PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 2, 1969

HOMOMORPHISMS OF B*-ALGEBRAS

JAMES D. STEIN, JR.

This paper is divided into two sections. The first deals with Banach algebra homomorphisms of a von Neumann algebra $\mathfrak{A}$, and extends the Bade-Curtis theory for commutative B*-algebras to von Neumann algebras, as well as characterizing the separating ideal in the closure of the range of the homomorphism. The second section concerns homomorphisms of B*-algebras; the chief result being the existence of an ideal $\mathscr{I}$ with cofinite closure such that the restriction of the homomorphism to any closed, two-sided ideal contained in $\mathscr{I}$ is continuous.

1. Homomorphisms of von Neumann algebras. Let $\mathfrak{A}$ be a von Neumann algebra, and let $\nu : \mathfrak{A} \rightarrow \mathfrak{B}$ be a Banach algebra homomorphism. The reduction theory enables us to write

$$\mathfrak{A} = \sum_{i=1}^{\infty} \oplus (C(X_i) \otimes B(\mathscr{H}_i)) \oplus \mathfrak{A}_1 ,$$

where $\mathfrak{A}_1$ is the direct sum of the type II and type III parts, X_i is a hyperstonian compact Hausdorff space, and $\mathscr{H}_i$ is Hilbert space of dimension i (∞ is an allowed index of i, $\mathscr{H}_\infty$ is separable Hilbert space). It was shown in [6] that there is an integer N such that

$$\nu \Big|_{\sum_{i=N+1}^{\infty} \oplus (C(X_i) \otimes B(\mathscr{H}_i)) \oplus \mathfrak{A}_1}$$

is continuous.

Some definitions are in order.

$$S(\nu, \mathfrak{B}) = \{z \in \mathfrak{B} \mid \exists \{x_n\} \subset \mathfrak{A} \ni x_n \rightarrow 0, \quad \nu(x_n) \rightarrow z\} ;$$

$S(\nu, \mathfrak{B})$ is a closed, 2-sided ideal in $\mathfrak{B}$ ([2]). If $f \in C(X_i)$, $T \in B(\mathscr{H}_i)$, then $\langle f \otimes T \rangle$ will denote $(x, y) \in \mathfrak{A}$, where $y = 0 \in \mathfrak{A}_1$ and

$$x \in \sum_{k=1}^{\infty} \oplus (C(X_k) \otimes B(\mathscr{H}_k))$$

has $f \otimes T$ in the i^{th} component and zero in all other components. Let $\varphi_i : C(X_i) \rightarrow \mathfrak{B}$ be defined by $\varphi_i(f) = \nu(\langle f \otimes I_i \rangle)$, where I_i is the identity of $B(\mathscr{H}_i)$, and let F_i be the Bade-Curtis [1] singularity set associated with φ_i. Let $M(F_i) = \{f \in C(X_i) \mid f(F_i) = 0\}$, let $T(F_i) = \{f \in C(X_i) \mid f$ vanishes on a neighborhood of $F_i\}$, and let $R(F_i) = \{f \in C(X_i) \mid f$ is constant in a neighborhood of each point of $F_i\}$. It was shown in [6] that ν is continuous on

$$\sum_{i=1}^{N} \oplus (R(F_i) \otimes B(\mathscr{H}_i)) \oplus \sum_{i=N+1}^{\infty} \oplus (C(X_i) \otimes B(\mathscr{H}_i)) \oplus \mathfrak{A}_1 ,$$

and that this sub-algebra, denoted by $\mathfrak{A}_0$, is dense in $\mathfrak{A}$. Let μ be the unique continuous extension of $\nu \mid \mathfrak{A}_0$ to $\mathfrak{A}$ and let $\lambda = \nu - \mu$. In this section the Bade-Curtis results ([1], Theorems 4.3 and 4.5) will be extended to $\mathfrak{A}$, and a complete characterization of $S(\nu, \mathfrak{B})$ will be obtained.

THEOREM 1.1. (a) *The range of μ is closed in $\mathfrak{B}$ and $\overline{\nu(\mathfrak{A})} = \mu(\mathfrak{A}) \oplus S(\nu, \mathfrak{B})$, the direct sum being topological.*

(b) $S(\nu, \mathfrak{B}) = \overline{\lambda(\mathfrak{A})}$.

(c) *Let*

$$M = \sum_{i=1}^{N} \oplus (M(F_i) \otimes B(\mathscr{H}_i)) \oplus \sum_{i=N+1}^{\infty} (C(X_i) \otimes B(\mathscr{H}_i)) \oplus \mathfrak{A}_1 .$$

Then $S(\nu, \mathfrak{B}) \cdot M = M \cdot S(\nu, \mathfrak{B}) = (0)$, and $\lambda \mid M$ is a homomorphism.

Proof. $\mu(\mathfrak{A})$ is closed by [2], Lemma 5.3. We first show $\lambda(\mathfrak{A}) \subseteqq S(\nu, \mathfrak{B})$. If $x \in \mathfrak{A}$, choose a sequence $\{x_n\}$ from the dense sub-algebra such that $\lim_{n \to \infty} x_n = x$. Since μ is continuous,

$$\mu(x) = \lim_{n \to \infty} \mu(x_n) = \lim_{n \to \infty} \nu(x_n) ,$$

and since $\lim_{n \to \infty}(x_n - x) = 0$,

$$\mu(x) - \nu(x) = \lim_{n \to \infty} (\nu(x_n) - \nu(x)) = \lim_{n \to \infty} \nu(x_n - x) = s \in S(\nu, \mathfrak{B}) .$$

But $\nu(x) = \mu(x) + \lambda(x)$ and $\nu(x) = \mu(x) - s$, so $\lambda(x) = -s \in S(\nu, \mathfrak{B})$.

If $s \in S(\nu, \mathfrak{B})$, there is a sequence $\{x_n\}$ in $\mathfrak{A}$ such that

$$\lim_{n \to \infty} x_n = 0, \qquad \lim_{n \to \infty} \nu(x_n) = s .$$

Now $\lim_{n \to \infty} \mu(x_n) = 0$, and $s = \lim_{n \to \infty} (\mu(x_n) + \lambda(x_n))$, so

$$\| s - \lambda(x_n) \| \leqq \| s - (\lambda(x_n) + \mu(x_n)) \| + \| \mu(x_n) \| \longrightarrow 0 ,$$

and so $S(\nu, \mathfrak{B}) = \overline{\lambda(\mathfrak{A})}$.

Let $U = \nu^{-1}(S(\nu, \mathfrak{B}))$. We now show $\mu(\mathfrak{A}) \cap S(\nu, \mathfrak{B}) = (0)$. If $\mu(x) \in S(\nu, \mathfrak{B})$, since $\nu(x) = \mu(x) + \lambda(x)$ and $\lambda(\mathfrak{A}) \subseteqq S(\nu, \mathfrak{B})$, we see that $\nu(x) \in S(\nu, \mathfrak{B})$, and so $x \in U$. But by [6], Theorem II. 5, and [7], Proposition 2.1, $U = \overline{\operatorname{Ker}(\nu)} = \operatorname{Ker}(\mu)$, so $\mu(x) = 0$.

To complete the proof of (a) and (b), all we need show is that any $z \in \overline{\nu(\mathfrak{A})}$ can be written $z = \mu(x) + s$, where $x \in \mathfrak{A}$, $s \in S(\nu, \mathfrak{B})$. Let $\hat{\nu} : \mathfrak{A}/U \longrightarrow \nu(\mathfrak{A})/S(\nu, \mathfrak{B})$ be defined by $\hat{\nu}(x + U) = \nu(x) + S(\nu, \mathfrak{B})$, by [2], Theorem 4.6, and [5], Theorem 4.9.2, this is a continuous isomorphism of a B^*-algebra and thus has closed range. So $z + S(\nu, \mathfrak{B}) = \nu(x) + S(\nu, \mathfrak{B})$, and so $\exists s \in S(\nu, \mathfrak{B})$ such that $z = \nu(x) + s = \mu(x) + (\lambda(x) + s)$. But $\lambda(x) + s \in S(\nu, \mathfrak{B})$.

Define T by substituting $T(F_i)$ for $M(F_i)$, $1 \leqq i \leqq N$, in the definition of M. The same proof as [6], Prop. II. 3, shows that T is dense in M, and by the continuity of μ to show $\mu(M)\cdot S(\nu, \mathfrak{B}) = (0)$ we need merely show $x \in \mathfrak{A}$, $z \in T \Rightarrow \mu(z)\lambda(x) = 0$ (the proof of $S(\nu, \mathfrak{B})\cdot\mu(M) = (0)$ is symmetric). Clearly $zx \in T$, and μ and ν agree on T, so $\mu(z)\lambda(x) = \mu(z)(\nu(x) - \mu(x)) = \mu(z)\nu(x) - \mu(z)\mu(x) = \nu(zx) - \mu(zx) = 0$. That $\lambda \mid M$ is a homomorphism follows from $\mu(M)\cdot S(\nu, \mathfrak{B}) = (0)$ and the arguments of Bade and Curtis ([1], p. 601).

The analogue of [1], Theorem 4.3d, will be stated but, once the definitions are made, the proofs precisely parallel the proofs given in [1], and so will be omitted. It should be noted, however, that the proofs carry over because, for $1 \leqq i \leqq N$, $C(X_i) \otimes B(\mathscr{H}_i)$ is actually the algebraic tensor product.

For $1 \leqq i \leqq N$, let $F_i = \{\omega_{i,k} \mid 1 \leqq k \leqq n_i\}$, and for each i, $1 \leqq i \leqq N$, choose functions $e_{i,k} \in C(X_i)$ such that $e_{i,k}$ is 1 in a neighborhood of $\omega_{i,k}$ and $e_{i,k}e_{i,j} = 0$, $k \neq j$. Let I_i denote the identity of $B(\mathscr{H}_i)$, and define $\lambda_{i,k}(x) = \lambda(\langle e_{i,k} \otimes I_i\rangle x)$ (note that this is equal to $\lambda(x\langle e_{i,k} \otimes I_i\rangle)$. Let $R_{i,k} = \overline{\lambda_{i,k}(\mathfrak{A})}$, let $M(\omega_{i,k})$ be all functions in $C(X_i)$ vanishing at $\omega_{i,k}$, and let $M_{i,k}$, be $\mathfrak{A}$ with $C(X_i) \otimes B(\mathscr{H}_i)$ replaced in the direct sum by $M(\omega_{i,k}) \otimes B(\mathscr{H}_i)$.

PROPOSITION 1.2.

(a) $$\lambda = \sum_{i=1}^{N} \sum_{k=1}^{n_i} \lambda_{i,k}$$

(b) $$S(\nu, \mathfrak{B}) = \sum_{i=1}^{N} \sum_{k=1}^{n_i} \oplus R_{i,k} ,$$

the direct sum being topological.

(c) $$(i, j) \neq (k, l) \Rightarrow R_{i,j}\cdot R_{k,l} = (0) ,$$

and

$$R_{i,k}\cdot\mu(M_{i,k}) = \mu(M_{i,k})\cdot R_{i,k} = (0) .$$

(d) *The restriction of* $\lambda_{i,k}$ *to* $M_{i,k}$ *is a homomorphism.*

It is possible to obtain a characterization of the ideal $S(\nu, \mathfrak{B})$ by examining the action of ν as related to the operator algebras $B(\mathscr{H}_i)$, rather than the function spaces $C(X_i)$. For $1 \leqq i \leqq N$, let e_i be the identity of $C(X_i)$, and let $\lambda_i(x) = \lambda(\langle e_i \otimes I_i\rangle x)$; then $\lambda(x) = \sum_{i=1}^{N}\lambda_i(x)$. Now

$$\begin{aligned}&\mu(\langle e_j \otimes I_j\rangle)\lambda_i(x)\\ &\quad = \mu(\langle e_j \otimes I_j\rangle)[\nu(\langle e_i \otimes I_i\rangle x) - \mu(\langle e_i \otimes I_i\rangle x)]\\ &\quad = \nu(\langle e_j \otimes I_j\rangle\langle e_i \otimes I_i\rangle x) - \mu(\langle e_j \otimes I_j\rangle\langle e_i \otimes I_i\rangle x)\\ &\quad = \delta_{ij}\lambda_i(x)\ ,\end{aligned}$$

and if $i \neq j$ then

$$\lim_{n\to\infty} \lambda_i(x_n) = \lim_{n\to\infty} \lambda_j(y_n)$$

yields the fact that both these limits are zero, and consequently

$$S(\nu, \mathfrak{B}) = \sum_{i=1}^{N} \oplus \overline{\lambda_i(\mathfrak{A})}\ ,$$

a topological direct sum. Now each of these components will be characterized.

Fix n such that $1 \leqq n \leqq N$, and let $\{T_{i,j} \mid 1 \leqq i, j \leqq n\}$ be a system of matrix units for $B(\mathscr{H}_n)$, i.e., $T_{i,j}T_{k,l} = \delta_{jk}T_{i,l}$. Define, for $1 \leqq i, j \leqq n$, maps $\nu_{i,j}$, $\mu_{i,j}$, and $\gamma_{i,j}$ of $C(X_n)$ into $\mathfrak{B}$ by $\nu_{i,j}(f) = \nu(\langle f \otimes T_{i,j}\rangle)$, $\mu_{i,j}(f) = \mu(\langle f \otimes T_{i,j}\rangle)$, and $\gamma_{i,j}(f) = \nu_{i,j}(f) - \mu_{i,j}(f)$. If

$$x = \left\langle \sum_{i,j=1}^{n} f_{i,j} \otimes T_{i,j} \right\rangle ,$$

we can clearly write

$$\nu(x) = \sum_{i,j=1}^{n} \nu_{i,j}(f_{i,j})\ ;$$

similar assertions hold for $\mu(x)$ and $\lambda(x)$. All maps are linear, but the "off-diagonal" maps (those for which $i \neq j$) are not necessarily homomorphisms.

Computational procedures similar to those already employed will show

$$\mu(\langle e_n \otimes T_{k,l}\rangle)\gamma_{i,j}(f) = \delta_{il}\gamma_{k,j}(f)$$

and

$$\gamma_{i,j}(f)\mu(\langle e_n \otimes T_{k,l}\rangle) = \delta_{jk}\gamma_{i,l}(f)\ ,$$

so if

$$\lim_{m\to\infty} \gamma_{i,j}(f_m) = \lim_{m\to\infty} \gamma_{k,l}(g_m)$$

and $i \neq k$, left multiplication by $\mu(\langle e_n \otimes T_{i,i}\rangle)$ shows that

$$\lim_{m\to\infty}\gamma_{i,j}(f_m) = 0\ ;$$

the same trick with right multiplication works if $j \neq l$, and so

$$\overline{\lambda_n(\mathfrak{A})} = \sum_{i,j=1}^{n} \oplus \overline{\gamma_{i,j}(\mathfrak{A})} ,$$

and this is a topological direct sum.

Since $T_{i,j} = T_{i,k}T_{k,j}$, we see that

$$\begin{aligned}\nu_{i,j}(fg) &= \nu(\langle fg \otimes T_{i,j}\rangle) \\ &= \nu(\langle f \otimes T_{i,k}\rangle\langle g \otimes T_{k,j}\rangle) = \nu_{i,k}(f)\nu_{k,j}(g) ;\end{aligned}$$

consequently $\nu_{i,i}$ is a homomorphism for $1 \leqq i \leqq n$ (let $j = k = i$) and so by [1], Th. 4.3b), $\overline{\gamma_{i,i}(\mathfrak{A})}$ is the Jacobson radical of $\overline{\nu_{i,i}(C(X_n))}$. Since $\mu(\langle e_n \otimes T_{i,j}\rangle)\gamma_{j,j}(f) = \gamma_{i,j}(f)$, it is clear that

$$\overline{\gamma_{i,j}(\mathfrak{A})} = \mu(\langle e_n \otimes T_{i,j}\rangle)\overline{\gamma_{j,j}(\mathfrak{A})} .$$ This yields

PROPOSITION 1.3. *$S(\nu, \mathfrak{B})$ is the direct sum of Jacobson radicals of commutative Banach algebras and "rotations" of these radicals.*

Note that $\nu_{i,j}(f) = \nu_{i,i}(f)\nu_{i,j}(e_n)$, and so the continuity of the $\nu_{i,j}$, and hence the continuity of ν, depends only on the continuity of the diagonal homomorphisms $\nu_{i,i}$. Coupling this fact with Theorem 4.5 of [1], we observe that if all the Jacobson radicals of the closures of the images of the diagonal homomorphisms are nil ideals, then the homomorphism is continuous.

2. Homomorphisms of B^*-algebras.

Let $\mathfrak{A}$ be a B^*-algebra, and let $\nu : \mathfrak{A} \to \mathfrak{B}$ be a Banach algebra homomorphism, with $S(\nu, \mathfrak{B})$ defined as in § 1.

DEFINITION 2.1.

$$\begin{aligned}\mathscr{T}_L &= \{x \in \mathfrak{A} \mid \nu(x) \cdot S(\nu, \mathfrak{B}) = (0)\} , \\ \mathscr{T}_R &= \{x \in \mathfrak{A} \mid S(\nu, \mathfrak{B}) \cdot \nu(x) = (0)\} .\end{aligned}$$

DEFINITION 2.2.

$$\begin{aligned}\mathscr{I}_L &= \{x \in \mathfrak{A} \mid \sup_{\|z\| \leqq 1} \| \nu(xz) \| < \infty\} , \\ \mathscr{I}_R &= \{x \in \mathfrak{A} \mid \sup_{\|z\| \leqq 1} \| \nu(zx) \| < \infty\} , \\ \mathscr{I} &= \mathscr{I}_L \cap \mathscr{I}_R .\end{aligned}$$

$\mathscr{T}_L$, $\mathscr{T}_R$, $\mathscr{I}_L$, $\mathscr{I}_R$, and $\mathscr{I}$ are all two-sided ideals in $\mathfrak{A}$ (see [4] and [6]), and in a recent paper [4] Johnson has shown that $\overline{\mathscr{T}_L}$ is a cofinite ideal in $\mathfrak{A}$, and observes that, if one could show $\nu \mid \overline{\mathscr{T}_L}$ is continuous, one would have a direct extension of the Bade-Curtis theory to arbitrary B^*-algebras. An examination of this problem, coupled with an analysis of these ideals, constitutes the body of this section.

We first note that $\mathscr{I}_L \subset \mathscr{T}_L$. For, if $x \notin \mathscr{T}_L$ then there is an $s \in S(\nu, \mathfrak{B})$ such that $\nu(x)s \neq 0$, and consequently $\exists \{x_n\} \subset \mathfrak{A}$ such that $x_n \to 0$, $\nu(x_n) \to s$, and so $\nu(xx_n) \to v(x)s \neq 0$. Given $M > 0$, choose x_n such that

$$\| x_n \| \leqq \frac{\| \nu(x)s \|}{2M}, \qquad \| \nu(xx_n) \| > \tfrac{1}{2} \| \nu(x)s \| .$$

Then

$$\frac{x_n}{\| x_n \|}$$

has norm one and

$$\left\| \nu\left(x \frac{x_n}{\| x_n \|}\right) \right\| > M ,$$

and so $x \notin \mathscr{I}_L$. Similarly $\mathscr{I}_R \subset \mathscr{T}_R$.

Repeated use throughout this section will be made of the following lemma and its corollaries.

LEMMA 2.1. *Let $\{f_n\}$, $\{g_n\}$ be sequences from $\mathfrak{A}$ such that $m \neq n \Rightarrow g_m g_n = 0$, $g_n f_m = 0$. Then there is an integer N such that $n \geqq N \Rightarrow g_n f_n \in \mathscr{I}_R$.*

Proof. Suppose not, and renumber to obtain a sequence such that $g_n f_n \notin I_R$ for any n. Then for each n choose $x_n \in \mathfrak{U}$ such that $\| x_n \| \leqq 1$,

$$\| \nu(x_n g_n f_n) \| > n 2^n \| g_n \| \| \nu(f_n) \| .$$

Let

$$x = \sum_{k=1}^{\infty} (1/2^k \| g_k \|) x_k g_k ;$$

then clearly $x \in \mathfrak{A}$. We also have

$$x f_n = \sum_{k=1}^{\infty} (1/2^k \| g_k \|) x_k g_k f_n = x_n g_n f_n / (2^n \| g_n \|) ,$$

and so

$$\begin{aligned} \| \nu(x) \| \| \nu(f_n) \| &\geqq \| \nu(x f_n) \| \\ &= \| \nu(x_n g_n f_n) \| / 2^n \| g_n \| > n \| \nu(f_n) \| , \end{aligned}$$

which implies $\| \nu(x) \| > n$, a contradiction.

COROLLARY 2.1.1. *If $\{g_n\}$, $\{f_n\} \subset \mathfrak{A}$ satisfy $g_m g_n = 0$, $g_n f_n = f_n$, then $\exists N$ such that $n \geqq N \Rightarrow f_n \in \mathscr{I}_R$.*

COROLLARY 2.1.2. *If* $\{f_n\} \in \mathfrak{A}$ *satisfies* $f_m f_n = 0$, *then* $\exists N$ *such that* $n \geqq N \Rightarrow f_n^2 \in I_R$.

COROLLARY 2.1.3. *If* $\{f_n\}, \{g_n\} \subset \mathfrak{A}$ *satisfy* $g_m g_n = 0, f_m g_n = 0$, *then* $\exists N$ *such that* $n \geqq N \Rightarrow f_n g_n \in I_L$.

We can now combine these results with those of Johnson ([4], Th. 2.1) to see that, if $\mathfrak{A}$ is a B^*-algebra, $\overline{\mathscr{F}}$ is a cofinite ideal. The advantage of using $\overline{\mathscr{F}}$ can be seen from the following.

PROPOSITION 2.1. *Let* $\nu : \mathfrak{A} \to \mathfrak{B}$ *be a Banach algebra homomorphism, and let* $\mathfrak{U}$ *be a closed linear subspace of* $\mathscr{F}$. *Then*

$$\sup\{\| \nu(xy) \| \mid x, y \in \mathfrak{U}, \quad \| x \| \leqq 1, \quad \| y \| \leqq 1\} < \infty .$$

Proof. For $z \in \mathfrak{A}$, let L_z and R_z map $\mathfrak{A}$ into $\mathfrak{B}$ and be defined by $L_z(x) = \nu(zx)$, $R_z(x) = \nu(xz)$; these are clearly linear. If $z \in \mathscr{F}$, then both L_z and R_z are continuous. For, if $x_n \to 0$ and $L_z(x_n) \nrightarrow 0$, we can assume $\| L_z(x_n) \| \geqq \delta > 0$. Given $M > 0$, choose x_n such that

$$\| x_n \| \leqq \frac{\delta}{M} ,$$

then

$$\left\| \frac{M}{\delta} x_n \right\| \leqq 1 , \qquad \left\| L_z\left(\frac{M}{\delta} x_n \right) \right\| \geqq M ;$$

since this can be done for any M it contradicts $z \in \mathscr{F}_L$. Now, for each $x \in \mathfrak{U}$,

$$\begin{aligned} \sup\{\| L_z(x) \| \mid z \in \mathfrak{U}, \| z \| \leqq 1\} &= \sup\{\| \nu(zx) \| \mid z \in \mathfrak{U}, \| z \| \leqq 1\} \\ &\leqq \sup\{\| \nu(zx) \| \mid z \in \mathfrak{A}, \| z \| \leqq 1\} < \infty \end{aligned}$$

since $x \in \mathscr{F}_R$. By the Uniform Boundedness Principle ([3], 2.3.21)

$$\sup\{\| L_z \| \mid z \in \mathfrak{U}, \| z \| \leqq 1\} < \infty$$

and so

$$\sup\{\| \nu(zx) \| \mid z, x \in \mathfrak{U}, \| z \| \leqq 1, \| x \| \leqq 1\} < \infty$$

completing the proof.

PROPOSITION 2.2. *Let* $\mathfrak{A}$ *be a* C^*-*algebra, and let* $\mathfrak{U} \subseteqq \mathscr{F}$ *be a closed two-sided ideal. Then* $\nu \mid \mathfrak{U}$ *is continuous.*

Proof. Let $U \in \mathfrak{U}$, and recall that $\mathfrak{U}$ is a *-ideal. Use the polar decomposition to write $U = TP$, where T is a partial isometry (hence $\| T \| = 1$) and P is a positive operator satisfying $P^2 = U^* U$. Assume $\| U \| = 1$, then since P is self-adjoint, $\| P \|^2 = \| P^* P \| = \| P^2 \| =$

$\| U^*U \| = \| U \|^2 = 1$, so $\| P \| = 1$. Since P is self-adjoint, it has a square root $Q \in \mathfrak{U}$, so we can write $U = (TQ)Q$, where $TQ, Q \in \mathfrak{U}$, $\| TQ \| \leqq \| T \| \| Q \| \leqq 1$, $\| Q \| \leqq 1$. So, by Proposition 2.1,

$$\begin{aligned}&\sup \{\| \nu(U) \| \mid U \in \mathfrak{U} \mid U \in \mathfrak{U}, \| U \| \leqq 1\} \\ &\qquad \leqq \sup \{\| \nu(xy) \| \mid x, y \in \mathfrak{U}, \| x \| \leqq 1, \| y \| \leqq 1\} < \infty ,\end{aligned}$$

and so $\nu \mid \mathfrak{U}$ is continuous.

If $\mathfrak{U}$ is a commutative B^*-algebra, Proposition 2.2 shows that, if N is a closed neighborhood of the Bade-Curtis singularity set, ν is continuous on the ideal of all functions vanishing on N, and Proposition 2.2 can be regarded as the analogue for B^*-algebras of that theorem, especially in view of the remarks following Corollary 2.1.3. However, it appears to be a difficult problem to obtain the full strength of the Bade-Curtis results using these methods, but if a method is found there is a good chance that it would generalize the Bade-Curtis results to arbitrary B^*-algebras.

We now turn our attention to $C(X)$, where X is a compact Hausdorff space. The notation of § 1 applies.

PROPOSITION 2.3. *$T(F) \subseteqq \mathscr{I}$, and if $\mathscr{I}$ is closed, ν is continuous.*

Proof. Let f vanish on a neighborhood of F. If $f \notin \mathscr{I}$, $\exists \{g_n\} \in C(X)$ such that $\| g_n \| \leqq 1$, $\| \nu(fg_n) \| \geqq n^2$. Let $h_n = 1/n\, g_n$, then $h_n f \to 0$, and since ν is continuous on $T(F)$, $\nu(h_n f) \to 0$. But

$$\| \nu(h_n f) \| = \frac{1}{n} \| \nu(g_n f) \| \geqq n ,$$

a contradiction.

If $\mathscr{I}$ is closed, $M(F) = \overline{T(F)} \subseteqq \mathscr{I}$, and by Proposition 2.2, $\nu \mid M(F)$ is continuous. Using the technique of Theorem 4.1 of [1], ν is continuous.

Since $T(F) \subseteqq \mathscr{I}$ and, if K denotes the kernel of ν, $\bar{K} \cap T(F) = K \cap T(F)$ ([7], 2.3), one might wish to show that $\bar{K} \cap \mathscr{I} = K$ (clearly $K \subseteqq \mathscr{I}$). If $f \in \bar{K} \cap \mathscr{I}$, then $g_n \to 0 \Rightarrow \nu(g_n f) \to 0$. Let $g \in M(F)$, and choose a sequence $\{g_n\}$ from $T(F)$ such that $g_n \to g$. Then $g_n f \in \bar{K} \cap T(F) \subseteqq K$, and so

$$\nu(gf) = \lim_{n \to \infty} \nu(g_n f) = 0 .$$

So $M(F) \cdot (\bar{K} \cap \mathscr{I}) \subseteqq K$.

If $\mathfrak{A} = C(X)$, Corollary 2.1.2 can be strengthened so the conclusion is $\exists N$ such that $n \geqq N \Rightarrow f_n \in \mathscr{I}$. If this integer N is independent of the sequence $\{f_n\}$, then the homomorphism is continuous, if X is such that every point is a G_δ. We first note that, if $\{E_n \mid n = 1, 2, \cdots\}$ is a disjoint sequence of open sets, then $n \geqq N$, $f(E_n') = 0 \Rightarrow f \in \mathscr{I}$; this is a clear consequence of Corollary 2.1.2. The goal will be to show that, if N is independent of sequence, then $M(F) \subseteqq \mathscr{I}$, as in Proposition 2.3 this will show ν is continuous. Choose open sets $E, G \subseteqq X$ such that $\bar{E} \cap \bar{G} = F$, and let $f \in M(F)$. Let

$$A_k = \left\{x \in X \mid |f(x)| \geqq \frac{1}{k}\right\},$$

and let $B_k = A_k \cap \bar{G}$; then B_k is closed and disjoint from $\bar{E}$ for all k. By Urysohn's Lemma, choose a function g_k such that $0 \leqq g_k \leqq 1$, $g_k(\bar{E}) = 1$, $g_k(B_k) = 0$. We assert that $\{g_n f \mid n = 1, 2, \cdots\}$ is Cauchy. Assume $n > m$, and look at $\| g_n f - g_m f \|$. This value is the maximum of the supremums of $| g_n f(x) - g_m f(x) |$ on the sets $\bar{E}$, B_m, and $K_m = X \sim (B_m \cup \bar{E})$. This supremum is clearly 0 on $\bar{E}$ (since $g_n(\bar{E}) = g_m(\bar{E}) = 1$) and on B_m (since $n > m \Rightarrow B_m \subseteqq B_n$), and clearly

$$\sup_{x \in K_m} | g_n f(x) - g_m f(x) | \leqq \frac{1}{n} + \frac{1}{m} < \frac{2}{m},$$

so the sequence is Cauchy, and there is an $h \in C(X)$ such that $\| g_n f - h \| \to 0$. $h(\bar{E}) = f(\bar{E})$, since $g_n(\bar{E}) = 1$ for all n. If $x \in \bar{G}$ and $|f(x)| > 0$, there is an integer K such that $k \geqq K \Rightarrow x \in A_k \Rightarrow x \in B_k \Rightarrow g_k f(x) = 0$; if $f(x) = 0$ $g_k f(x) = 0$ for all k, and so $h(\bar{G}) = 0$.

Now choose sequences of disjoint open sets $\{E_n\}$, $\{G_n\}$ (the E_n are not necessarily disjoint from the G_n) such that $F \subseteqq \bar{E}_n \cap \bar{G}_n$, $\bar{E} \supseteqq E_N'$, and $\bar{G} \supseteqq G_N'$. If $g \in C(X)$, $g(G_N') = 0 \Rightarrow g \in \mathscr{I}$, or $g(E_N') = 0 \Rightarrow g \in \mathscr{I}$, so $h(\bar{G}) = 0 \Rightarrow h \in \mathscr{I}$; similarly $(h - f)(\bar{E}) = 0 \Rightarrow h - f \in \mathscr{I}$, so $f = h + (f - h) \in \mathscr{I}$. Thus $M(F) \subseteqq \mathscr{I}$, completing the proof. A similar idea also works for von Neumann algebras by reducing it to a consideration of $\varphi_i : C(X) \to \mathfrak{B}$ defined by $\varphi_i(f) = \nu(\langle f \otimes I_i \rangle)$.

References

1. W. G. Bade and P. C. Curtis, Jr., *Homomorphisms of commutative banach algebras*, Amer. J. Math. **82** (1960), 589-608.

2. S. B. Cleveland, *Homomorphisms of non-commutative *-algebras*, Pacific J. Math. **13** (1963), 1097-1109.

3. N. Dunford and J. Schwartz, *Linear operators*, Interscience, New York, 1958.

4. B. E. Johnson, *Continuity of homomorphisms of algebras of operators* (II) (to appear)

5. C. E. Rickart, *Banach algebras*, Van Nostrand, New York, 1960.

6. J. D. Stein, Jr., *Homomorphisms of von Neumann algebras* (to appear Amer. J. Math.)

7. ———, *Homomorphisms of semi-simple algebras*, Pacific J. Math. **26** (1968), 589-594.

Received March 8, 1968.

UNIVERSITY OF CALIFORNIA, LOS ANGELES