PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 3, 1969

TRIPLES OF OPERATOR-VALUED FUNCTIONS RELATED TO THE UNIT CIRCLE

A. G. GIBSON

In this paper various triples of operator-valued functions acting in a Hilbert space are characterized, and the members are shown to be connected by a one-to-one-to-one correspondence. The elements of the triples are operator measures, generalized resolvents, and positive definite sequences which are related to the unit circle. The relationships between operator measures and positive definite sequences were first obtained by M. A. Naĭmark and B.Sz.-Nagy in their dilation and moment theorems. The main contribution of this paper is a characterization of the interrelated resolvent classes. By exploiting the correspondence between the various classes, a unified development of the theory is obtained.

R. McKelvey [8] developed the interrelations among three classes of operator-valued functions $\{E_t, R_\lambda, V_s\}$ related to operators with spectrum in a half-plane. In the prototype for the general situation these functions were associated with a self-adjoint operator T, and E_t was its spectral function, R_λ its resolvent, and V_s the unitary group e^{-isT}. In more general cases, the three classes were associated with dissipative and symmetric operators.

In this study we investigate the interrelations between three analogous classes of operator-valued functions $\{E_\theta, R_z, T^{(k)}\}$ related to operators with spectrum in the unit circle. In the prototype these functions are associated with a unitary operator T, and E_θ is its spectral function, R_z its resolvent, and $T^{(k)}$ the cyclic group of its integral powers. Generalizations here include the triples associated with contraction, isometric, and partially unitary operators. In our most general case, the triple $\{E_\theta, R_z, T^{(k)}\}$ belongs to $\mathscr{S} = \{\mathscr{E}, \mathscr{R}, \mathscr{T}\}$, where the classes in the triple $\mathscr{S}$ are called *operator distribution functions*, *generalized resolvents*, and *positive definite operator-valued sequences* respectively. In saying that the triple of operator-valued functions $\{E_\theta, R_z, T^{(k)}\}$ belongs to the triple of classes $\mathscr{S} = \{\mathscr{E}, \mathscr{R}, \mathscr{T}\}$ we shall mean $E_\theta \in \mathscr{E}$, $R_z \in \mathscr{R}$, and $T^{(k)} \in \mathscr{T}$, i.e., a triple of classes is to be understood in the dual sense of a triple and as the Cartesian product of the classes $\mathscr{E}, \mathscr{R}, \mathscr{T}$.

The classes $\mathscr{E}$, $\mathscr{R}$, and $\mathscr{T}$ of bounded linear operators on the complex Hilbert space $\mathscr{H}$ to $\mathscr{H}$ are defined as follows:

(α) E_θ $(0 \leqq \theta \leqq 2\pi)$ belongs to the class $\mathscr{E} = \mathscr{E}(\mathscr{H})$ whenever

(a) $0 \leqq (E_{\theta_1}x, x) \leqq (E_{\theta_2}x, x) \leqq (x, x)$ for $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$ and

$x \in \mathscr{H}$.

(b) $E_\theta = E_{\theta+0} \equiv \operatorname*{Lim}_{\varphi\to\theta+} E_\varphi,\ 0 \leqq \theta < 2\pi$.

(c) $E_0 = 0$.

(β) $R_z\,(|z| \neq 1)$ belongs to the class $\mathscr{R} = \mathscr{R}(\mathscr{H})$ whenever

(a) R_z is a holomorphic function of z for $|z| \neq 1$.

(b) $$\|R_z x\|^2 \leqq \frac{2\,\mathrm{Re}\,(R_z x, x) - (R_0 x, x)}{1 - |z|^2}\,, \qquad |z| \neq 1,\ x \in \mathscr{H}\,,$$

i.e., $(R_0 x, x)$ is real and this inequality holds.

(c) $R_{z*} = R_0 - R_z^*,\ 0 < |z| < 1$ or $|z| > 1$, where $z^* = 1/\bar{z}$.

(γ) $T^{(k)}\,(\pm k = 0, 1, 2, \cdots)$ belongs to the class $\mathscr{T} = \mathscr{T}(\mathscr{H})$ whenever

(a) $T^{(0)} \leqq I$.

(b) The sequence $\{T^{(k)}\}$ is positive definite, i.e.,

$$\sum_{j,k=-n}^{n}\sum (T^{(j-k)}x_j, x_k) \geqq 0$$

for $n = 1, 2, \cdots$, and arbitrary sequences $\{x_k\}_{-n}^{n}$ of vectors in $\mathscr{H}$.

In § 2 a functional calculus will be presented wich is based on a mapping from a class of functions bounded and measurable with respect to the operator distribution function $E_\theta \in \mathscr{E}$ to a class of bounded operators. It is similar to the functional calculus previously developed by M. Schreiber [12, 13], and to that known for the unitary operator [10, § 109]. We then prove the main theorem of this paper which is the following:

THEOREM A. *Each function $E_\theta \in \mathscr{E}$, $R_z \in \mathscr{R}$, or $T^{(k)} \in \mathscr{T}$ belongs to a unique triple of functions $\{E_\theta, R_z, T^{(k)}\}$ in $\mathscr{S} \equiv \{\mathscr{E}, \mathscr{R}, \mathscr{T}\}$, such that the members of the triple are interrelated by formulas* (1)-(6).

Formulas (1)-(6) are the following representations which will be seen in § 2.1 to make sense in either the strong or weak topology.

(1) $$R_z = \int_0^{2\pi} \frac{1}{1 - e^{i\theta}z}\,dE_\theta \qquad (|z| \neq 1)\,.$$

(2) $$T^{(k)} = \int_0^{2\pi} e^{ik\theta}\,dE_\theta \qquad (\pm k = 0, 1, 2, \cdots)\,.$$

(3) $$\begin{aligned} R_z &= \sum_{k=0}^{\infty} z^k T^{(k)} && (|z| < 1) \\ &= -\sum_{k=1}^{\infty} z^{-k} T^{(-k)} && (|z| > 1)\,. \end{aligned}$$

$$(4) \qquad E_{\theta_2} - E_{\theta_1} = \frac{1}{\pi} \operatorname*{Lim}_{r \to 1-} \int_{\theta_1}^{\theta_2} \operatorname{Re} R_{r\exp(-i\theta)} d\theta - \frac{(\theta_2 - \theta_1)}{2\pi} R_0$$

$$(\theta_1, \theta_2 \text{ points of continuity of } E_\theta) .$$

$$(5) \qquad E_{\theta_2} - E_{\theta_1} = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{e^{-ik\theta_1} - e^{-ik\theta_2}}{ik} T^{(k)}$$

$$(\theta_1, \theta_2 \text{ points of continuity of } E_\theta) .$$

$$(6) \qquad T^{(k)} = \left. \frac{d^k R_z}{k! dz^k} \right|_{z=0} \qquad (k = 0, 1, 2, \cdots)$$

$$T^{(-k)} = -\frac{1}{k!} \operatorname*{Lim}_{z \to 0} \frac{d^k R_{1/z}}{dz^k} \qquad (k = 1, 2, \cdots) .$$

DEFINITION. A triple of functions $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}$, whose members are related as in Theorem A, will be called a *matched triple of functions.*

The new parts of Theorem A are the characterization (β) of the class $\mathscr{R}$, and the inversion formulas (4) and (5). The relationship between $\mathscr{E}$ and $\mathscr{T}$ is essentially a generalization of Naĭmarks moment theorem obtained by S. K. Berberian in [3, Ths. 3,4]. In Theorem B it is shown that those generalized resolvents $R_z \in \mathscr{R}$ which satisfy the additional condition $R_0 = I$ can be represented as the ordinary resolvent of a variable contraction operator T_z on $\mathscr{H}$ which is a holomorphic function of z for $|z| < 1$.

DEFINITION. A triple of classes $\mathscr{S}' = \{\mathscr{E}', \mathscr{R}', \mathscr{T}'\}$ will be called a *matched triple of classes* if and only if

(i) $\mathscr{S}' \subseteqq \mathscr{S}$; i.e., $\mathscr{E}' \subseteqq \mathscr{E}$, $\mathscr{R}' \subseteqq \mathscr{R}$ and $\mathscr{T}' \subseteqq \mathscr{T}$.

(ii) When $\{E_\theta, R_z, T^{(k)}\}$ is a matched triple of functions in $\mathscr{S}$, then any one of the relations $E_\theta \in \mathscr{E}'$, $R_z \in \mathscr{R}'$, $T^{(k)} \in \mathscr{T}'$ implies all three, i.e., implies that the triple is in $\mathscr{S}'$.

A characterization of the matched triple of classes $\mathscr{S}_0 = \{\mathscr{E}_0, \mathscr{R}_0, \mathscr{T}_0\}$ associated with a unitary operator is given in § 4. Also, we shall develop the matched triple of classes $\mathscr{S}_1 = \{\mathscr{E}_1, \mathscr{R}_1, \mathscr{T}_1\}$ related to a partially unitary operator (an operator which can be written as the direct sum of a unitary operator and the zero operator). It will be shown that every matched triple of functions in $\mathscr{S}(\mathscr{H})$ is a projection of a matched triple of functions in $\mathscr{S}_1(\mathscr{H}^+)$, where $\mathscr{H}^+$ is a Hilbert space extension of $\mathscr{H}$.

In § 5 we prove a characterization of the matched triple of classes $\mathscr{S}_2 = \{\mathscr{E}_2, \mathscr{R}_2, \mathscr{T}_2\}$ associated with a contraction operator on $\mathscr{H}$, and show that a multiplicative functional calculus may be obtained in this case. The relationship between the classes $\mathscr{E}_2$ and $\mathscr{T}_2$ was first established by Sz.-Nagy [17] and M. Schreiber [12]. The characterization of the interrelated class $\mathscr{R}_2$ appears to be new.

The isometric operator case is discussed in § 6. Equivalent characterizations of the class of generalized resolvents of an isometric operator are obtained, and Theorem B is used to obtain a new characterization which leads to a simple and direct proof of a formula for the form of all generalized resolvents of a closed isometric operator.

2. The general theory of the triple $\mathscr{S}$.

2.1. A study of the relationships between the classes $\mathscr{E}$, $\mathscr{R}$, and $\mathscr{T}$ depends in large part upon the exploitation of a functional calculus analogous to that for a unitary operator [10, § 109]. We begin by showing that we may integrate a class of complex-valued functions with respect to operator distribution functions E_θ.

Suppose $E_\theta \in \mathscr{E}(\mathscr{H})$, i.e., E_θ satisfies the conditions (α) of § 1. If Δ is the left-open-right-closed interval $(\theta_1, \theta_2] \subset [0, 2\pi]$, define

$$E(\Delta) \equiv E_{\theta_2} - E_{\theta_1} .$$

Since $(E_\theta x, x)$ is real valued, nondecreasing, and continuous on the right, there exists a unique Borel measure μ_x (fixed x) such that

$$\mu_x(\Delta) = (E_{\theta_2}x, x) - (E_{\theta_1}x, x)$$

[11, p. 227]. If $\chi(\sigma; \theta)$ is the characteristic function of the Borel set $\sigma \subseteqq [0, 2\pi]$, then the Borel measure μ_x may be obtained from the cumulative distribution function $(E_\theta x, x)$ by the formula

$$\mu_x(\sigma) = \int_0^{2\pi} \chi(\sigma; \theta) d(E_\theta x, x) .$$

We now consider $\mu_x(\sigma)$ as a functional in x with σ fixed, and uniquely define another functional dependent on a pair of vectors $x, y, \in \mathscr{H}$ by the polarization formula

$$\mu_{x,y}(\sigma) \equiv \tfrac{1}{4}[\mu_{x+y}(\sigma) - \mu_{x-y}(\sigma) + i\mu_{x+iy}(\sigma) - i\mu_{x-iy}(\sigma)] .$$

As the bilinear form $(E_\theta x, y)$ and the quadradic form $(E_\theta x, x)$ are also related by the polarization formula [20, p. 322], it follows that

$$\mu_{x,y}(\sigma) = \int_0^{2\pi} \chi(\sigma; \theta) d(E_\theta x, y) .$$

It is easy to see that $\mu_{x,y}(\sigma)$ is a symmetric bilinear functional of x and y such that $|\mu_{x,x}(\sigma)| \leqq \|x\|^2$. Therefore, $\mu_{x,y}(\sigma) = (E(\sigma)x, y)$, where $E(\sigma)$ is a uniquely determined linear operator with domain $\mathscr{H}$ and $\|E(\sigma)\| \leqq 1$ [1, § 21]. Also $E(\sigma)$ is countable additive in the strong or weak operator topology, and $E([0, 2\pi]) = E_{2\pi} \leqq I$ (cf. [12, p. 580]). $E(\sigma)$ is called an *operator measure*. M. Schreiber considered the case $E_{2\pi} = I$ [12, 13], and S. K. Berberian [3] considered nonnormalized operator measures.

Operator measures are a generalization of the concept of spectral measures or resolutions of the identity associated with normal operators to the case where the orthogonality condition $E(\sigma)^2 = E(\sigma)$ is not required. Since $E_\theta \in \mathscr{E}$ is the cumulative distribution for the operator measure $E(\sigma)$, and one determines the other, we call the class $\mathscr{E}$ the class of *operator distribution functions*. The *support* of $E(\sigma)$, denoted $\Lambda(E)$, is the complement of the union of all open sets where $E(\sigma)$ vanishes.

Denote by $f(e^{i\theta})$, $0 \leqq \theta \leqq 2\pi$, a complex-valued function which is defined on the unit circle in the complex plane, and denote by $B(dE)$ the class of functions $f(e^{i\theta})$ which, when considered as a function of θ, are bounded and Borel measurable on $\Lambda(E)$. It has been shown by M. Schreiber [12, p. 580] that if $f(e^{i\theta}) \in B(dE)$, then one may define the operator F by the integral

$$F \equiv \int_0^{2\pi} f(e^{i\theta}) dE_\theta \tag{7}$$

and this integral is well-defined in either the strong or weak topology. We shall indicate the functional correspondence (7) of functions $f(e^{i\theta})$ in the class $B(dE)$ to operators F on $\mathscr{H}$ by the notation $f \sim F$.

2.2. We define the class $\mathscr{E}_1$ of bounded linear operators on $\mathscr{H}$ as follows:

(α_1) $E_\theta (0 \leqq \theta \leqq 2\pi)$ belongs to the class $\mathscr{E}_1 = \mathscr{E}_1(\mathscr{H})$ whenever

(a) $E_\theta = E_\theta^*$, $0 \leqq \theta \leqq 2\pi$.

(b) $E_{\theta_1} E_{\theta_2} = E_{\min(\theta_1, \theta_2)}$, $\theta_1, \theta_2 \in [0, 2\pi]$.

(c) $E_{\theta+0} = E_\theta$, $0 \leqq \theta < 2\pi$.

(d) $E_0 = 0$.

Note that (a) and (b) imply $E_{2\pi} \leqq I$. If we also require $E_{2\pi} = I$, then we obtain a subclass $\mathscr{E}_0$ of $\mathscr{E}_1$ which is the usual orthogonal spectral family or resolution of the identity for a unitary operator given by the spectral theorem (cf. [10, p. 281] or [20, p. 357]). Likewise, we note that (αa) implies $E_{2\pi} \leqq I$ for $E_\theta \in \mathscr{E}$, and if $E_{2\pi} = I$, then $\mathscr{E}$ is the class of functions called a generalized spectral family [16, p. 6] or a generalized resolution of the identity [2, p. 121].

A theorem due to Naimark [16, p. 6] asserts that every generalized spectral family, can be represented as the projection of an orthogonal spectral family. Naimark's theorem has been extended to nonnormalized cases by McKelvey [8] and Berberian [3, Th. 1]. For our classes $\mathscr{E}$ and $\mathscr{E}_1$ it may be stated:

THEOREM 1. (*Naimark*). *Let $E_\theta \in \mathscr{E}(\mathscr{H})$. There exists a Hilbert* *space* $\mathscr{H}^+ \supseteqq \mathscr{H}$ *and a projector valued* $E_\theta^+ \in \mathscr{E}_1(\mathscr{H}^+)$ *such that*

(i) $E_\theta x = PE_\theta^+ x$, *for all* $x \in \mathscr{H}$, *where* P *is the orthogonal projector onto the subspace* $\mathscr{H}$ *of* $\mathscr{H}^+$.

(ii) $\mathscr{H}^+$ *is spanned by* $\mathscr{H} \cup \{E_\theta^+ x : x \in \mathscr{H},\ 0 \leqq \theta \leqq 2\pi\}$, *and*

(iii) $E^+(\sigma) = 0$ *if and only if* $E(\sigma) = 0$, *where* σ *is any Borel set on* $[0, 2\pi]$ *and* $E(\sigma)$ *is the operator measure related to* E_θ. *Also,* $E_{2\pi}^+ = I$ *if and only if* $E_{2\pi} = I$.

If $E_\theta^+ \in \mathscr{E}_1$ is the minimal dilation of $E_\theta \in \mathscr{E}$, then (iii) of the Naimark dilation theorem implies that $\Lambda(E^+) = \Lambda(E)$, and hence

$$B(dE^+) = B(dE) .$$

For $f \sim F$ we define F^+ by

$$F^+ = \int_0^{2\pi} f(e^{i\theta}) dE_\theta^+ . \tag{7$^+$}$$

The Naimark theorem and equation (7) then give

$$Fx = P\int_0^{2\pi} f(e^{i\theta}) dE_\theta^+ x = PF^+ x , \tag{8}$$

for $x \in \mathscr{H}$, and

$$(Fx, y) = (F^+ x, y) , \tag{9}$$

for $x, y \in \mathscr{H}$.

2.3. The functional correspondence (7) has the following properties (cf. [8]): If $f \sim F$, $g \sim G$, $f_n \sim F_n$, and $g_n \sim G_n$, then

(i) *Linear*: $c_1 f + c_2 g \sim c_1 F + c_2 G$, where c_1, c_2 are scalar constants.

(ii) *Preserves conjugates*: $\bar{f} \sim F^*$.

(iii) *Positive*: $f(e^{i\theta}) \geqq 0$ on $\Lambda(E)$ implies $F \geqq 0$.

(iv) When E_θ is *projector-valued* ($E_\theta = E_\theta^* = E_\theta^2$), then the correspondence is *multicative*: $fg \sim FG$.

(v)
$$\| Fx \|^2 \leqq \int_0^{2\pi} | f(e^{i\theta}) |^2 d(E_\theta x, x) ,$$

for $x \in \mathscr{H}$, with equality when E_θ is projector-valued.

(vi) *Norm-decreasing*: $\| F \| \leqq \operatorname{ess\,sup} | f(e^{i\theta}) |$ on $\Lambda(E)$.

(vii) *Strong convergence*: When $f_n(e^{i\theta}) \to f(e^{i\theta})$ boundedly a.e. (dE), i.e., when $| f_n(e^{i\theta}) | \leqq M$ and $f_n(e^{i\theta}) \to f(e^{i\theta})$ as $n \to \infty$ a.e. (dE), then $F_n \to F$ strongly, i.e., $\| F_n x - Fx \| \to 0$ for all $x \in \mathscr{H}$.

(viii) *Uniform convergence*: When $f_n(e^{i\theta}) \to f(e^{i\theta})$ uniformly on $\Lambda(E)$ as $n \to \infty$, then $F_n \to F$ uniformly, i.e., $\| F_n - F \| \to 0$.

(ix) $1 \sim E_{2\pi}$.

2.4. Suppose that an $E_\theta \in \mathscr{E}$ is given. Define R_z and $T^{(k)}$ by means of formulas (1) and (2), i.e., in terms of the functional correspondence (7)

$$\frac{1}{1-e^{i\theta}z} \sim R_z \quad \text{and} \quad e^{ik\theta} \sim T^{(k)} .$$

LEMMA 1. *The operators R_z and $T^{(k)}$ defined above belong to the classes $\mathscr{R}$ and $\mathscr{J}$ respectively. Furthermore, the functions E_θ, R_z, and $T^{(k)}$ are interrelated by formulas* (1)-(6).

Proof. (i) To show $R_z \in \mathscr{R}$ we verify conditions (a)-(c) of (β).

(a) Since the integrand in formula (1) is holomorphic in z and continuous in θ on a bounded contour, $R_z x$ is holomorphic for $|z| \neq 1$, $x \in \mathscr{H}$. Then R_z is holomorphic for $|z| \neq 1$ (cf. [20, p. 206]).

(b) Note that

$$\frac{1}{1-e^{i\theta}z} = \frac{1}{2} + \frac{1}{2}\frac{1+e^{i\theta}z}{1-e^{i\theta}z} = \frac{1}{2} + \frac{1}{2}\frac{1+2i\operatorname{Im} e^{i\theta}z - |z|^2}{|1-e^{i\theta}z|^2} ,$$

so

$$\operatorname{Re}\frac{1}{1-e^{i\theta}z} = \frac{1}{2} + \frac{1}{2}\frac{1-|z|^2}{|1-e^{i\theta}z|^2} .$$

Integrating this identity with respect to $d(E_\theta x, x)$, $0 \leqq \theta \leqq 2\pi$, and using the linearity, one obtains

$$\operatorname{Re}(R_z x, x) = \tfrac{1}{2}\int_0^{2\pi} d(E_\theta x, x) + \tfrac{1}{2}(1-|z|^2)\int_0^{2\pi}\frac{1}{|1-e^{i\theta}z|^2}d(E_\theta x, x) .$$

Now the first integral on the right is $(R_0 x, x)$ by formula (1) for $z = 0$. Then

$$\frac{2\operatorname{Re}(R_z x, x) - (R_0 x, x)}{1-|z|^2} = \int_0^{2\pi}\frac{1}{|1-e^{i\theta}z|^2}d(E_\theta x, x) \geqq \|R_z x\|^2 ,$$

for $|z| \neq 1$, $x \in \mathscr{H}$, by property (v) of § 2.3.

(c) Suppose that $z \neq 0$, $|z| \neq 1$, and $x, y \in \mathscr{H}$. By equation (1)

$$(R_{z^*}x, y) = \int_0^{2\pi}\frac{1}{1-e^{i\theta}z^*}d(E_\theta x, y) ,$$

where $z^* = 1/\bar{z}$. Then using properties (i) and (ii) of § 2.3

$$\begin{aligned}(R_z^* x, y) &= \int_0^{2\pi}\overline{\left[\frac{1}{1-e^{i\theta}z}\right]}d(E_\theta x, y) = \int_0^{2\pi}\frac{1}{1-e^{-i\theta}\bar{z}}d(E_\theta x, y) \\ &= \int_0^{2\pi}d(E_\theta x, y) - \int_0^{2\pi}\frac{1}{1-e^{i\theta}z^*}d(E_\theta x, y) \\ &= (R_0 x, y) - (R_{z^*}x, y) = ([R_0 - R_{z^*}]x, y) ,\end{aligned}$$

i.e., $R_{z^*} = R_0 - R_z^*$.

(ii) $T^{(k)} \in \mathscr{T}$:

By (2) for $k = 0$ and (ix) of § 2.3 we have $T^{(0)} = E_{2\pi} \leqq I$. That formula (2) satisfies (γb) is not new (cf. [3, Th. 4]).

(iii) Related by formulas (1)-(6):

(a) Since the function $1/(1 - e^{i\theta}z)$ is holomorphic for $|z| \neq 1$, it has Laurent series expansions for $|z| < 1$ and $|z| > 1$. Using properties (i) and (viii) of § 2.3 and formulas (1) and (2) on these Laurent expansions, we obtain formula (3). Clearly, formula (6) is an inversion of formula (3).

(b) To prove (4), let $0 < \varphi \leqq 2\pi$, $0 \leqq r < 1$, and let θ_1, θ_2 be points of continuity of E_θ. Define

$$\begin{aligned} f(r, e^{i\varphi}) &\equiv \frac{1}{\pi}\int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{1}{1 - e^{i\varphi}(re^{-i\theta})}d\theta - \frac{(\theta_2 - \theta_1)}{2\pi} \\ &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{2}{1 - re^{i(\varphi-\theta)}} - 1\right\}d\theta \\ &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \operatorname{Re}\frac{1 + re^{i(\varphi-\theta)}}{1 - re^{i(\varphi-\theta)}}d\theta = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} P_r(\varphi - \theta)d\theta , \end{aligned}$$

where P_r is Poisson's kernel [7, p. 30]. Then $|f(r, e^{i\varphi})| \leqq 1$, and by Fatou's theorem (cf. [7, p. 34])

$$\begin{aligned} \operatorname*{Lim}_{r \to 1^-} f(r, e^{i\varphi}) &= 1 \text{ if } \varphi \in (\theta_1, \theta_2) \\ &= 0 \text{ if } \varphi \notin [\theta_1, \theta_2] , \end{aligned}$$

i.e., f approaches $\chi(\Delta; \varphi)$ a.e., $0 < \varphi \leqq 2\pi$, where $\chi(\Delta; \varphi)$ is the characteristic function of the interval $\Delta = (\theta_1, \theta_2]$.

Define

$$F(r) \equiv \int_0^{2\pi} f(r, e^{i\varphi})dE_\varphi .$$

Then by property (vii) of § 2.3

$$\operatorname*{Lim}_{r \to 1^-} F(r) = \int_0^{2\pi} \chi(\Delta; \varphi)dE_\varphi = E_{\theta_2} - E_{\theta_1} .$$

But for $0 \leqq r < 1$

$$\begin{aligned} F(r) &= \int_0^{2\pi} \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{2}{1 - re^{i(\varphi-\theta)}} - 1\right\}d\theta dE_\varphi \\ &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\int_0^{2\pi} \operatorname{Re}\left\{\frac{2}{1 - re^{-i\theta}e^{i\varphi}} - 1\right\}dE_\varphi d\theta \\ &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2}(2 \operatorname{Re} R_{r\exp(-i\theta)} - R_0)d\theta \end{aligned}$$

which implies (4). The interchanging of the order of integration is justified because the integrand is continuous with respect to θ, φ.

(c) Similarly, (5) may be seen to be an inversion of (2) by showing

$$\begin{aligned}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{e^{-ik\theta_1}-e^{-ik\theta_2}}{ik}T^{(k)} &= \frac{1}{2\pi}\operatorname*{Lim}_{r\to 1^-}\int_0^{2\pi}\int_{\theta_1}^{\theta_2}P_r(\varphi-\theta)d\theta dE_\varphi \\ &= E_{\theta_2}-E_{\theta_1}\,.\end{aligned}$$

2.5 THEOREM A. *Each function* $E_\theta \in \mathscr{E}$, $R_z \in \mathscr{R}$, *or* $T^{(k)} \in \mathscr{T}$ *belongs to a unique triple of functions* $\{E_\theta, R_z, T^{(k)}\}$ *in* $\mathscr{S} \equiv \{\mathscr{E}, \mathscr{R}, \mathscr{T}\}$, *such that the members of the triple are interrelated by formulas* (1)-(6).

Proof. Due to Lemma 1 we need only show that each $R_z \in \mathscr{R}$ can be expressed by equation (1), and each $T^{(k)} \in \mathscr{T}$ by equation (2), in terms of some unique $E_\theta \in \mathscr{E}$. This is done in (i) and (ii) below.

(i) Suppose $R_z \in \mathscr{R}$, i.e., R_z satisfies the conditions (β). Then R_z satisfies the following "weak" properties:

(β^*) (a) $(R_z x, x)$ is a holomorphic function of z for $|z| \neq 1$, $x \in \mathscr{H}$.
(b) $\operatorname{Re}(R_z x, x) \gtreqless \frac{1}{2}(R_0 x, x)$, $|z| \lesseqgtr 1$, $x \in \mathscr{H}$.
(c) $(R_0 x, x) \leqq (x, x)$, $x \in \mathscr{H}$.
(d) $([R_0 - R_{z^*}]x, x) = \overline{(R_z x, x)}$, $z \neq 0$, $|z| \neq 1$, $x \in \mathscr{H}$.

(a), (b), and (d) are immediate from (β) and the fact that

$$\tfrac{1}{2}(1-|z|^2)\,\|R_z x\|^2 \gtreqless 0 \quad \text{for} \quad |z| \lesseqgtr 1,\ x \in \mathscr{H}\ .$$

By (βb) with $z = 0$

$$\|R_0 x\|^2 \leqq (R_0 x, x) \leqq |(R_0 x, x)| \leqq \|R_0 x\|\,\|x\|\ .$$

Hence $\|R_0 x\| \leqq \|x\|$, and $(R_0 x, x) \leqq \|x\|^2$.

Define

$$\varphi(z) \equiv -(R_0 x, x) + 2(R_z x, x)\ , \quad \text{for} \quad |z| \neq 1\ .$$

Since $(R_z x, x)$ is holomorphic in $|z| \neq 1$, and $(R_0 x, x)$ is a constant with respect to z, it is clear that $\varphi(z)$ is holomorphic in $|z| \neq 1$. Now

$$\operatorname{Re}\varphi(z) = -(R_0 x, x) + 2\operatorname{Re}(R_z x, x)\ ,$$

so by (β^*b), $\operatorname{Re}\varphi(z) \geqq 0$ for $|z| < 1$, and $\operatorname{Re}\varphi(z) \leqq 0$ for $|z| > 1$. Also, using (β^*c), $\varphi(0) = (R_0 x, x) \leqq (x, x)$; and by ($\beta^*$d)

$$\varphi(z^*) = -(R_0 x, x) + 2(R_{z^*} x, x) = (\mathrm{R}_0 x, x) - 2\overline{(R_z x, x)} = -\overline{\varphi(z)}\ ,$$

for $z \neq 0$ and $|z| \neq 1$. Thus if $R_z \in \mathscr{R}$, then $\varphi(z)$ satisfies
(β') (a) $\varphi(z)$ is holomorphic in $|z| \neq 1$.

(b) $\operatorname{Re}\varphi(z) \gtreqqless 0$ for $|z| \gtrless 1$.
(c) $\varphi(0) \leqq (x, x)$ (i.e., $\varphi(0)$ is real and $\varphi(0) \leqq (x, x)$).
(d) $\varphi(z^*) = -\overline{\varphi(z)}$, $z \neq 0$, $|z| \neq 1$.

Now the conditions (β') are sufficient to apply a theorem due to Herglotz [5] (see also [9, pp. 58–60], or [2, p. 5] with t replaced by $2\pi - \theta$) which says that $\varphi(z)$ is of the form

$$(10) \qquad \varphi(z) = \int_0^{2\pi} \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} d\sigma(\theta) + i \operatorname{Im}\varphi(0) ,$$

$|z| < 1$, where $\sigma(\theta)$, $0 \leqq \theta \leqq 2\pi$, is a real nondecreasing function of bounded variation, and $\operatorname{Im}\varphi(0) = 0$, since $\varphi(0)$ is real.

If $|z| < 1$, then by (10) and ($\beta'd$)

$$\varphi(z^*) = -\overline{\varphi(z)} = -\int_0^{2\pi} \frac{1 + e^{-i\theta}\bar{z}}{1 - e^{-i\theta}\bar{z}} d\sigma(\theta) = \int_0^{2\pi} \frac{1 + e^{i\theta}z^*}{1 - e^{i\theta}z^*} d\sigma(\theta) .$$

If $|z| < 1$, then $|z^*| > 1$. Therefore, formula (10) is valid for all $|z| \neq 1$.

From the definition of $\varphi(z)$

$$(R_z x, x) = \tfrac{1}{2}(R_0 x, x) + \tfrac{1}{2}\varphi(z) = \tfrac{1}{2}\varphi(0) + \tfrac{1}{2}\varphi(z) , \qquad |z| \neq 1 ,$$

so

$$(R_z x, x) = \tfrac{1}{2}\int_0^{2\pi} d\sigma(\theta) + \tfrac{1}{2}\int_0^{2\pi} \frac{1 + e^{i\theta}z}{1 - e^{i\theta}z} d\sigma(\theta) = \int_0^{2\pi} \frac{1}{1 - e^{i\theta}z} d\sigma(\theta) ,$$

for $|z| \neq 1$, $x \in \mathscr{H}$.

Under the normalization conditions

$$(11) \qquad \begin{aligned} &\sigma(0) = 0 \\ &\sigma(\theta) = \sigma(\theta + 0),\ 0 \leqq \theta < 2\pi , \end{aligned}$$

where $\sigma(\theta + 0) \equiv \operatorname{Lim} \sigma(\varphi)$ as $\varphi \to \theta^+$, the real nondecreasing function $\sigma(\theta) = \sigma(\theta; x)$ in (10) is uniquely determined by $\varphi(z)$ or $(R_z x, x)$ (cf. [2, pp. 3–7]).

Defining the function $\sigma(\theta; x, y)$ for $x, y \in \mathscr{H}$ by the polarization formula

$$\begin{aligned} \sigma(\theta; x, y) \equiv \tfrac{1}{4}[&\sigma(\theta; x + y) - \sigma(\theta; x - y) \\ &+ i\sigma(\theta; x + iy) - i\sigma(\theta; x - iy)] , \end{aligned}$$

and using the analogous formula for the bilinear form $(R_z x, y)$, one obtains

$$(12) \qquad (R_z x, y) = \int_0^{2\pi} \frac{1}{1 - e^{i\theta}z} d\sigma(\theta; x, y) , \qquad |z| \neq 1,\ x, y \in \mathscr{H} .$$

The complex-valued functions $\sigma(\theta; x, y)$ also satisfy the normalization conditions (11) and are uniquely determined by equation (12). In particular, one has $\sigma(\theta; x, x) = \sigma(\theta; x)$. It follows by an elementary argument (cf. [2, § 65]) that $\sigma(\theta; x, y)$ is a symmetric bilinear functional of x, y and

$$|\sigma(\theta; x, x)| \leqq \sigma(2\pi; x, x) = \int_0^{2\pi} d\sigma(\theta; x, x) = (R_0 x, x) \leqq (x, x) .$$

By the theorem on the general form of a symmetric bilinear functional [1, § 21], there exists a uniquely determined family of bounded self-adjoint linear operators E_θ which depend on the parameter $\theta\,(0 \leqq \theta \leqq 2\pi)$ such that $\sigma(\theta; x, y) = (E_\theta x, y)$ for all $x, y \in \mathscr{H}$. It remains to show $E_\theta \in \mathscr{E}$, i.e., E_θ satisfies conditions (a)-(c) of (α). But since $(E_\theta x, x) = \sigma(\theta; x, x) = \sigma(\theta)$, this follows from the normalization conditions (11) and a simple argument showing that if $(E_\theta x, x)$ is weakly continuous from the right, then it is strongly continuous from the right.

Equation (12) becomes equation (1) in the weak sense, hence in the strong sense, since both interpretations of the integral make sense and define the same operator R_z. Then E_θ is completely defined by the operator R_z, and in turn E_θ completely defines R_z by (1). This proves (i).

Since the construction of E_θ only depends upon the "weak" properties (β^*), we obtain as a byproduct of the proof the result:

THEOREM 2. *The class $\mathscr{R}$ may be characterized by the properties of* (β^*) *as well as those of* (β).

(ii) Suppose $T^{(k)} \in \mathscr{T}$, i.e., $T^{(k)}$ satisfies the conditions (γ). Then by a generalization of Naimarks moment theorem [3, Ths. 3 and 4], or by an argument similar to [2, § 62] we obtain

$$T^{(k)} = \int_0^{2\pi} e^{ik\theta} dE'_\theta \qquad (\pm k = 0, 1, 2, \cdots) ,$$

where $E'_\theta \in \mathscr{E}$.

It remains to show $E_\theta \equiv E'_\theta$. But this follows from formulas (1)-(6). For example, by substituting (1) and (2) into (3) and using the normalization conditions. This proves Theorem A.

3. The generalized resolvent.

3.1 REMARKS. If $\{E_\theta, R_z, T^{(k)}\}$ is a matched triple of functions in $\mathscr{S}$, then

(i) $R_0 = T^{(0)} = E_{2\pi} \leqq I$.

(ii) $T^{(-k)} = [T^{(k)}]^*$, $\pm k = 0, 1, 2, \cdots$.

(iii) $|(T^{(k)}x, x)| \leqq (T^{(0)}x, x)$, $x \in \mathscr{H}$, $\pm k = 0, 1, 2, \cdots$.

(iv) $\| T^{(k)} \| \leqq \| T^{(0)} \| \leqq 1$, $\pm k = 0, 1, 2, \cdots$.

Proof. (i) and (iii) follow from equations (1) and (2). (ii) follows by the functional calculus from $e^{-ik\theta} = \overline{e^{ik\theta}}$. (iv) is proved in [3, corollary to Th. 2].

3.2 THEOREM 3. *Let $\{E_\theta, R_z, T^{(k)}\}$ be a matched triple of functions in $\mathscr{S}$, and let $N = N(E_{2\pi})$ be the null-space of $E_{2\pi}$. Then*

(i) $N(E_{\theta_1}) \supseteqq N(E_{\theta_2}) \supseteqq N$ *and* $\overline{E_{\theta_1}\mathscr{H}} \subseteqq \overline{E_{\theta_2}\mathscr{H}} \subseteqq \overline{E_{2\pi}\mathscr{H}} = N^\perp$, *for* $0 \leqq \theta_1 < \theta_2 \leqq 2\pi$.

(ii) $N(T^{(k)}) \supseteqq N(T^{(0)}) = N$ *and* $\overline{T^{(k)}\mathscr{H}} \subseteqq \overline{T^{(0)}\mathscr{H}} = N^\perp$, $\pm k = 0, 1 \cdots$.

(iii) $N(R_{z^*}) \supseteqq N(R_z) = \{x: (R_z x, x) = 0\} = N$ *and* $\overline{R_{z^*}\mathscr{H}} \subseteqq \overline{R_z\mathscr{H}} = N^\perp$, *for* $|z| < 1$.

In particular, the decomposition $\mathscr{H} = N^\perp + N$ is reducing for all values of the functions $E_\theta, R_z, T^{(k)}$, and these functions vanish identically on N.

Proof. (i) Since E_θ is nonnegative, the Cauchy-Schwartz inequality,

$$|(E_\theta x, y)|^2 \leqq (E_\theta x, x)(E_\theta y, y) ,$$

is valid and shows that $N(E_\theta) = \{x: (E_\theta x, x) = 0\}$. Also, $\overline{E_\theta\mathscr{H}} = N(E_\theta)^\perp$. The assertions then follow from (αa).

(ii) If $T^{(0)}x = 0$, then $E_{2\pi}x = 0$ by remark (i) of § 3.1. Hence $E_\theta x \equiv 0$, $0 \leqq \theta \leqq 2\pi$, by (i) above. Then equation (2) gives $T^{(k)} = 0$, $\pm k = 0, 1, 2, \cdots$. Also,

$$\overline{T^{(k)}\mathscr{H}} = N(T^{(k)*})^\perp = N(T^{(-k)})^\perp \subseteqq N(T^{(0)})^\perp = \overline{T^{(0)}\mathscr{H}} .$$

(iii) Suppose $|z| < 1$. By (βb) $\mathrm{Re}\,(R_z x, x) \geqq \frac{1}{2}(R_0 x, x)$. Therefore, if $(R_{z_0}x, x) = 0$ for some z_0, $|z_0| < 1$, then $\mathrm{Re}\,(R_{z_0}x, x) = 0$, and $(R_0 x, x) = (E_{2\pi}x, x) = 0$. Thus $E_{2\pi}x = 0$, and $E_\theta x \equiv 0$, $0 \leqq \theta \leqq 2\pi$, by (i) above. But then equation (1) gives $R_z x = 0$, $|z| < 1$. Clearly $R_z x = 0$ implies $(R_z x, x) = 0$. It follows that $N(R_z) = \{x: (R_z x, x) = 0\} = N$, $|z| < 1$.

Since $\mathrm{Re}\,(R_z^* x, x) = \mathrm{Re}\,(R_z x, x)$, the above argument shows that if $(R_{z_0}^* x, x) = 0$ for some z_0, $|z_0| < 1$, then $E_\theta x \equiv 0$, $0 \leqq \theta \leqq 2\pi$. But then the adjoint of formula (1) gives $R_z^* x = 0$, $|z| < 1$. Hence $N(R_z^*) = N$, $|z| < 1$.

By (βc) $R_z x = 0$, $|z| < 1$, implies $R_{z^*}x = 0$, $|z| < 1$, i.e., $N(R_{z^*}) \supseteqq N(R_z)\,|z| < 1$. It is possible for $N(R_{z^*}) \supset N(R_z)$, $|z| < 1$. For example,

$$\begin{aligned} R_z &\equiv R_0 , \qquad |z| < 1 \\ &\equiv 0 , \qquad |z| > 1 \end{aligned}$$

belongs to the class $\mathscr{R}$, and $N(R_{z^*}) = \mathscr{H} \supset N(R_z)$, $|z| < 1$, if $R_0 \neq 0$.

Finally, for $|z| < 1$

$$\overline{R_{z^*}\mathscr{H}} = N(R_{z^*}^*)^\perp = N(R_0 - R_z)^\perp \subseteqq N(R_z)^\perp = N(R_z^*)^\perp = \overline{R_z\mathscr{H}} = N^\perp .$$

By (i)-(iii) $N^\perp$ and N and invariant subspaces for E_θ, R_z, $T^{(k)}$, and, consequently, the decomposition is reducing.

DEFINITION. A matched triple of functions $\{E_\theta, R_z, T^{(k)}\}$ is said to be *proper* whenever $N = \{0\}$.

REMARKS. 1. Any matched triple generates a proper triple on the reducing subspace.

2. For $E_\theta \in \mathscr{E}_1$, the corresponding matched triple is proper if and only if $E_{2\pi} = I$.

3.3 DEFINITION. The linear operator T of the Hilbert space $\mathscr{H}$ is said to be a *contraction operator* if $\|Tx\| \leqq \|x\|$ for all $x \in D_T$, where $D_T =$ domain of $T = \mathscr{H}$, i.e., $\|T\| \leqq 1$.

DEFINITION. The linear operator T is said to be an *isometric operator* if $(Tx, Ty) = (x, y)$ for all $x, y \in D_T$. If, in addition, $D_T = \Delta_T \equiv$ range of $T = \mathscr{H}$, then T is said to be *unitary*.

Suppose T is a contraction operator on the Hilbert space $\mathscr{H}$. Then the *resolvent operator*

$$r(z) \equiv (I - zT)^{-1} , \qquad |z| < 1 ,$$

exists as a bounded operator with domain $\Delta_T(z) \equiv$ range $(I - zT) = (I - zT)D_T = \mathscr{H}$.

LEMMA 2. (a) *If $\|T\| \leqq 1$, $D_T = \mathscr{H}$, then T has resolvent $r(z)$ which satisfies*

$$\text{(13)} \qquad \operatorname{Re}(r(z)x, x) \geqq \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}(1 - |z|^2)\|r(z)x\|^2 ,$$

$|z| < 1$, $x \in \mathscr{H}$, *or the equivalent*

$$\text{(13')} \qquad \|[r(z) - I]x\| \leqq |z|\,\|r(z)x\| , \qquad |z| < 1, \ x \in \mathscr{H} .$$

Furthermore, $r(z)$ satisfies

$$\text{(14)} \qquad (1 - |z|)\|r(z) - I\| \leqq |z| , \qquad |z| < 1 .$$

(b) *If T has resolvent $r(z)$ which satisfies equation* (13) (*or* (13')) *for a single z_0 in $0 < |z_0| < 1$ and $x \in \mathscr{H}$, then $\|T\| \leqq 1$,*

$D_T = \mathscr{H}$.

Proof. (a) For $y \in D_T = \mathscr{H}$

$$\| [I - zT]y \|^2 = 2 \operatorname{Re}(y, [I - zT]y) + |z|^2 \| Ty \|^2 - \| y \|^2 . \tag{15}$$

If T is a contraction operator, then

$$\| [I - zT]y \|^2 \leqq 2 \operatorname{Re}(y, [I - zT)y) + (|z|^2 - 1) \| y \|^2$$

for $|z| < 1$. Letting $x = [I - zT]y$, $y = r(z)x$ in this equation, we obtain (13). Now

$$\| r(z)x - Ix \|^2 = \| r(z)x \|^2 - 2 \operatorname{Re}(r(z)x, x) + \| x \|^2 ,$$

so (13) is equivalent to (13′).

Nothing that $\| T \| \leqq 1$, $D_T = \mathscr{H}$, implies

$$\| [I - zT]y \| \geqq (1 - |z|) \| y \| ,$$

and letting $x = [I - zT]y$, $y = r(z)x$ in this inequality, we obtain $(1 - |z|) \| r(z)x \| \leqq \| x \|$ for all $x \in \mathscr{H} = \Delta_T(z)$, i.e.,

$$(1 - |z|) \| r(z) \| \leqq 1 . \tag{16}$$

Using (16) in (13′), we have (14).

(b) Suppose an operator T has resolvent $r(z)$ which satisfies (13′) (or (13)) for some z_0 in $0 < |z_0| < 1$. Now

$$\| r(z_0)x - Ix \| = \| r(z_0)x - (I - z_0T)r(z_0)x \| = |z_0| \| Tr(z_0)x \| ,$$

so (13′) gives

$$|z_0| \| Tr(z_0)x \| \leqq |z_0| \| r(z_0)x \|$$

for $0 < |z_0| < 1$, $x \in \mathscr{H}$. Hence $\| T \| \leqq 1$.

3.4 THEOREM B. *$R_z \in \mathscr{R}$ and $R_0 = I$ if and only if*

$$\begin{aligned} R_z &= (I - zT_z)^{-1} , \qquad |z| < 1 \\ &= I - (I - z^{-1}T^*_{z^*})^{-1} , \qquad |z| > 1 , \end{aligned}$$

where

(i) $\| T_z \| \leqq 1$, $|z| < 1$, $D(T_z) = \mathscr{H}$

(ii) *T_z is a holomorphic function of z for $|z| < 1$.*

Proof. Suppose

$$\begin{aligned} R_z &= (I - zT_z)^{-1} , \qquad |z| < 1 \\ &= I - (I - z^{-1}T^*_{z^*})^{-1} , \qquad |z| > 1 , \end{aligned}$$

where T_z satisfies (i) and (ii). Then clearly, $D(R_z) = \mathscr{H}$, and $R_0 = I^{-1} = I$.

(a) By (i)

$$\|[I - zT_z]x\| \geqq \|x\| - |z|\,\|T_zx\| \geqq (1 - |z|)\|x\|\,, \qquad |z| < 1\,,$$

so R_z not only exists but is bounded for $|z| < 1$. Then $R_z = (I - zT_z)^{-1}$ is holomorphic for $|z| < 1$, $x \in \mathscr{H}$, because T_z is holomorphic by (ii). Now $\|T_z^*\| = \|T_z\| \leqq 1$, $|z| < 1$, $D(T_{z^*}) = \mathscr{H}$, so

$$\|[I - z^{-1}T_{z^*}^*]x\| \geqq \|x\| - |z^{-1}|\,\|T_{z^*}^*x\| \geqq (1 - |z^{-1}|)\|x\|\,, \qquad |z| > 1\,.$$

Thus $[\mathrm{I} - z^{-1}T_{z^*}^*]^{-1}$ exists and is bounded for $|z| > 1$ with domain $\mathscr{H}$. Also, it follows from (ii) that $T_{z^*}^*$ is holomorphic for $|z| > 1$. Then $R_z = I - (I - z^{-1}T_{z^*}^*)^{-1}$ is holomorphic for $|z| > 1$, $x \in \mathscr{H}$. We have shown that R_z is a holomorphic function of z for $|z| \neq 1$.

(b) Since $R_z^* = [(I - zT_z)^{-1}]^* = (I - \bar{z}T_z^*)^{-1}$, $|z| < 1$, we have

$$I - R_z^* = I - (I - \bar{z}T_z^*)^{-1} = R_{z^*}\,, \qquad |z| < 1,\ z \neq 0\,.$$

Similarily, for $|z| > 1$

$$R_z^* = [I - (I - z^{-1}T_{z^*}^*)^{-1}]^* = I - (I - z^*T_{z^*})^{-1}\,,$$

so

$$I - R_z^* = (I - z^*T_{z^*})^{-1} = R_{z^*}\,, \qquad |z| > 1\,.$$

We have proved that $R_{z^*} = I - R_z^*$, $|z| \neq 1$, $z \neq 0$.

(c) By (i) and Lemma 2(a), T_z has resolvent R_z satisfying (13) and (13′) for all $|z| < 1$, $x \in \mathscr{H}$. Now $\|T_{\bar{z}}^*\| \leqq 1$, $|z| < 1$, and $D(T_{\bar{z}}^*) = \mathscr{H}$, so that Lemma 2(a) also applies to $T_{\bar{z}}^*$. Then $(I - zT_{\bar{z}}^*)^{-1}$ satisfies (13′) for $|z| < 1$, i.e.,

$$\|[(I - zT_{\bar{z}}^*)^{-1} - I]x\| \leqq |z|\,\|(I - zT_{\bar{z}}^*)^{-1}x\|\,, \qquad |z| < 1\,.$$

Replacing z by z^{-1} in this relation and nothing that $(I - z^{-1}T_{z^*}^*)^{-1} = R_{z^*}^*$, $|z| > 1$, we obtain

$$\|[R_{z^*}^* - I]x\| \leqq |z^{-1}|\,\|R_{z^*}^*x\|\,, \qquad |z| > 1\,.$$

Thus $R_{z^*}^*$ satisfies (13′) (and (13)) for all $|z| > 1$. But $R_{z^*}^* = I - R_z$, $|z| > 1$, by (b) above. Hence (13′) becomes

$$\|[(I - R_z) - I]x\| \leqq |z^{-1}|\,\|[I - R_z]x\|\,, \qquad |z| > 1\,,$$

or

$$\|[R_z - I]x\| \geqq |z|\,\|R_zx\|\,, \qquad |z| > 1\,, \tag{13″}$$

which is equivalent to

$$\mathrm{Re}\,(R_z x, x) \leqq \tfrac{1}{2}\,\|x\|^2 + \tfrac{1}{2}(1 - |z|^2)\,\|R_z x\|^2\,, \qquad |z| > 1\,.$$

We have shown that R_z satisfies (βb) for all $|z| \neq 1$.

Combining (a)-(c) above, we have $R_z \in \mathscr{R}$ and $R_0 = I$.

Conversely, suppose $R_z \in \mathscr{R}$ and $R_0 = I$. By Theorem 3 (iii), $N(R_z) = N(R_0) = \{0\}$, so R_z is proper, and R_z^{-1} exists, $|z| < 1$. In addition $N(R_z^*) = N(R_z) = \{0\}$, so $(R_z^*)^{-1}$ exists, $|z| < 1$. By (βb) for $|z| < 1$ we have

$$\tfrac{1}{2}\|x\|^2 \leqq \mathrm{Re}\,(R_z x, x) \leqq |(R_z x, x)| \leqq \|R_z x\|\,\|x\|\,,$$

so that

$$\|R_z\| \geqq \tfrac{1}{2}\,\|x\|\,, \qquad \text{for all } x \in \mathscr{H},\ |z| < 1\,.$$

This implies that R_z^{-1} exists and is bounded for $|z| < 1$. Therefore, $(I - R_z^{-1}) = (R_z - I)R_z^{-1}$ is holomorphic in z for $|z| < 1$ because R_z is. Also,

$$\lim_{z \to 0}\,(R_z - I)R_z^{-1} = 0\,.$$

Then the function $z^{-1}(R_z - I)R_z^{-1}$ is holomorphic in z for $|z| < 1$, because the apparent singularity at $z = 0$ is removable by making it continuous at $z = 0$. Consequently, one may define T_z to be the following function holomorphic for $|z| < 1$:

$$T_z \equiv \frac{1}{z}(I - R_z^{-1}) \quad \text{for} \quad 0 < |z| < 1$$
$$T_0 \equiv \lim_{z \to 0} T_z\,.$$

Then

$$R_z = (I - zT_z)^{-1}\,, \qquad |z| < 1\,.$$

Since $(R_z^*)^{-1}$ exists we have

$$T_z^* = \frac{1}{\bar{z}}[I - (R_z^*)^{-1}]\,, \qquad |z| < 1\,.$$

Hence

$$\begin{aligned} R_z^* &= (I - \bar{z}T_z^*)^{-1}\,, \qquad |z| < 1 \\ &= I - R_{z^*}\,, \qquad |z| < 1\,, \end{aligned}$$

by (βc), and

$$R_{z^*} = I - (I - \bar{z}T_z^*)^{-1}\,, \qquad |z| < 1\,,$$

i.e.,

$$R_z = I - (I - z^{-1}T_{z^*}^*)^{-1}, \qquad |z| > 1 .$$

Since (βb) with $R_0 = I$ and $|z| < 1$ is (13), we see that T_z has resolvent R_z which satisfies (13) for all $|z| < 1$ and $x \in \mathscr{H}$. Then Lemma 2(b) implies $\|T_z\| \leqq 1$, $|z| < 1$, $D(T_z) = \mathscr{H}$, which is (i). This proves Theorem B.

COROLLARY. *$R_z \in \mathscr{R}$, $R_0 = I$, and $N(R_z) = \{0\}$ for $|z| > 1$ if and only if*

$$R_z = (I - zT_z)^{-1}, \qquad |z| \neq 1 ,$$

where

(i) $\|T_z\| \leqq 1$, $|z| < 1$, $D(T_z) = \mathscr{H}$.

(ii) *T_z is a holomorphic function of z for $|z| < 1$.*

(iii) *T_{z^*} exists and $T_{z^*}^{-1} = T_z^*$, $0 < |z| < 1$.*

Proof. If $N(R_z) = \{0\}$, $|z| > 1$, then one may define

$$T_z \equiv \frac{1}{z}(I - R_z^{-1})$$

for all $0 < |z| < 1$ *and* $|z| > 1$. Then

$$T_{z^*} = \bar{z}(I - R_{z^*}^{-1}) = \bar{z}(R_{z^*} - I)R_{z^*}^{-1}$$

exists for $0 < |z| < 1$. Also, since $N(R_{z^*} - I) = N(R_z^*) = N(R_z) = \{0\}$ for $0 < |z| < 1$, $(R_{z^*} - I)^{-1}$ exists. Then $T_{z^*}^{-1}$ exists and

$$\begin{aligned} T_{z^*}^{-1} &= z^* R_{z^*}(R_{z^*} - I)^{-1} = z^*(R_z^* - I)(R_z^*)^{-1} \\ &= z^*[I - (R_z^*)^{-1}] = T_z^*, \qquad 0 < |z| < 1 . \end{aligned}$$

Hence for $|z| > 1$

$$\begin{aligned} R_z &= I - (I - z^{-1}T_{z^*}^*)^{-1} = I - (I - z^{-1}T_z^{-1})^{-1} \\ &= I + [(I - zT_z)z^{-1}T_z^{-1}]^{-1} = I + zT_z(I - zT_z)^{-1} \\ &= I + (zT_z - I + I)(I - zT_z)^{-1} = I - I + (I - zT_z)^{-1} \\ &= (I - zT_z)^{-1} . \end{aligned}$$

Conversely, if T_{z^*} exists and $R_z = (I - zT_z)^{-1}$, $|z| \neq 1$, then

$$R_z^{-1} = (I - zT_z)$$

exists for $|z| \neq 1$. Therefore, $N(R_z) = \{0\}$ for *all* $|z| \neq 1$.

4. Triples related to unitary and partially unitary operators.

4.1. Consider the triple of classes $\mathscr{S}_0 = \{\mathscr{E}_0, \mathscr{R}_0, \mathscr{T}_0\}$ defined as follows:

(α_0) E_θ $(0 \leqq \theta \leqq 2\pi)$ belongs to the class $\mathscr{E}_0 = \mathscr{E}_0(\mathscr{H})$ whenever it is an orthogonal resolution of the identity, i.e., satisfies
 (a) $E_\theta = E_\theta^*$, $0 \leqq 0 \leqq 2\pi$.
 (b) $E_{\theta_1} E_{\theta_2} = E_{\min(\theta_1, \theta_2)}$, $\theta_1, \theta_2 \in [0, 2\pi]$.
 (c) $E_{\theta+0} = E_\theta$, $0 \leqq \theta < 2\pi$.
 (d) $E_0 = 0$, $E_{2\pi} = I$.

(β_0) R_z $(|z| \neq 1)$ belongs to the class $\mathscr{R}_0 = \mathscr{R}_0(\mathscr{H})$ whenever
 (a) $zR_z - z'R_{z'} = (z - z')R_z R_{z'}$, $|z| \neq 1$, $|z'| \neq 1$.
 (b) If $R_z x = 0$, then $x = 0$. $(|z| \neq 1$, $x \in \mathscr{H})$.
 (c) $R_z^* = I - R_{z^*}$, $z \neq 0$ and $|z| \neq 1$.

(γ_0) $T^{(k)}$ $(\pm k = 0, 1, 2, \cdots)$ belongs to the class $\mathscr{T}_0 = \mathscr{T}_0(\mathscr{H})$ whenever $T^{(1)} \equiv T$ is a unitary operator, and $T^{(k)} \equiv T^k$, $\pm k = 0, 1, 2, \cdots$.

LEMMA 3. *Let T be a unitary operator acting in $\mathscr{H}$. Then T has associated with it a unique triple $\{E_\theta, R_z, T^{(k)}\}$ of functions in classes $\mathscr{E}_0$, $\mathscr{R}_0$, $\mathscr{T}_0$ respectively, determined as follows:*

(α_0') *E_θ is the spectral function of T, i.e.,*

$$T = \int_0^{2\pi} e^{i\theta} dE_\theta \ .$$

(β_0') *R_z is the resolvent of T, i.e.,*

$$R_z = (I - zT)^{-1} \ , \qquad |z| \neq 1 \ .$$

(γ_0') *$T^{(k)}$ is the cyclic group of powers of T, i.e.,*

$$T^{(k)} = T^k \ , \qquad \pm k = 0, 1, 2, \cdots \ .$$

Conversely, each function of class $\mathscr{E}_0$, $\mathscr{R}_0$, or $\mathscr{T}_0$ is associated with precisely one unitary T in the manner just described.

Proof. These characterizations are elementary facts and their proofs will be omitted.

4.2. The functional correspondence (7) is multiplicative when E_θ is projector-valued, i.e., $E_\theta^2 = E_\theta$. In the most general case $E_{2\pi}$ is also a projector, but not necessarily the identity, i.e., $E_{2\pi} \leqq I$. We have defined this class $\mathscr{E}_1$ in § 2.2. Now define the triple of classes $\mathscr{S}_1 = \{\mathscr{E}_1, \mathscr{R}_1, \mathscr{T}_1\}$ of bounded linear operators as follows:

(α_1) E_θ $(0 \leqq \theta \leqq 2\pi)$ belongs to the class $\mathscr{E}_1 = \mathscr{E}_1(\mathscr{H})$ whenever
 (a) $E_\theta = E_\theta^*$, $0 \leqq \theta \leqq 2\pi$.
 (b) $E_{\theta_1} E_{\theta_2} = E_{\min(\theta_1, \theta_2)}$, $\theta_1, \theta_2 \in [0, 2\pi]$.
 (c) $E_{\theta+0} = E_\theta$, $0 \leqq \theta < 2\pi$.
 (d) $E_0 = 0$.

(β_1) R_z ($|z| \neq 1$) belongs to the class $\mathscr{R}_1 = \mathscr{R}_1(\mathscr{H})$ whenever
 (a) $zR_z - z'R_{z'} = (z - z')R_zR_{z'}$, $|z| \neq 1$, $|z'| \neq 1$.
 (b) $R_z^* = R_0 - R_{z^*}$, $z \neq 0$ and $|z| \neq 1$.

(γ_1) $T^{(k)}$ ($\pm k = 0, 1, 2, \cdots$) belongs to the class $\mathscr{T}_1 = \mathscr{T}_1(\mathscr{H})$ whenever $T^{(1)} \equiv T$ is a partially unitary operator, and $T^{(k)} \equiv T^k$, $\pm k = 0, 1, 2, \cdots$; i.e.,

$$T^{(k)} = T^k = U^k \oplus 0 \,, \qquad \pm k = 0, 1, 2, \cdots,$$

where $\mathscr{H} = \mathscr{H}_U \oplus \mathscr{H}_0$, U is a unitary operator on $\mathscr{H}_U$, and 0 is the zero operator on $\mathscr{H}_0$.

REMARKS. (i) The class $\mathscr{R}_1$ is similar to the class of pseudo-resolvents discussed by Hille and Phillips [6, § 5.8–5.10].
 (ii) $T^{(k)} \in \mathscr{T}_1$ satisfies
  (a) $T^{(k)*} = T^{(-k)}$, $\pm k = 0, 1, 2, \cdots$.
  (b) $T^{(k)}T^{(m)} = T^{(k+m)}$, $\pm k, \pm m = 0, 1, 2, \cdots$.

LEMMA 4. $\mathscr{S}_0 \subset \mathscr{S}_1 \subset \mathscr{S}$.

Proof. The only new part of this lemma is $\mathscr{R}_1 \subset \mathscr{R}$. Suppose $R_z \in \mathscr{R}_1$. (β_1a) implies that R_z is a holomorphic function of z for $|z| \neq 1$. Note that $\mathrm{Re}\,(R_zx, x) = \frac{1}{2}[(R_zx, x) + (R_z^*x, x)]$. If $0 < |z| < 1$, or $|z| > 1$, then successively substituting (β_1b) for R_z^* and (β_1a) for R_{z^*} in this equality, and simplifying, we obtain

$$\mathrm{Re}\,(R_zx, x) = \frac{1}{2}(R_0x, x) + \frac{1}{2}(1 - |z|^2)\,||R_zx||^2 \,.$$

Furthermore, this equality shows that (R_0x, x) is real, hence $R_0 = R_0^*$. Also, (β_1a) for $z' = 0$ and the continuity of R_z at $z = 0$ implies $R_0 = R_0^2$. Then $||R_0x||^2 = (R_0^2x, x) = (R_0x, x)$, thus the inequality in (βb) is an equality for all $|z| \neq 1$. Hence $R_z \in \mathscr{R}$.

4.3 THEOREM A$_0$. *The triple of classes $\mathscr{S}_0 = \{\mathscr{E}_0, \mathscr{R}_0, \mathscr{T}_0\}$ is a matched triple of classes.*

Proof. We need to show that $\mathscr{S}_0$ satisfies the requirements (i) and (ii) of the definition of a matched triple of classes given in § 1. But (i) was proved in Lemma 4, and (ii) follows from Lemma 3.

THEOREM A$_1$. *The triple of classes $\mathscr{S}_1 = \{\mathscr{E}_1, \mathscr{R}_1, \mathscr{T}_1\}$ is a matched triple of classes.*

Proof. By Lemma 4 $\mathscr{S}_1 \subset \mathscr{S}$. In Theorem 3 of § 3.2 it was shown that $N = N(E_{2\pi}) = N(T^{(0)}) = N(R_z)$ for $|z| < 1$, and that the decomposition $\mathscr{H} = N^\perp \oplus N$ is reducing for all values of E_θ, R_z, and $T^{(k)}$. By Theorem A_0, $\mathscr{S}_1(N^\perp)$ is a matched triple of classes, and trivially, $S_1(N)$ is a matched triple of classes. It follows that $\mathscr{S}_1(\mathscr{H})$ is a matched triple of classes.

4.4. The following theorem asserts the existence of a type $\mathscr{S}_1$ minimal dilation of an arbitrary matched triple of functions.

THEOREM C. *Let $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}(\mathscr{H})$ be a matched triple of functions. There exists a Hilbert space $\mathscr{H}^+ \supseteqq \mathscr{H}$ and a matched triple of functions $\{E_\theta^+, R_z^+, T^{(k)+}\} \in \mathscr{S}_1(\mathscr{H}^+)$ such that*

(i) *for $x \in \mathscr{H}$*

$$E_\theta x = PE_\theta^+ x\ , \quad R_z x = PR_z^+ x\ , \quad T^{(k)}x = PT^{(k)+}x\ ,$$

where P is the orthogonal projector onto the subspace $\mathscr{H}$ of $\mathscr{H}^+$.

(ii) $\mathscr{H}^+ = \operatorname{span}(\mathscr{H}, \mathscr{H}_0)$, *where*

$$\begin{aligned}\mathscr{H}_0 &= \operatorname{span}\{E_\theta^+ x: x \in \mathscr{H},\ 0 \leqq \theta \leqq 2\pi\}\\ &= \operatorname{span}\{R_z^+ x: x \in \mathscr{H},\ |z| \neq 1\}\\ &= \operatorname{span}\{T^{(k)+}x: x \in \mathscr{H},\ \pm k = 0, 1, 2, \cdots\}\ .\end{aligned}$$

(iii) $E^+(\sigma) = 0$ *if and only if* $E(\sigma) = 0$, *where σ is any Borel set on $[0, 2\pi]$ and $E(\sigma)$ is the operator measure related to E_θ. Also, $E_{2\pi}^+ = I$ if and only if $E_{2\pi} = I$.*

Proof. The assertions involving E_θ are contained in the Naimark diation theorem. The other two relations in (i) follow immediately from Theorems A and A_1. The equivalence of the spans in (ii) follows from formulas (1)-(6).

Note that in the special case of $T^{(k)} = T^k$ for $k \geqq 0$ and $T^{(k)} = (T^*)^{|k|}$ for $k < 0$, Theorem C contains the fundamental Sz.-Nagy Unitary Dilation Theorem [17, Th. 1] and [16, Th. III]. Also, for arbitrary $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}$, the minimal dilation $\{E_\theta^+, R_z^+, T^{(k)+}\} \in \mathscr{S}_1$ will be proper if and only if $E_{2\pi} = I$. In particular, a proper triple need not have a proper minimal dilation.

5. Triples related to contraction operators.

5.1 THEOREM 4. *The following conditions on a bounded operator-valued function R_z acting on $\mathscr{H}$ are equivalent:*

(i)
$$\begin{aligned}R_z &= \sum_{k=0}^{\infty} z^k T^k \qquad (|z| < 1)\\ &= -\sum_{k=1}^{\infty} z^{-k} T^{*k} \qquad (|z| > 1)\end{aligned}$$

where T is a fixed contraction operator on $\mathscr{H}$.

(ii) $$R_z = (I - zT)^{-1} \qquad (|z|) < 1)$$
$$= I - (I - z^{-1}T^*)^{-1} \qquad (|z| > 1)$$

where T is a fixed contraction operator on $\mathscr{H}$.

(iii) $$R_z = (I - zT)^{-1} \qquad (|z| < 1)$$
$$= I - (I - z^{-1}T^*)^{-1} \qquad (|z| > 1) ,$$

and

$$(1 - |z|) \| R_z - I \| \leqq |z| \qquad (|z| < 1) ,$$

where T is some fixed operator with $\overline{D_T} = \mathscr{H}$.

(iv) (a) $zR_z - \zeta R_\zeta = (z - \zeta)R_z R_\zeta$ *for* $|z|, |\zeta| < 1$.
(b) $(1 - |z|) \| R_z - I \| \leqq |z|, \quad |z| < 1$.
(c) $R_{z^*} = I - R_z^*, \quad 0 < |z| < 1$.

Under these conditions $R_z \in \mathscr{R}$ and $R_0 = I$.

Proof. (i) $\Leftrightarrow$ (ii): This follows by showing that the series in (i) are the expansions of the resolvents in (ii) (cf. [20, p. 261]).

(ii) $\Leftrightarrow$ (iii): That (ii) implies (iii) follows from Lemma 2 (a).

Assume (iii). The inequality for $z = 0$ gives $R_0 = I$. Define

$$T(z) \equiv \frac{1}{z}(R_z - I) , \qquad 0 < |z| < 1 ,$$

and

$$T(0) \equiv \operatorname*{Lim}_{z \to 0} T(z) .$$

Then for $x \in \mathscr{H}$.

$$T(z)x = \frac{1}{z}(R_z x - Ix) = \frac{1}{z}[R_z x - R_z(I - zT)x] = R_z Tx .$$

Hence

$$T(0)x = \operatorname*{Lim}_{z \to 0} R_z Tx = R_0 Tx = Tx .$$

Using the inequality in (iii) we obtain

$$\| T(z)x \| = \frac{1}{|z|} \| [R_z - I]x \| \leqq \frac{1}{1 - |z|} \| x \| ,$$

for $|z| < 1$, $x \in \mathscr{H}$. In particular,

$$\| Tx \| = \| T(0)x \| \leqq \| x \| ,$$

for all $x \in \mathscr{H}$, i.e., T is a contraction operator on $\mathscr{H}$.

(iii) $\Leftrightarrow$ (iv): It is easy to show that $R_z = (I - zT)^{-1}$, $|z| < 1$, satisfies the resolvent equation (iva). Conversely, (iva) implies R_z^{-1} exists for all $|z| < 1$. Therefore, one may define $T = (I - R_z^{-1})/z$, $0 < |z| < 1$, where (iva) implies that T is independent of z. Then $R_z = (I - zT)^{-1}$, $|z| < 1$. By taking adjoints we obtain the equivalence of (iva) and the form of R_z in (iii) for $|z| > 1$.

Condition (ii) and Theorem B imply $R_z \in \mathscr{R}$ and $R_0 = I$.

COROLLARY. *If T is a contraction operator on $\mathscr{H}$, and $T^{(k)}$ is defined by $T^{(k)} = T^k$ for $k = 0, 1, 2, \cdots$, and $T^{(k)} = (T^*)^{|k|}$ for $k = -1, -2, \cdots$, then $T^{(k)} \in \mathscr{T}$.*

Proof. If $T^{(k)}$ is defined as in the statement of the corollary, then the corresponding R_z defined by formula (3) satisfies condition (i) of the theorem. Therefore, $R_z \in \mathscr{R}$, and $T^{(k)} \in \mathscr{T}$ by Theorem A.

REMARKS. 1. A direct proof of the above corollary has been given previously by Sz.-Nagy [16, § 9].

2. Under the conditions of Theorem 4, R_z also satisfies the resolvent equation (iva) for $|z|, |\zeta| > 1$. Then $N(R_z)$ is constant for $|z| > 1$, but in general, we do not have $N(R_z) = \{0\}$ for $|z| > 1$.

3. If the resolvent equation (iva) is satisfied for z and ζ on opposite sides of the unit circle, i.e., for all $|z|, |\zeta| \neq 1$, then $N(R_z) = \{0\}$ for all $|z| \neq 1$, and $R_z \in \mathscr{R}_0$. In this case $R_z = (I - zT)^{-1}$ for all $|z| \neq 1$, where T is necessarily unitary by Lemma 3 of § 4.1.

5.2. Using the terminology of M. Schreiber [12, 13], we make the following:

DEFINITION. An operator distribution function $E_\theta \in \mathscr{E}$ is called *strong* (or *Nagy*) if

$$\int_0^{2\pi} e^{ik\theta} dE_\theta = \left[\int_0^{2\pi} e^{i\theta} dE_\theta\right]^k, \qquad k = 0, 1, 2, \cdots$$
$$= \left[\int_0^{2\pi} e^{i\theta} dE_\theta\right]^{*|k|}, \qquad k = -1, -2, \cdots .$$

REMARK. Necessarily $E_{2\pi} = I$ for strong operator distribution functions.

We consider the classes $\mathscr{E}_2$, $\mathscr{R}_2$, and $\mathscr{T}_2$ defined as follows:

(α_2) E_θ $(0 \leqq \theta \leqq 2\pi)$ belongs to the class $\mathscr{E}_2 = \mathscr{E}_2(\mathscr{H})$ whenever $E_\theta \in \mathscr{E}$ is a strong operator distribution function.

(β_2) R_z $(|z| \neq 1)$ belongs to the class $\mathscr{R}_2 = \mathscr{R}_2(\mathscr{H})$ whenever
(a) $zR_z - \zeta R_\zeta = (z - \zeta)R_z R_\zeta$ for $|z|, |\zeta| < 1$.

(b) $(1-|z|)\|R_z - I\| \leqq |z|, |z|<1$.
(c) $R_{z^*} = I - R_z^*,\ 0<|z|<1$.

(γ_2) $T^{(k)}$ ($\pm k = 0, 1, 2, \cdots$) belongs to the class $\mathscr{T}_2 = \mathscr{T}_2(\mathscr{H})$ whenever $T^{(1)} \equiv T$ is a contraction operator on $\mathscr{H}$ and

$$T^{(k)} \equiv T^k, \qquad k = 0, 1, 2, \cdots$$
$$\equiv T^{*|k|}, \qquad k = -1, -2, \cdots .$$

THEOREM D. *The triple of classes $\mathscr{S}_2 = \{\mathscr{E}_2, \mathscr{R}_2, \mathscr{T}_2\}$ is a matched triple of classes.*

Proof. (i) $\mathscr{S}_2 \subset \mathscr{S}$: Suppose $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}_2$. By ($\alpha_2$) $E_\theta \in \mathscr{E}$. Theorem 4 implies $R_z \in \mathscr{R}$ and its corollary implies $T^{(k)} \in \mathscr{T}$. Therefore, $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}$. It is clear that $\mathscr{S}_2 \neq \mathscr{S}$, so $\mathscr{S}_2 \subset \mathscr{S}$.

(ii) If $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}$, then any one of the relations $E_\theta \in \mathscr{E}_2$, $R_z \in \mathscr{R}_2$, $T^{(k)} \in \mathscr{T}_2$ implies all three, i.e., implies that $\{E_\theta, R_z, T^{(k)}\} \in \mathscr{S}$:

$E_\theta \in \mathscr{E}_2 \Leftrightarrow T^{(k)} \in \mathscr{T}_2$ is established by Sz.-Nagy in [17, p. 90, (3)-(5)]. See also [12, Th. 2.2] and [3, Th. 4].

If $T^{(k)} \in \mathscr{T}_2$, then the corresponding R_z defined by formula (3) satisfies condition (i) of Theorem 4, which is equivalent to the conditions of (β_2). Hence $R_z \in \mathscr{R}_2$. Conversely, if $R_z \in \mathscr{R}_2$, then equating coefficients in the series of formula (3) and Theorem 4(i), we see that $T^{(k)}$ is of the necessary form for $T^{(k)} \in \mathscr{T}_2$. This proves Theorem D.

We remark that the new part of Theorem D is the characterization (β_2) of the interrelated resolvent class $\mathscr{R}_2$. We also note that $\mathscr{S}_0 \subset \mathscr{S}_2 \subset \mathscr{S}$.

5.3. In general, the functional correspondence (7) is not multiplicative for $f \in B(dE)$. However, if we require $E_\theta \in \mathscr{E}_2$, then for certain subclasses of $B(dE)$ the functional calculus is multiplicative.

DEFINITION. The *Hardy H_∞ class of functions* is the algebra of bounded holomorphic functions in the unit disc [7].

By Fatou's theorem [7, p. 34] the limit $f(e^{i\theta})$ of $f(re^{i\theta})$ as $r \to 1^-$ exists almost everywhere with respect to θ, i.e., everywhere except on a set C_f of Lebesgue measure zero.

DEFINITION. By $H_\infty(dE)$ we mean the subalgebra of H_∞ such that

$$E(C_f) = \int_0^{2\pi} \chi(C_f; \theta)dE_\theta = 0 ,$$

where C_f is that set of Lebesgue measure zero given in Fatou's theorem, $\chi(C_f; \theta)$ is the characteristic function of C_f, and $E_\theta \in \mathscr{E}_2$.

Note that $H_\infty(dE) \subset B(dE)$ since the above definition implies that the set C_f does not belong to $\Lambda(E)$.

Assuming $E_\theta \in \mathscr{E}_2$, the functional correspondence given in § 2.3 has the following additional properties (cf. [19]):

(x) If f, g are polynomials in $e^{i\theta}$ (or in $e^{-i\theta}$), then $fg \sim FG$.

(xi) $f_n \to f$ and $g_n \to g$ boundedly a.e. (dE) and $f_n g_n \sim F_n G_n$ implies $fg \sim FG$.

(xii) When $f, g \in H_\infty(dE)$, then $fg \sim FG$.

(xiii) If $f \in H_\infty(dE)$, then $f \sim F = F(T)$, T a contraction operator, and

$$\| F(T) \| \leqq \sup_{|z|<1} | f(z) | \, .$$

It follows that the mapping $f \to F$ given by (7) is a homomorphism from the algebra $H_\infty(dE)$ into the algebra of bounded linear transformations of the Hilbert space $\mathscr{H}$. Further properties of this mapping have been studied by Schreiber [13] and Sz.-Nagy [18, 19].

6. The isometric operator case.

6.1. Let $R_z \in \mathscr{R}$ and suppose that for a certain operator T the relation

$$R_z(I - zT)x = x \qquad (x \in D_T) \tag{17}$$

holds for some z such that $|z| \neq 1$.

REMARKS. (i) Formula (17) is equivalent to the statement that $(I - zT)$ has a bounded inverse satisfying $(I - zT)^{-1} \subseteqq R_z$.

(ii) If $R_0 = I$ and (17) is valid for any point z_0 such that $0 < |z_0| < 1$, then $\| Tx \| \leqq \| x \|$ for all $x \in D_T$, i.e., $\| T \| \leqq 1$. Furthermore, if $R_0 = I$ and (17) is valid for any point z_0 such that $|z_0| > 1$, then $\| Tx \| \geqq \| x \|$ for all $x \in D_T$.

(iii) If $R_0 = I$ and (17) holds for any two points z_1, z_2 such that $0 < |z_1| < 1$ and $|z_2| > 1$, then T is isometric.

(iv) If $R_z \in \mathscr{R}$ and $R_0 = I$, then relation (17) is equivalent to $T \subseteqq T_z$, $|z| < 1$; and $T^{-1} \subseteqq T^*_{z^*}$, $|z| > 1$, where T_z and $T^*_{z^*}$ are the operators introduced in Theorem B.

(v) If $R_z \in \mathscr{R}$, $R_0 = I$, and $N(R_z) = \{0\}$ for $|z| > 1$, then relation (17) is equivalent to $T \subseteqq T_z$, $|z| \neq 1$, where T_z is as in the corollary of Theorem B.

(vi) If relation (17) holds in an open set contained in $|z| < 1$ (or $|z| > 1$), then by analytical continuation it holds throughout $|z| < 1$ (or $|z| > 1$).

(vii) If relation (17) holds for $|z| < 1$, then $E_{2\pi}x = x$ for $x \in \overline{D_T}$.

In particular, if $\overline{D_T} = \mathscr{H}$, then $E_{2\pi} = I$.

Proof. (i) is clear. (ii) follows by substituting (17) into formulas (13′) and (13″). (iii) follows from (ii) since an operator T is isometric if and only if $\|Tx\| = \|x\|$ for all $x \in D_T$ [14, Th. 2.46]. Property (iv) follows from the resolvent representations in Theorem B. For $|z| < 1$ we may substitute $R_z = (I - zT_z)^{-1}$ into (17) and obtain the equivalent equation $Tx = T_z x$, $x \in D_T$. For $|z| > 1$, we may substitute $R_z = I - (I - z^{-1}T_{z^*}^*)^{-1}$ into (17) and obtain the equivalent equation $Ix = T_{z^*}^* Tx$, $x \in D_T$, which is equivalent to T having an inverse satisfying $T^{-1} \subseteqq T_{z^*}^*$. Property (v) is obtained from (iv) and the corollary to Theorem B. (vi) is clear. (vii) follows by letting $z = 0$ in (17) and using Remark (i) of § 3.1.

THEOREM E 6.2. *Suppose that $\{E_\theta, R_z, T^{(k)}\}$ is a matched triple of functions in S with $E_{2\pi} = I$, that R_z is represented by T_z, and that E_θ^+ is the minimal dilation of E_θ. Let T be a certain operator with $D_T \subseteqq \mathscr{H}$. Then these conditions are equivalent*:

(i) *T is an isometric operator such that $T \subseteqq T^{(1)}$.*

(ii) $$T \subseteqq T^{(1)+} = \int_0^{2\pi} e^{i\theta} dE_\theta^+ .$$

(iii) *$R_z(I - zT)x = x$, for $|z| \neq 1$ and $x \in D_T$.*

(iv) *$T \subseteqq T_z$ for $|z| < 1$, T^{-1} exists, and $T^{-1} \subseteqq T_{z^*}^*$ for $|z| > 1$.*

Proof. (i) $\Rightarrow$ (ii): Assume (i). Then $Tx = T^{(1)}x = PT^{(1)+}x$ for $x \in D_T$. But for $x \in D_T$

$$\|Tx\| = \|x\| = \|T^{(1)+}x\| ,$$

since T is isometric, and $T^{(1)+}$ is unitary by Lemma 3. Therefore, $Tx = T^{(1)+}x$ for all $x \in D_T$, i.e., $T \subseteqq T^{(1)+}$.

Suppose (ii). $T \subseteqq T^{(1)+}$ implies $Tx = T^{(1)+}x$ for all $x \in D_T$. Then by property (v) of § 2.3 and equation (9)

$$\begin{aligned} \|Tx\|^2 = \|T^{(1)+}x\|^2 &= \int_0^{2\pi} |e^{i\theta}|^2 d(E_\theta^+ x, x) = \int_0^{2\pi} d(E_\theta^+ x, x) = (E_{2\pi}^+ x, x) \\ &= (E_{2\pi}x, x) = \|x\|^2 \qquad \text{for all } x \in D_T , \end{aligned}$$

i.e., T is an isometric operator. Also, $T \subseteqq T^{(1)+}$ implies

$$Tx = T^{(1)+}x = PT^{(1)+}x = T^{(1)}x$$

for $x \in D_T$, i.e., $T \subseteqq T^{(1)}$.

(ii) $\Rightarrow$ (iii): Since $E_\theta^+ \in \mathscr{E}_0$, Lemma 3 implies that $T^{(1)+}$ is unitary and $R_z = (I - zT^{(1)+})^{-1}$, $|z| \neq 1$. Therefore,

$$R_z(I - zT^{(1)+})x = x$$

for all $x \in \mathscr{H}$, and in particular, for all $x \in D_T$. But for $x \in D_T$, $T^{(1)} + x = Tx$ by (ii), hence this becomes (iii).

(iii) $\Rightarrow$ (ii): For $x \in D_T$, set $y = (I - zT)x$. Using (iii) and an argument similar to [8, § 5.2] one may show successively that R_z satisfies the resolvent equation, $R_{z^*}R_z y = PR_{z^*}^+ R_z^+ y$, and $R_z y = R_z^+ y$ for all $y \in \varDelta_T(z)$. Then $Tx = T^{(1)+}x$ for all $x \in D_T$, i.e., $T \subseteqq T^{(1)+}$.

(iii) $\Leftrightarrow$ (iv): This follows from Remark (iv) of § 6.1.

REMARKS. 1. When $\overline{\varDelta_T} = \mathscr{H}$, condition (iv) of Theorem E takes the form

(iv′) T^{-1} exists, and $T \subseteqq T_z = (T^{-1})^*$, for $|z| < 1$.

2. When $\overline{D_T} = \mathscr{H}$, condition (iv) of Theorem E takes the form

(iv″) T^{-1} exists, and $T^{-1} \subseteqq T_z^* = T^*$, for $|z| < 1$.

3. By Remark (vii) of § 6.1, the condition $E_{2\pi} = I$ in Theorem E may be omitted in (iv″), and weakened in parts (iii) and (iv) to $E_{2\pi}x = x$ for all $x \in \mathscr{H} \ominus \overline{D_T}$.

Condition (iii) (or (iv)) of Theorem E characterizes those $R_z \in \mathscr{R}$ with $R_0 = I$ which are generalized resolvents of a given isometric operator T. Using Remark 3, we obtain the following:

COROLLARY. *In order that a set of bounded linear operators R_z in $\mathscr{H}$ (with $D(R_z) = \mathscr{H}$ and $|z| \neq 1$) be a generalized resolvent of the isometric operator T in $\mathscr{H}$, it is necessary and sufficient that the following conditions be satisfied:*

(a) $R_z \in \mathscr{R} = \mathscr{R}(\mathscr{H})$, $|z| \neq 1$

(b) $R_z(I - zT)x = x$, *for all* $x \in D_T$ *and* $|z| \neq 1$

(c) $R_0 x = x$ *for all* $x \in \mathscr{H} \ominus \overline{D_T}$.

We remark that for T a closed isometric operator, this corollary is equivalent to Theorem 2 of [4]. The generalized resolvent of a closed isometric operator T was defined in [4] as a set R_z ($|z| \neq 1$) of operators in $\mathscr{H}$ satisfying

$$R_z = P(I - zT^+)^{-1}x\ , \qquad x \in \mathscr{H}\ ,$$

where T^+ is a unitary extension of T in a Hilbert space $\mathscr{H}^+ \supseteqq \mathscr{H}$, and P is the orthogonal projection of $\mathscr{H}^+$ onto $\mathscr{H}$. We note that Theorem C implies that this definition is equivalent to the conditions given in the corollary.

6.3 Condition (iv) of Theorem E leads to a simple and direct proof of a formula for all generalized resolvents of a closed isometric operator which is analogous to Štraus's formula for symmetric operators [15, Th. 7]. The formula was announced by M. E. Čumakin, a student of Štraus, in 1964 [4, Th. 3]

We shall need the following definitions and lemma which were originally introduced by McKelvey in [8].

DEFINITION. A pair of operators L_1, L_2 acting in $\mathscr{H}$ are said to be *formal adjoints* whenever

$$(L_1x, y) = (x, L_2y)$$

for $x \in D(L_1)$, $y \in D(L_2)$.

DEFINITION. A pair of formally adjoint contraction operators B_+, B_- will be called a **-pair* between closed subspaces $\mathscr{H}_+, \mathscr{H}_-$ of $\mathscr{H}$ whenever

$$D(B_\pm) = \mathscr{H}_\pm \,, \qquad B_\pm \mathscr{H}_\pm \subseteqq \mathscr{H}_\mp \,.$$

DEFINITION. A *-pair $\tilde{B}_+, \tilde{B}_-$ will be called a *maximal *-extension* of the *-pair B_+, B_- whenever

$$\tilde{B}_\pm \supseteqq B_\pm \quad \text{and} \quad D(\tilde{B}_\pm) = \mathscr{H} \,.$$

LEMMA 5. *Let B_+, B_- be a given *-pair between the subspaces $\mathscr{H}_+, \mathscr{H}_-$, and let B'_+, B'_- be any *-pair between the subspaces $\mathscr{H} \ominus \mathscr{H}_+$, $\mathscr{H} \ominus \mathscr{H}_-$. Then*

$$\tilde{B}_\pm = B_\pm \oplus B'_\pm$$

*is a maximal *-extension of B_+, B_-. Conversely, every maximal *-extension of B_+, B_- has this form.*

Suppose that T is a closed isometric operator acting in $\mathscr{H}$. Then $D_T = \overline{D_T} \subseteqq \mathscr{H}$. Set $\mathscr{H}_+ = D_T$ and $\mathscr{H}_- = \varDelta_T$. Then $\mathscr{H}_- = T\mathscr{H}_+$ and $\mathscr{H}_+ = T^{-1}\mathscr{H}_-$.

THEOREM 5. (*Čumakin*). *If T is a closed isometric operator, then a necessary and sufficient condition that the operator-valued functions T_z, $|z| < 1$, and $T^*_{z^*}x$, $|z| > 1$, shall represent a generalized resolvent R_z of T in the sense of Theorem E(iv) is that*

$$T_z = T \oplus \varPhi_+(z) \,, \qquad |z| < 1$$

and

$$T^*_z = T^{-1} \oplus \varPhi_-(z) \,, \qquad |z| < 1 \,,$$

where

(a) $\Phi_\pm(z)$ *is a formally adjoint pair of linear operators mapping* $\mathscr{H} \ominus \mathscr{H}_\pm$ *into* $\mathscr{H} \ominus \mathscr{H}_\mp$.

(b) $\|\Phi_\pm(z)\| \leqq 1$.

(c) $\Phi_+(z)$ *is a holomorphic operator-valued function of* z *for* $|z| < 1$, *and* $\Phi_-(z^*)$ *is a holomorphic operator-valued function of* z *for* $|z| > 1$.

Proof. Note that T, T^{-1} form a *-pair between the closed subspaces $\mathscr{H}_+, \mathscr{H}_-$.

Suppose T_z, $|z| < 1$, and $T^*_{z^*}$, $|z| > 1$, represent a generalized resolvent of T, i.e., $T \subseteqq T_z$, $|z| < 1$, and $T^{-1} \subseteqq T^*_{z^*}$, $|z| > 1$. Recall that the second condition is equivalent to $T^{-1} \subseteqq T^*_z$, $|z| < 1$. Now T_z and its adjoint T^*_z, $|z| < 1$, form a maximal *-extension of T, T^{-1}. By Lemma 5, T_z and T^*_z have the form

$$T_z = T \oplus \Phi_+(z) \quad \text{and} \quad T^*_z = T^{-1} \oplus \Phi_-(z) ,$$

where $\Phi_+(z), \Phi_-(z)$ is a *-pair between the subspaces $\mathscr{H} \ominus \mathscr{H}_+$, and $\mathscr{H} \ominus \mathscr{H}_-$. Then (a) and (b) are satisfied. Now $\Phi_+(z)$ is holomorphic for $|z| < 1$ because T_z is, and $\Phi_-(z^*)$ is holomorphic for $|z| > 1$ because $T^*_{z^*}$ is, i.e., (c) is also fulfilled.

Conversely, suppose

$$T_z = T \oplus \Phi_+(z) \quad \text{and} \quad T^*_z = T^{-1} \oplus \Phi_-(z) ,$$

for $|z| < 1$, where $\Phi_\pm(z)$ satisfy (a)-(c). By Lemma 5, T_z, T^*_z form a maximal *-extension of T, T^{-1}. Then $T \subseteqq T_z$ and $T^{-1} \subseteqq T^*_z$, for $|z| < 1$; hence $T^{-1} \subseteqq T^*_{z^*}$, $|z| > 1$. Also, (c) implies T_z is holomorphic for $|z| < 1$, and $T^*_{z^*}$ is holomorphic for $|z| > 1$. Therefore, T_z, $|z| < 1$, and $T^*_{z^*}$, $|z| > 1$, represent a generalized resolvent R_z of T. This proves Theorem 5.

The author wishes to express his sincere appreciation to Professor R. W. McKelvey for his guidance and encouragement in this work.

Bibliography

1. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Vol. 1, Ungar, New York, 1961.

2. ———, *Theory of linear operators in Hilbert space*, Vol. 2, Ungar, New York, 1963.

3. S. K. Berberian, *Naĭmark's moment theorem*, Michigan Math. J. **13** (1966) 171–184.

4. M. E. Cŭmakin, *Generalized resolvents of an isometric operator*, Dokl. Akad. Nauk SSSR **154** (1964), 791–794; Soviet Math. Dokl. **5** (1964), 193–196.

5. G. Herglotz, *Über potenzreihen mit positivem, reellem teil im einheitskreis*, Ber. Sächs. Ges. d. Wiss. Leipzig **63** (1911), 501–511.

6. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, revised ed.,

Amer. Math. Soc. Colloq. Pub. vol. 31, R. I., 1957.

7. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, N. J., 1962.

8. R. McKelvey, *Spectral measures, generalized resolvents, and functions of positive type*, J. Math. Anal. Appl. **11** (1965), 447-477.

9. F. Riesz, *Sur certains systèmes singuliers d'équations intégrales*, Ann. Sci. École Norm. Sup. (3) **28** (1911), 33-62.

10. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Ungar, New York, 1955.

11. H. L. Royden, *Real analysis*, Macmillan, New York, 1963.

12. M. Schreiber, *Unitary dilations of operators*, Duke Math. J. **23** (1956), 579-594.

13. ———, *A functional calculus for general operators in Hilbert space*, Trans. Amer. Math. Soc. **87** (1958), 108-118.

14. M. H. Stone, *Linear transformations in Hilbert space*, Amer. Math. Soc. Colloq. Pub. vol. 15, New York, 1932.

15. A. V. Štraus, *Generalized resolvents of symmetric operators*, Izv. Akad. Nauk SSSR Ser. Mat. **18** (1954) 51-86.

16. B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space Which Extend Beyond This Space*, Ungar, New York, 1960 (Appendix to [**10**]).

17. ———, *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. Szeged **15** (1953), 87-92.

18. B. Sz.-Nagy and C. Foias, *Sur les contractions de l'espace de Hilbert*, III, Acta Sci. Math. Szeged **19** (1958), 26-45.

19. B. Sz.-Nagy and C. Foias, *Sur les contractions de l'espace de Hilbert*, VI, *Calcul functionnel*, Acta Sci. Math. Szeged **23** (1962), 130-167.

20. A. E. Taylor, *Introduction to functional analysis*, Wiley & Sons, New York, 1958.

Received June 27, 1967. This work is part of a doctoral thesis submitted to the University of Colorado, and was supported by a National Science Foundation Cooperative Graduate Fellowship.

UNIVERSITY OF NEW MEXICO