \alpha \leqq 1$. Then $(t^\alpha)''' \geqq 0$ which by (ii) means that $(p(t)t^\alpha)' \leqq 0$. This is equivalent to $p'(t) \leqq -\alpha p(t)/t \leqq 0$. Therefore Theorem 3.8 can be applied, which gives $\liminf_{t\to\infty} y(t) = 0$. Suppose that $\limsup_{t\to\infty} y(t) > 0$. Then there is a sequence $\{t_n\} \to \infty$ such that $y''(t_n) \geqq 0, y'(t_n) = 0$, and $y(t_n) \to 0$ as $n \to \infty$. Therefore $\lim_{n\to\infty} H(y(t_n)) \geqq 0$ which implies that $H(y(t)) \geqq 0$ for $t \geqq t_0$ since $H(y(t))$ is nonincreasing. By (3.8) it follows that

$$\int_{t_0}^{\infty} (y(s))^r ds = \frac{H(y(t_0))}{M} < \infty .$$

But this contradicts Lemma 3.10, since $(y'(t))^2 \leqq Ky(t)$ eventually.

Now suppose that $1 < \alpha$. Recall that by hypothesis it is not possible that $y'(t) \geqq 0$ eventually. Suppose that $y'(t) \leqq 0$ eventually. Then $y''(t) < 0$ eventually is impossible since $y(t) > 0$ eventually. Thus, let $\{t_n\} \to \infty$ be such that $y''(t_n) \geqq 0$. Then $H(y(t_n)) > 0$ and the theorem follows as in the preceding paragraph. Now suppose that $y'(t)$ has arbitrarily large zeros. Then there is a sequence $\{s_n\}$ such that $y''(s_n) \geqq 0$ and $y'(s_n) = 0$. Thus $H(y(s_n)) > 0$ and the rest of the proof is as above. This proves that $\lim_{t\to\infty} y(t) = 0$.

It follows that $\lim_{t\to\infty} y'(t) = 0$ since $(y'(t))^2 \leqq Ky(t)$. To see that $\lim_{t\to\infty} y''(t) = 0$, proceed as follows. Since

$$y''(s) + p(s)y(s)\Big|_{t_0}^{t} = -\int_{t_0}^{t} q(s)(y(s))^r ds + \int_{t_0}^{t} p'(s)y(s)ds ,$$

it follows that $y''(t) + p(t)y(t)$ is nonincreasing. Therefore

$$\lim_{t\to\infty} y''(t) + p(t)y(t) = L .$$

Since $\lim_{t\to\infty} p(t)y(t) = 0$ and $\limsup_{t\to\infty} |y''(t)| = 0$, it follows that $L = 0$ and therefore $\lim_{t\to\infty} y''(t) = 0$.

REMARK. For $\alpha > 2$ condition (ii) in the preceding theorem may be replaced by the weaker condition

(ii)′ $u'' + p(t)u = 0$ has no oscillatory solutions in $[a, \infty)$, $p(t)t^2$ is bounded, and $p'(t) \leqq 0$.

The sufficiency of condition (ii)′ is shown as follows. Application of Theorems 3.6 and 3.7 shows that $\lim_{t\to\infty} y(t) = 0$. $\operatorname{Lim}_{t\to\infty} y''(t) = \lim_{t\to\infty} y'(t) = 0$ is established in the same way as before. It will now be shown that (ii) is stronger than (ii)′ if $\alpha > 2$. Suppose that (ii) holds. Then $(p(t)t^\alpha)' \leqq 0$ which implies that $p'(t) \leqq -(\alpha/t)p(t)$. Therefore by a comparison theorem (Birkhoff and Rota, [1, p. 22]) it follows that $p(t) \leqq -At^\alpha$ for some $A > 0$. Therefore, by Kneser's criterion

(Hartman [6, p. 362]), $u'' + p(t)u = 0$ has no oscillatory solutions and clearly $p(t)t^2$ is bounded for large t. Therefore (ii) implies (ii)$'$ if $\alpha > 2$.

The following corollary is merely the refinement which the proof of Theorem 3.11 yields in the linear case. If $\alpha = 0$ below, the condition is $q(t) - p'(t) \geqq \varepsilon > 0$. Thus Corollary 3.12 is a supplement to a theorem of Lazer [10, p. 462] which allows $q(t) - p'(t) \geqq 0$ but requires $q(t) \geqq \varepsilon > 0$.

COROLLARY 3.12. *Suppose that the conditions of Theorem* 3.2 *for* $r = 1$ *are satisfied and in addition that*

$$t^{\alpha}q(t) - (t^{\alpha}p(t))' \geqq \varepsilon > 0$$

for some $0 \leqq \alpha < 3$ *and* $\varepsilon > 0$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1) *with* $r = 1$, *then* $\lim_{t\to\infty} y''(t) = \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} y(t) = 0$.

The next theorem gives some information about the nonoscillatory solutions of (1.1) under different hypotheses than in Theorem 3.11.

THEOREM 3.12. *Let* $p(t) > 0, q(t) \geqq 0, p'(t) \leqq 0$, *and* $\int^{\infty} q(s)ds = \infty$. *If* $y(t)$ *is a nonoscillatory solution of* (1.1), *then* $y(t) = 0((p(t))^{-1})$.

Proof. Suppose that $y(t) > 0$ eventually. If $\lim_{t\to\infty} y(t) = 0$, there is nothing to prove. Therefore, suppose that $\limsup_{t\to\infty} y(t) > 0$.

Note that $\lim_{t\to\infty} F(y(t)) \leqq 0$ by Corollary 3.4 and the fact that $F(y(t))$ is nondecreasing in t. It is asserted that $\lim_{t\to\infty} F(y(t)) = 0$. Since $\liminf_{t\to\infty} y(t) = 0$ (by Theorem 3.8), there is a sequence $\{t_n\} \to \infty$ such that $y''(t_n) \geqq 0, y'(t_n) = 0$ and $y(t_n) \to 0$ as $n \to \infty$. Since $y''(t)$ is bounded above (see the proof of Lemma 3.9), $F(y(t_n)) \to 0$ as $n \to \infty$. Thus $\lim_{t\to\infty} F(y(t)) = 0$.

It is now asserted that $y''(t) + p(t)y(t)$ has the limit $0 \leqq A < \infty$ as $t \to \infty$. Since $y''(t) + p(t)y(t)$ is nonincreasing in t, the limit $A < \infty$ exists, and since $y''(t)$ has arbitrarily large zeros, $A \geqq 0$. Now let $\{s_n\}$ be such that $y'(s_n) = 0$ and $y(s_n) \geqq B > 0$ for all $n = 1, 2, \cdots$. Since $\lim_{t\to\infty} F(y(t)) = 0$, it follows that

$$-[2y''(s_n) + p(s_n)y(s_n)]y(s_n) \longrightarrow 0$$

as $n \to \infty$. Since $y(s_n) \geqq B > 0$, it follows that $2y''(s_n) + p(s_n)y(s_n) \to 0$ as $n \to \infty$. Since $y''(s_n) + p(s_n)y(s_n) \to A$, it follows that $y''(s_n) \to -A$. Therefore, $p(s_n)y(s_n) \to 2A$ as $n \to \infty$. Since $\{s_n\}$ is an arbitrary sequence of relative maxima of $y(t)$, $p(t)y(t)$ is bounded. This proves the theorem.

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