PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 2, 1969

A NOTE ON EXTREMAL LENGTH AND MODULUS

RICHARD KATZ

The equality of extremal length and modulus is shown for general annuli in a Riemannian manifold.

In a recent paper, [2], Ow showed that the modulus and extremal length of an annulus in a Riemannian manifold are equal under the assumption that the set of critical points of the harmonic measure of the annulus has capacity zero. The purpose of this note is to show that this condition can be dispensed with. We refer the reader to [2] for definitions.

THEOREM. *The extremal length* (λ) *and modulus* (μ) *of an annulus* (Ω, α, β) *are equal.*

Proof. The inequality $\lambda \geqq \mu$ was shown in [2].

To show the opposite inequality, let u be the harmonic measure of (Ω, α, β), and h be a function such that

$$h \mid \alpha = 0, h \mid \beta = 1, \mid \nabla h^2 - \nabla u^2 \mid < \varepsilon ,$$

and h has only a finite number of critical points in Ω. For the existence of such a function see Milnor [1]. Let Γ_0 be the set of integral curves of ∇h which do not meet a critical point and P the set of admissible densities on Ω. It is immediate that $|\nabla h| \in P$. Now for $\gamma \in \Gamma_0$, $\rho \in P$

$$\int_\Omega \rho^2 dV = \int_\alpha \Big(\int_\gamma \frac{\rho^2}{\nabla h^2} dh\Big) * dh \geqq \int_\alpha \Big(\int_\gamma \Big|\frac{\rho}{\nabla h}\Big| dh\Big)^2 * dh$$

and

$$\int_\alpha \Big(\int_\gamma \Big|\frac{\rho}{\nabla h}\Big| dh\Big)^2 * dh = \int_\alpha \Big(\int_\gamma \rho dl\Big)^2 * dh \geqq \inf_{\Gamma_0} \Big(\int_\gamma \rho dl\Big)^2 \int_\alpha * dh .$$

Since $|\nabla h^2 - \nabla u^2| < \varepsilon$, it follows that

$$\Big| \int_\alpha * dh - \mu^{-1} \Big| < \varepsilon V(\Omega)$$

and therefore,

$$\lambda \leqq \sup_P \inf_{\Gamma_0} \frac{\Big(\int_\gamma \rho dl\Big)^2}{\int_\Omega \rho^2 dV} \leqq \mu + K\varepsilon$$

for a suitable constant K. Since ε was arbitrary, we have $\lambda \leqq \mu$ which completes the proof.

REFERENCES

1. J. Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, N. J., 1965.
2. W. Ow, *An extremal length criterion for the parabolicity of Riemannian spaces*, Pacific J. Math. **23** (1967), 585-590.

Received March 4, 1968.

CALIFORNIA STATE COLLEGE
LOS ANGELES, CALIFORNIA