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UNCONDITIONAL AND SHRINKING BASES IN LOCALLY CONVEX SPACES

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Let E be a locally convex space with an unconditional Schauder basis $\{x_k\}$ and let $\{f_k\}$ be the sequence of coefficient functionals biorthogonal to $\{x_k\}$. Owing to works of R. C. James and S. Karlin it is known that if E is a Banach space then each of the three conditions which follow is necessary and sufficient for $\{f_k\}$ to be a basis for E^* in the strong (norm) topology.

(1) E has no subspace topologically isomorphic to the space l^1.

(2) E^* is separable in the strong topology.

(3) E^* is weakly $(w(E^*, E^{**}))$ sequentially complete.

The primary purpose of this paper is to show that in certain spaces which are more general than Frechet spaces and hence than Banach spaces, each of the above three conditions is necessary and sufficient for

(0) $\{f_k\}$ is a strong basis for E^*.

Specifically if E is a complete barrelled space, each of (1) and (2) is sufficient for (0). In any locally convex space (2) implies (1) (even if E has no basis) and so each of (1) and (2) is necessary for (0). If E is a space having the property that weak* bounded subsets of E^* are strongly bounded (complete locally convex spaces and barrelled spaces have this property) then (3) is sufficient for (0). (3) is necessary for (0) if E is both barrelled and metrizable.

Besides the papers of James [8] and Karlin [9], related matter of importance is contained in the works of M. M. Day [2, Ch. 4] and C. Bessaga and A. Pelczynski [1]. E. L. Dubinsky and J. R. Retherford [5] using Köthe sequence space techniques have proved the (1) $\leftrightarrow$ (2) part of Theorem 2.12.

1. Preliminarie and fundamental theorems. Since the main results of this paper depend upon many theorems which are not widely known in their more general settings, the author thought it wise to include this section which introduces, in addition to basic theorems, some difinitions, terminology, and notation. No proofs are given for known results.

If E is a locally convex space, then π will be used for the canonical mapping from E to E^{**}, the space of strongly continuous functions on E^*. For convenience, a subscript 2 will be added to the word "space" to designate that the Hausdorff axiom is satisfied.

A sequence $\{x_k\}$ in a linear topological space$_2$ is called a basis if for each element x in the space there is a unique sequence of scalars $\{f_k(x)\}$ such that $x = \sum_{k=1}^{\infty} f_k(x)x_k$. The resulting linear functionals $\{f_k\}$ are called the coefficient functionals. If the coefficient functionals are continuous, then $\{x_k\}$ (or $\{x_k, f_k\}$) is called a Schauder basis. The n-th partial sum operator s_n is defined by $s_n(x) = \sum_{k=1}^{n} f_k(x)x_k$; the n-th remainder is defined by $R_n(x) = \sum_{k=n+1}^{\infty} f_k(x)x_k$. We shall use Φ for the collection of all nonempty finite subsets of the positive integers ω. If $\sigma \in \Phi$, then the σ-partial sum operator S_σ is defined by $S_\sigma(x) = \sum_{k \in \sigma} f_k(x)x_k$; the σ-remainder is defined by $R_\sigma(x) = x - S_\sigma(x)$. Note that $R_\rho \circ R_\sigma(x) = R_{\rho \cap \sigma}(x)$, and if $\rho \cap \sigma \neq \emptyset$ then $S_\rho \circ S_\sigma(x) = S_{\rho \cap \sigma}(x)$; if $\rho \cap \sigma = \emptyset$, then $S_\rho \circ S_\sigma(x) = \theta$ for every x.

PROPOSITION 1.1. *Let E be a linear topological space$_2$. If $\{x_k, f_k\}$ is a Schauder basis or a weak Schauder basis for E, then $\{f_k, \pi(x_k)\}$ is a weak* Schauder basis for E^*. It follows that if $\{f_k\}$ is also a strong basis then it is a strong Schauder basis with $\{\pi(x_k)\}$ as the coefficient functionals. The σ-partial sum for $\{f_k\}$ is then the adjoint of S_σ; i.e., $S_\sigma^*(f) = \sum_{k \in \sigma} \pi(x_k)(f)f_k = \sum_{k \in \sigma} f(x_k)f_k$. The σ-remainder is R_σ^* and of course $R_\sigma^* f = f - S_\sigma^* f$.*

As a kind of converse we have the following.

PROPOSITION 1.2. *Let E be a locally convex space$_2$. If $\{f_k F_k\}$ is a weak* Schauder basis for E^*, then $\{x_k, f_k\}$ is a weak Schauder basis for E, where for each k, x_k is that unique element of E such that $\pi(x_k) = F_k$.*

The next two propositions require the Barrel Theorem for their proofs.

PROPOSITION 1.3. *Let E be a barrelled space$_2$. If $\{x_k, f_k\}$ is a Schauder basis for E then $\{f_k\}$ is a strong basis for the closed linear span, $\overline{[f_k]}$, of $\{f_k\}$.*

Actually, in Proposition 1.3, "Schauder basis" may be replaced by "weak Schauder basis" as the following reveals.

PROPOSITION 1.4. *In a barrelled space a weak Schauder basis is a Schauder basis.*

A Schauder basis $\{x_k, f_k\}$ is called a shrinking basis if $\{f_k\}$ is a

basis for E^* (in the strong topology). A basis $\{x_k\}$ is boundedly complete if for each sequence $\{t_k\}$ of scalars such that $\{\sum_{k=1}^{n} t_k x_k\}_{n=1}^{\infty}$ is bounded, the series $\sum_{k=1}^{\infty} t_k x_k$ is convergent. The next proposition due to J. Dieudonné [3, Prop. 6] strengthens Proposition 1.1 in the case where E is barrelled.

PROPOSITION 1.5. *If $\{x_k, f_k\}$ is a Schauder basis in a barrelled space$_2$ E, then $\{f_k\}$ is a weak* boundedly complete basis for E^*. Hence, if $\{x_k, f_k\}$ is a shrinking basis then $\{f_k\}$ is a boundedly complete basis for E^*.*

The next proposition was proved for Banach spaces by R. C. James [8] and generalized by J. R. Retherford [13].

PROPOSITION 1.6. *Let $\{x_k, f_k\}$ be a Schauder basis for a barrelled space$_2$ E. Then E is reflexive if and only if $\{x_k\}$ is both shrinking and boundedly complete.*

If $(E, \|\cdot\|)$ is a Banach space with basis $\{x_k\}$, then a new norm related to the basis can be formed: $\|x\|' = \sup_n \{\|S_n(x)\|\}$. Both norms have the same topology but the new norm is more useful. A more general device has been developed by C. W. McArthur [14]. In the fundamental lemma which follows a part of this device is abstracted even further in order that the result may be applied to Theorem 1.18.

LEMMA 1.7. *Let (E, τ) be a locally convex space and $\{g_\alpha : \alpha \in A\}$ a net* [10, p. 28] *of pointwise bounded linear operators on E. Let $\mathscr{V}$ be the local base consisting of all barrelled neighborhoods of E. For $V \in \mathscr{V}$, define V' to be $\{x \in E : g_\alpha(x) \in V$ for all $\alpha \in A\}$. Then we have the following.*

(1) *$\mathscr{V}' \equiv \{V' : V \in \mathscr{V}\}$ is a local base for a locally convex topology τ' on E.*

(2) *If there is a subnet $\{g_{n(\beta)} : \beta \in B\}$ such that, for each $x \in E$, $\{g_{n(\beta)}(x) : \beta \in B\}$ is convergent to x then τ' is stronger than τ.*

(3) *If in addition to* (2) *each g_α is continuous and E is barrelled, then $\tau' = \tau$.*

Proof. By the criterion 6.5 of [10, p. 47], $\mathscr{V}'$ forms a local base for a locally convex topology if the three conditions below are met.

(i) Each $V' \in \mathscr{V}'$ is convex, circled, and radial at zero.

(ii) For each U' and V' of $\mathscr{V}'$ there is a $W' \in \mathscr{V}'$ such that

$W' \subset U' \cap V'$.

(iii) tV' belongs to $\mathscr{V}'$ for each $V' \in \mathscr{V}'$ and each scalar t.

Proof of (i). Let s and t be nonnegative scalars such that $s + t = 1$. If $x, y \in V$; then $g_\alpha(x)$, $g_\alpha(y) \in V$ for all $\alpha \in A$. Since V is convex, $sg_\alpha(x) + tg_\alpha(y) = g_\alpha(sx + ty) \in V$ for all $\alpha \in A$ and hence $sx + ty \in V'$ so V' is convex. Let $|t| \leqq 1$ $x \in V'$. $g_\alpha(x) \in V$ for all α and since V is circled, $tg_\alpha(x) = g_\alpha(tx) \in V$ for all α; hence $tx \in V'$. Therefore V' is circled. Let $x \in E$. Since g_α is pointwise bounded, there is a positive scalar s such that if $0 < t < s$ then $tg_\alpha(x) \in V$ for all α. Hence $tx \in V'$ so V' is radial at zero.

Proof of (ii). There exists a $W \in \mathscr{V}$ such that $W \cup U \cap V$. Hence $W' \subset (U \cap V)' = U' \cap V'$.

Proof of (iii). For each scalar t, $tV \in \mathscr{V}$. Hence $(tV)' = tV' \in \mathscr{V}$.

To prove (2) we show that $V' \subset V$ for each $V \in \mathscr{V}$. If $x \in V'$, then $g_\alpha(x) \in V$ for all α and in particular $g_{n(\beta)}(x) \in V$ for all $\beta \in B$. Thus since V is closed V is closed $x \in V$.

To prove (3) we show that V' is τ-closed and hence a τ-barrel. Let $\{y_\beta : \beta \in B\}$ be a net in V' converging to y. Let $\alpha \in A$; then since g_α is continuous, $g_\alpha(y_\beta)$ converges with β to $g_\alpha(y)$. Since $g_\alpha(y_\beta) \in V$ for each $\beta \in B$, $g_\alpha(y) \in V$; α was arbitrary, so $y \in V'$.

Part (1) and (2) of the next proposition follow from the lemma.

PROPOSITION 1.8. *Let (E, τ) be a locally convex space$_2$ with a basis $\{x_k\}$ and let $\mathscr{V}$ be as in Lemma 1.7. For $V \in \mathscr{V}$, define $V' = \{x : S_n(x) \in V$ for each $n \in \omega\}$.*

(1) *$\mathscr{V}' = \{V' : V \in V\}$ is a local base for a locally convex topology τ' which is stronger than τ.*

(2) *If E is barrelled and $\{x_k\}$ is a Schauder basis then $\tau = \tau'$.*

(3) *If (E, τ) is complete, then so is (E, τ').*

The last propositions of this section are slight modifications of results due to Retherford and McArthur [14],

PROPOSITION 1.9. *Let E be a complete locally convex space$_2$ and $\{x_k\}$ a sequence of nonzero elements of E. If for each continuous seminorm P on E there is a continuous seminorm Q such that $P(\sum_{k=1}^{p} a_k x_k) \leqq Q(\sum_{k=1}^{q} a_k x_k)$ for $p \leqq q$ and arbitrary scalars $a_1, \cdots, a_q$, then $\{x_k\}$ is a Schauder basic sequence (i.e., $\{x_k\}$ is a Schauder basis for its closed linear span $\overline{[x_k]}$).*

PROPOSITION 1.10. *Let E be a barrelled space$_2$ with a Schauder basis $\{x_k\}$. Then for each continuous seminorm P there is a continuous seminorm Q such that for $p \leqq q$ and arbitrary scalars*

$a_1, \cdots, a_q$, *we have* $P(\sum_{k=1}^{p} a_k x_k) \leqq Q(\sum_{k=1}^{q} a_k x_k)$.

Let $\sum_{k=1}^{\infty} y_k$ be a (formal) series in a linear topological space E. The series is called unordered bounded if the set of unordered partial sums $\{\sum_{k \infty \sigma} y_k : \sigma \in \phi\}$ is bounded. The series is called unconditionally (or unordered) convergent if the net of unordered partial sums (with inclusion as the order relation in ϕ) is convergent to an element of E. It is well known that a series $\sum_{k=1}^{\infty} y_k$ in a linear topological space$_2$ is unconditionally convergent if and only if for each permutation p of ω, the set of positive integers, the series $\sum_{k=1}^{\infty} y_p(k)$ is convergent (each series converges to the same element regardless of the permutation).

The series $\sum_{k=1}^{\infty} y_k$ is subseries convergent if for every subsequence $\{y_{n_k}\}$ of $\{y_k\}$ the series $\sum_{k=1}^{\infty} y_{n_k}$ is convergent. $\sum_{k=1}^{\infty} y_k$ is bounded-multiplier convergent if for each bounded sequences of scalars $\{t_k\}$ the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent. $\sum_{k=1}^{\infty} y_k$ is absolutely convergent in a locally convex space E if the series is convergent and for each continuous seminorm P on E, the series $\sum_{k=1}^{\infty} P(y_k)$ is convergent. In any locally convex space, absolute convergence implies unconditional convergence.

PROPOSITION 1.11. *In a locally convex space a series is weakly absolutely convergent if and only if it is weakly unconditionally convergent.*

Define a to be the set of sequences $\{\alpha_m\}$ such that $\alpha_m = \pm 1$ for each m; define b to be the set of sequences $\{\beta_m\}$ such that $\beta_m = 0$ or 1 for each m; and define e to be the set of all complex sequences $\{\varepsilon_m\}$ such that $|\varepsilon_m| \leqq 1$ for each m.

PROPOSITION 1.12. *If* $\sum_{k=1}^{\infty} y_k$ *is a series in a locally convex space* E, *then the statements below are equivalent.*

(1) $\sum y_k$ *is unordered bounded.*
(2) $\sum y_k$ *is weakly unordered Cauchy.*
(3) *The set* $\{\sum_{k=1}^{n} \alpha_k y_k : \{\alpha_k\} \in a$ *and* $n \in \omega\}$ *is bounded.*
(4) *The set* $\{\sum_{k=1}^{n} \beta_k y_k : \{\beta_k\} \in b$ *and* $n \in \omega\}$ *is bounded.*
(5) *The set* $\{\sum_{k=1}^{n} \varepsilon_k y_k : \{\varepsilon_k\} \in e$ *and* $n \in \omega\}$ *is bounded.*

COROLLARY 1.13. *In a weakly sequentially complete locally convex space a series is unordered bounded if and only if it is weakly unconditionally convergent (weakly absolutely convergent).*

The Orlicz-Pettis theorem for Banach spaces on subseries convergence has been shown valid for locally convex spaces by Grothendieck

and McArthur. For a proof of the theorem, which follows, see [12].

THEOREM 1.14. *In a locally convex space*$_2$ *a series is subseries convergent if and only if it is weakly subseries convergent.*

It is well known that in a sequentially complete locally convex space, bounded-multiplier convergence and subseries convergence are each equivalent to unconditional convergence [2]. The next theorem adds three other conditions which are equivalent to unconditional convergence. A proof of the equivalence of (1), (2), (3), and (4) can be found in [15]. (1) is proved equivalent to (2) in [11] by use of a certain convergence criterion.

THEOREM 1.15. *Let* $\{y_k\}$ *be a sequence in a sequentially complete locally convex space* E. *The following six statements are equivalent.*

(1) *The series* $\sum_{k=1}^{\infty} y_k$ *converges unconditionally.*

(2) *For any equicontinuous subset* A *of* E^*, *the series* $\sum_{k=1}^{\infty} |f(y_k)|$ *converges uniformly for* $f \in A$.

(3) *The series* $\sum_{k=1}^{\infty} \alpha_k y_k$ *converges uniformly for* $\{\alpha_k\} \in a$.

(4) *The series* $\sum_{k=1}^{\infty} \varepsilon_k y_k$ *converges uniformly for* $\{\varepsilon_k\} \in e$.

(5) *The series* $\sum_{k=1}^{\infty} y_k$ *is bounded-multiplier convergent.*

(6) *The series* $\sum_{k=1}^{\infty} y_k$ *is subseries convergent.*

Since a weakly (sequentially) complete locally convex space is (sequentially) complete, Theorem 1.14 and 1.15 imply that in a weakly sequentially complete locally convex space a series is unconditionally convergent if and only if it is weakly unconditionally convergent. Thus similarly to Corollary 1.13 we have the following.

COROLLARY 1.16. *In a weakly sequentially complete locally convex space*$_2$, *a series is unordered bounded if and only if it is unconditionally convergent.*

A basis $\{x_k, f_k\}$ in a linear topological space$_2$ is called an unconditional basis if for each x in the space the series $\sum_{k=1}^{\infty} f_k(x) x_k$ is unconditionally convergent to x. Hence, $\{x_k, f_k\}$ is an unconditional Schauder basis if and only if for each permutation p of ω, $\{x_{p(k)}, f_{p(k)}\}$ is a Schauder basis. It follows that each of Propositions 1.1 and 1.5 has an analog for unconditional bases. For instance we have the following.

THEOREM 1.17. *If* $\{x_k, f_k\}$ *is an unconditional Schauder basis for a linear topological space*$_2$ E, *then* $\{f_k, \pi(x_k)\}$ *is a weak** *unconditional Schauder basis for* E^*.

If $\{x_k, f_k\}$ is a basis and $\alpha = \{\alpha_k\}$ is a sequence of scalars, define $S_{n,\alpha}(x) = \sum_{k=1}^{n} \alpha_k f_k(x) x_k$. Let (E, τ) be a locally convex space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. Let $\mathscr{V}^{\wedge}$ be the local base consisting of all barrelled neighborhoods in E. If $V \in \mathscr{V}^{\wedge}$, define $V_b = \{x \in E: S_\sigma(x) \in V$ for all $\sigma \in \Phi\}$, $V_a = \{x \in E: S_{n,\alpha}(x) \in V$ for all $n \in \omega$ and all $\alpha \in a\}$, and $V_e = \{x \in E: S_{n,\varepsilon}(x) \in V$ for all $n \in \omega$ and all $\varepsilon \in e\}$. Let $\mathscr{V}^{\wedge}{}_b = \{V_b : V \in \mathscr{V}^{\wedge}\}$, $\mathscr{V}^{\wedge}{}_a = \{V_a : V \in \mathscr{V}^{\wedge}\}$, and $\mathscr{V}^{\wedge}{}_e = \{V_e : V \in \mathscr{V}^{\wedge}\}$.

THEOREM 1.18. *Le (E, τ) be as above. The families $\mathscr{V}^{\wedge}{}_b$, $\mathscr{V}^{\wedge}{}_a$, and $\mathscr{V}^{\wedge}{}_e$ form local bases for locally convex Hausdorff topologies on E, say τ_b, τ_a, and τ_e. Each of these topologies is stronger than τ and if E is barrelled then $\tau = \tau_b = \tau_a = \tau_e$.*

Proof. According to Lemma 1.7, $\mathscr{V}^{\wedge}{}_b$, $\mathscr{V}^{\wedge}{}_a$, and $\mathscr{V}^{\wedge}{}_e$ form local bases if the families of operators $\{S_\sigma : \sigma \in \Phi\}$, $\{S_{n,\alpha} : (n, \alpha) \in \omega \times a\}$, and $\{S_{n,\varepsilon} : (n, \varepsilon) \in \omega \times e\}$ are pointwise bounded. Since $\{x_k, f_k\}$ is an unconditional Schauder basis it is a weak unconditional Schauder basis. Thus $\sum_{k=1}^{\infty} f_k(x) x_k$ is weakly unordered Cauchy for each x and hence by Proposition 1.12 the three families above are pointwise bounded.

Let d be either the family of sequences a or e. The set $\omega \times d$ is a directed set under the order relation $<$ defined by: $(n, \alpha) < (m, \beta)$ if and only if $n \leqq m$. Thus $\{S_{n,\alpha} : (n, \alpha) \in \omega \times d\}$ is a net. Let α^1 be the sequence $(1, 1, 1, \cdots)$ and not that the subnet $\{S_{n,\alpha^1}(x) : n \in \omega\}$ converges to x for each $x \in E$. Thus by part (2) of 1.7. τ_b, τ_a, and τ_e are stronger than τ.

Let $\mathscr{V}'$ be $\mathscr{V}^{\wedge}{}_b$, $\mathscr{V}^{\wedge}{}_a$, or $\mathscr{V}^{\wedge}{}_e$ then $\bigcap_{V' \in \mathscr{V}'} V' \subset \bigcap_{V \in \mathscr{V}} V$. Since τ is a Hausdorff topology, the latter intersection is empty and hence the other topologies are also Hausdorff.

For each $n \in \omega$ and α in b, a, or e, the operator $S_{n,\alpha}$ is continuous so if E is barrelled then $\tau = \tau_b = \tau_a = \tau_e$.

The next result, its variations, and its corollary shall be referred to as the Generalized Gurevic Theorem (see [7]). We use d to mean either of the family of sequences b, a, or e (actually the theorem is valid if d satisfies $a \subset d \subset e$ or $b \subset d \subset e$).

THEOREM 1.19. *Let $\{x_k\}$ be a sequence of nonzero elements in a locally convex space$_2$ E. Call the following condition*: given a neighborhood of zero U there is a neighborhood of zero V such that for an arbitrary sequence $\{a_n\}$ of scalars arbitrary p and q in $\{k : k$ is a positive integer or $k = \infty\}$ with $q \geqq p$, and arbitrary sequence $\{\delta_n\} \in d$, we have $\sum_{n=1}^{q} a_n x_n \in V$ implies $\sum_{n=1}^{p} \delta_n a_n x_n \in U$.*

(1) *If E is sequentially complete, then $*$ implies $\{x_k\}$ is unconditionally Schauder basic.*

(2) *If E is barrelled and $\{x_k\}$ is an unconditional Schauder basis for E, then $*$ holds.*

VARIATION 1.20. In the theorem, condition $*$ can be replaced by condition $*_1$: given an equicontinuous set $A \subset E^*$, there is an equicontinuous set $B \subset E^*$ such that for an arbitrary positive number M and arbitrary sequence $\{a_n\}$ of scalars arbitrary p and q in $\{k: k$ is a positive interger or $k = \infty\}$ with $q \geqq p$, and arbitrary sequence $\{\delta_n\} \in d$, we have $|\sum_{n=1}^{q} a_n g(x_n)| \leqq M$ for all $g \in B$ implies $|\sum_{n=1}^{p} \delta_n a_n f(x_n)| \leqq M$ for all $f \in A$.

VARIATION 1.21. In the theorem condition $*$ can be replaced by condition $*_2$: given a continuous seminorm P there is a continuous seminorm Q such that for arbitrary sequence $\{a_n\}$ of scarlars arbitrary p and q in $\{k: k$ is a positive integer or $K = \infty\}$ with $q \geqq p$, and arbitrary sequence $\{\delta_n\} \in d$ we have $P(\sum_{n=1}^{p} \delta_n a_n x_n) \leqq Q(\sum_{n=1}^{q} a_n x_n)$.

Proof. First we show that $*$ and $*_1$ are equivalent. Since $\{A_0$: A equicontinuous$\}$ forms a local base in E, $*$ is easily seen to be equivalent to the following. Given an equicontinuous A there is an equicontinuous B such that for $\cdots$ we have $\sum_{n=1}^{q} a_n x_n \in B_0$ implies $\sum_{n=1}^{p} \delta_n a_n x_n \in A_0$. But this is equivalent to $*_1$.

Now we show that $*$ and $*_2$ are equivalent. Suppose that $*_2$ holds. Let U be a barrelled neighborhood of zero and let P be the Minkowski functional of U (U equals U_P, the closed unit ball of P). In accordance with $*_2$ corresponding to P there is a continuous seminorm Q such that $P(\sum^p) \leqq Q(\sum^q)$. If $\sum^q \in U_Q$ then $Q(\sum^q) \leqq 1$ and hence $P(\sum^p) \leqq 1$; that is, $\sum^p \in U_P$. To show that $*$ implies $*_2$ let P be a continuous seminorm and U_P its unit ball. There exists a U_Q (i.e., a barrelled neighborhood of zero with Minkowski functional Q) such that $\sum^q \in U_Q$ implies $\sum^p \in U_P$; or $Q\sum^q \leqq \varepsilon$ implies $P\sum^p \leqq \varepsilon$. If $Q\sum^q \equiv Q\sum_{k=1}^{q} a_{n_k} x_{n_k}$ is zero then $P\sum^p \leqq \varepsilon$ for every $\varepsilon > 0$ so that $P\sum^p = 0$. Hence $Q\sum^q = P\sum^p = 0$. If $Q\sum^q \neq 0$ then let $t = 1/Q\sum^q$. $Q(t\sum^q) = 1$; hence $P(t\sum^p) \leqq 1$. Upon substitution for t we get $P(\sum^p) \leqq Q(\sum^q)$.

To prove part (1) of the theorem, observe that $*_2$ is a stronger condition than that in Proposition 1.9 and hence $\{x_k\}$ is a Schauder basic sequence. To show that $\{x_k\}$ is unconditional, let U be a neighborhood of zero in E. In accordance with $*$ let V be the neighborhood corresponding to U. Let $\{f_k\} \subset \overline{[x_k]}^*$ be the family of coefficient functionals for $\{x_k\}$. Let $x \in \overline{[x_k]}$; then for n and m large enough $\sum_{k=m}^{n} f_k(x) x_k \in V$. Hence if $\{\delta_k\}$ is any element of d,

$\sum_{k=m}^{n} \delta_k f_k(x) x_k \in U$. It follows from Theorem 1.15 and the completeness of E that $\sum_{k=1}^{\infty} f_k(x) x_k$ is unconditionally convergent.

Part (2) is an immediate consequence of Theorem 1.18. Given a barrelled neighborhood of zero U, then $U_d \equiv \{x \in E: S_{n,\delta}(x) \in U$ for all $n \in \omega$ and $\delta \in d\}$ is also a neighborhood of zero, and U_d satisfies the requirements for V in $*$.

COROLLARY 1.22. *Let E be a barrelled space$_2$ with unconditional Schauder basis $\{x_k, f_k\}$ and let $\{y_k\}$ be any sequence in E and $\{\sigma_k\}$ a sequence in Φ such that $\sigma_n \cap \sigma_m = \emptyset$ if $m \neq n$. Define w_k to be the σ_k-partial sum of y_k; i.e., $w_k = \sum_{j \in \sigma_k} f_j(y_k) x_j$. Then given a neighborhood of zero U there is a neighborhood of zero V such that for arbitrary scalars t_k, arbitrary $\{\delta_k\} \in d$, and arbitrary n and m in $\{k: k \in \omega$ or $k = \infty\}$ with $n \geqq m$ we have $\sum_{k=1}^{n} t_k w_k \in V$ implies $\sum_{k=1}^{m} \delta_k t_k w_k \in U$. Thus if E is also sequentially complete and for each k, $w_k \neq 0$ then the sequence $\{w_k\}$ is unconditionally basic in E.*

Proof. Simply observe that if $a_{jk} = t_k f_j(y_k)$, then $\sum_{k=1}^{n} t_k w_k = \sum_{j \in \sigma_k, k \in \{1,2,\cdots,n\}} a_{jk} x_j$ and in the expansion the j's are all distinct by the null intersection requirement of the hypothesis. Then apply Theorem 1.19.

The Generalized Gurevic Theorem, which we have just proved, is of fundamental importance. We shall apply it often in the next section.

2. Unconditional and shrinking bases.

As pointed out in the introductory remarks, we shall show that most of the results of James and Karlin are valid for complete barrelled spaces, barrelled spaces, or complete locally convex spaces. Even unqualified locally convex spaces are sufficient for a few of the more simple results. The first theorem follows easily from the remarks made after the definition of unconditional basis, page 12, and Proposition 1.3.

THEOREM 2.1. *Let E be a barrelled space with an unconditional basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then $\{f_k \pi(x_k)\}$ is an unconditional Schauder basis for E^*.*

Hereafter we shall use "ω-complete" for "sequentially complete".

LEMMA 2.2. *Let E be a locally convex space$_2$ and $\{y_k\}$ a bounded sequence in E.*

(1) *If E is ω-complete, then the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent for each $t = \{t_k\} \in l^1$.*

(2) *If F is a subspace of l^1 such that for each $t \in F$ the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent, then the linear mapping $T: F \to E$ defined by $T(t) = \sum_{k=1}^{\infty} t_k y_k$ is continuous.*

Proof. To see that $\sum_{k=1}^{\infty} t_k y_k$ is convergent if E is ω-complete, let A_0 be a neighborhood of zero (where A is an equicontinuous subset of E^*). Since $\{y_k\}$ is bounded there is a positive scalar s such that $y_k \in sA_0$ for each k. For n and m large enough $\sum_{k=m}^{n} |t_k| < 1/s$. Since $|f(y_k)| \leqq s$ for each $f \in A$ and $k \in \omega$ it follows that $\sum_{k=m}^{n} t_k y_k \in A_0$. Hence the series is convergent.

Let A be equicontinuous and define $M = \sup\{|f(y_k)| : f \in A, k \in \omega\}$. If $t \in F$ and $\|t\| < 1/M + 1$ then for $f \in A$, $|fT(t)| = \sum_{k=1}^{\infty} t_k f(y_k)| \leqq \sum_{k=1}^{\infty} |t_k| \cdot |f(y_k)| < 1$. Therefore $T(t) \in A_0$ so T is continuous.

LEMMA 2.3. *Let T be a linear transformation of a normed space E into a locally convex space F. If there exists a continuous functional f on F such that $f(\theta) = 0$ and a positive number m such that $|fT(x)| \geqq m\|x\|$ for each $x \in E$, then T is one-to-one and relatively open.*

Proof. If $T(x)$ is zero then so is $f(T(x))$ and hence by the given inequality, $x = \theta$; so T is one-to-one. To prove that T is relatively open we show that the inverse map $T^{-1}: T(E) \to E$ is bounded on a neighborhood of zero, vis. $f^{-1}[-1, 1] \cap T(E)$. Let $x \in T^{-1}(f^{-1}[-1, 1])$ then $T(x) \in f^{-1}[-1, 1]$ so that $|fT(x)| \leqq 1$. Using the given inequality we get $m\|x\| \leqq |fT(x)| \leqq 1$. Thus $\|x\| \leqq 1/m$ for all $x \in T^{-1}(f^{-1}[-1, 1])$; i.e., $T^{-1}(f^{-1}[-1, 1])$ is bounded and hence $T^{-1}: T(E) \to E$ is continuous.

LEMMA 2.4. *Let E be a locally convex space and F a subspace of E. If E^* is separable, then so is F^* (in the $s(F^*, F)$ topology).*

Proof. Let $F^1 = \{f \in E^* : f(x) = 0 \text{ for each } x \in F\}$. The continuous image of a separable topological space is separable. Thus each quotient space (with quotient topology) of a separable space is separable. Hence E^*/F^1 is separable. The canonical map $[f] \to f/F$ is an isomorphism mapping E^*/F^1 onto F^* (see [10, p. 120]). Furthermore this mapping is continuous (where E^*/F^1 has the quotient topology and F^* the $s(F^*, F)$ topology) as we now show. It suffices to show that the mapping $\phi: E^* \to F^*$ defined by $\phi(f) = f/F$ is continuous since this map is the composition of the quotient map $f \to [f]$ and the mapping $[f] \to f/F$ (see [10, Th. 5.7, p. 39]). If $A \subset F$ then denote by $A^{\circ F^*}$ the polar of A in F^* and use $A^{\circ E^*}$ for the polar of A in E^*. Let B be a bounded subset of F. Then it is not hard to see that $\phi^{-1}[B^{\circ F^*}] = B^{\circ E^*}$ and hence ϕ is continuous. Thus the canonical

mapping from E^*/F^1 onto F^* is continuous and therefore F^* is separable.

Define a non-l^1 space (non-c_0 space) to be a locally convex space which has no subspace topologically isomorphic to l^1 (c_0).

COROLLARY 2.5. *If E is a locally convex space and E^* is separable, then E is a non-l^1 space.*

Proof. If E contains a subspace F topologically isomorphic to l^1, then F^* is topologically isomorphic to m, the nonseparable space of bounded sequences. Thus by Lemma 2.4, E^* could not be separable.

The following is an immediate consequence of the corollary.

THEOREM 2.6. *Let E be a locally convex space$_2$ with a basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then E is a non-l^1 space.*

To get a converse of Theorem 2.6, the space and basis must be strengthened.

THEOREM 2.7. *Let E be a ω-complete barrelled space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. If E is a non-l^1 space, then $\{x_k, f_k\}$ is a shrinking basis.*

Proof. Suppose that $\{x_k, f_k\}$ is not a shrinking basis. Then for some $g \in E^*$ the series $\sum_{k=1}^{\infty}[\pi(x_k)](g)f_k = \sum_{k=1}^{\infty} g(x_k)f_k$ is not strongly convergent. Thus since the series is weak* convergent (Proposition 1.1) it can not be strongly Cauchy. Therefore there is a strong neighborhood of zero B° (B is bounded in E) such that for each positive integer j there are integers n_j and m_j, $n_j > m_j > j$, with the property that $\sum_{k=m_j}^{n_j} g(x_k)f_k \notin B^0$; furthermore, the sequence $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ can be chosen so that $m_{j+1} > n_j$ for each j. This means that for each j there is a $z_j \in B$ with the property that $|\sum_{k=m_j}^{n_j} g(x_k)f_k(z_j)| > 1$; or, if we let $[m_j, n_j]$ denote just the integers between m_j and n_j then $|gS(z_j)_{[m_j,n_j]}| > 1$. Since the family $\{S_{[m_j,n_j]}\}_{j=1}^{\infty}$ is pointwise bounded (Proposition 1.12) it is equicontinuous because E is barrelled. Hence since $\{z_j\}$ is bounded, the set $\{S(z_j)_{[m_j,n_j]}\}_{j=1}^{\infty}$ is bounded in E. If we let $y_j = S(z_j)_{[m_j,n_j]}$ then we have shown that for each j, $|g(y_j)| > 1$ and the sequence $\{y_j\}$ is bounded. Let $t = \{t_j\} \in l^1$ and $t \neq 0$ and chooose the scalars ε_j so that $|t_j g(y_j)| = \varepsilon_j t_j g(y_j)$. By Lemma 2.2 the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent and by the Generalized Gurevic Theorem (Corollary 1.22) the

sequence $\{y_j\}$ is unconditionally basic. Therefore the series $\sum_{j=1}^{\infty}\varepsilon_j t_j y_j$ is convergent (Theorem 1.12). Now we have $|g\sum_{j=1}^{\infty}\varepsilon_j t_j y_j| = \sum_{j=1}^{\infty}|t_j|\cdot|g(y_j)| > \sum_{j=1}^{\infty}|t_j|$. In accordance with Variation 1.20 let A be the equicontinuous subset of E^* which depends upon the given equicontinuous set $\{g\}$. Then via Variation 1.20 the inequality just shown implies that $|f_t\sum_{j=1}^{\infty}t_j y_j| > \sum_{j=1}^{\infty}|t_j|$ for some $f_t \in A$. If we define $T(t) = \sum_{j=1}^{\infty}t_j y_j$ then we have

(i) for each $t \in l^1$ there is an $f_t \in A$ such that $|f_t T(t)| \geqq \|t\|_{l^1}$.

If Q is the Minkowski functional of A_0 then $|f(x)| \leqq Q(x)$ for each $x \in E$ and all $f \in A$. Now from (i) it follows that $|QT(t)| \geqq \|t\|_{l^1}$. T is continuous by Lemma 2.2 and relatively open and one-to-one by Lemma 2.3. Thus the subspace $T(l^1)$ of E is topologically isomorphic to l^1.

From Theorem 2.7 and Corollary 2.5 we get the following.

COROLLARY 2.8. *Let E be a ω-complete barrelled space$_2$ with an unconditional Schauder basis. Then if E^* is separable, the basis is shrinking.*

THEOREM 2.9. *Let E be either a ω-complete locally convex space$_2$ or a barrelled space$_2$ (or any locally convex space$_2$ with the property that weak* bounded subsets of E^* are strongly bounded) and let E have an unconditional Schauder basis $\{x_k, f_k\}$. If E^* is weakly ω-complete (i.e., $w(E^*, E^{**})$-ω-complete) then $\{x_k, f_k\}$ is a shrinking basis.*

Proof. $\{f_k\}$ is a weak* unconditional basis so for each $f \in E^*$ the set of unordered partial sums $\{S_\sigma^*(f): \sigma \in \Phi\}$ is weak* bounded and consequently strongly bounded because of the conditions on E. E^* is weakly ω-complete so by Corollary 1.16, $\sum_{k=1}^{\infty}f(x_k)f_k$ is strongly unconditionally convergent; and since we know the series is weak* convergent to f, it must be strongly unconditionally convergent to f.

To get a converse (Theorem 2.12 or 2.13) of Theorem 2.9 we impose on E (or E^*) more restrictions which are provided by Theorem 2.11, a generalization of Day's Theorem 2 [2, p. 74].

LEMMA 2.10. *Let E be a locally convex space$_2$. If E is weakly sequentially complete, then E is a non-c_0 space.*

Proof. Suppose that E is weakly ω-complete but that there is a subspace F which is topologically isomorphic to c_0. As a complete subset of the Hausdorff space E, F is closed and, since it is convex,

weakly closed. Thus F is weakly ω-complete which is impossible since c_0 is not weakly ω-complete.

THEOREM 2.11. *Let E be a barrelled space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. Then the statements below are equivalent.*

(1) *The basis is boundedly complete.*

(2) *E is weakly ω-complete.*

(3) *E is a ω-complete non-c_0 space.*

Proof. (1) $\rightarrow$ (2). Let $\{y_n\}$ be a weakly Cauchy sequence in E. Then $\{y_n\}$ is bounded and $\{f(y_n)\}_{n=1}^{\infty}$ is convergent for every $f \in E^*$. For each $k \in \omega$ let $a_k = \lim_{n\to\infty} f_k(y_n)$. Since $S_\sigma(y_n) - \sum_{k\in\sigma} a_k x_k = \sum_{k\in\sigma}(f_k(y_n) - a_k)x_k$, $s_\sigma(y_n)$ converges with n to $\sum_{k\in\sigma} a_k x_k$ (given a neighborhood of zero V choose the circled neighborhood W such that $W + \cdots + W \subset V$ where there are as many W's as the cardinal number of σ, and then pick $\varepsilon > 0$ so that $\varepsilon x_k \in W$ for each $k \in \sigma$; for n large enough, $|f_k(y_n) - a_k| < \varepsilon$ for all $k \in \sigma$ and so for such n, $\sum_{k\in\sigma}[f_k(y_n) - a_k] \cdot x_k \in W + \cdots + W \subset V$). Thus if U is a neighborhood of zero and V a barrel such that $V + V \subset U$ we have for each $\sigma \in \Phi$, an integer N_σ such that $\sum_{k\in\sigma} a_k x_k - S_\sigma(y_{N_\sigma}) \in V$. V_b is a neighborhood of zero (see Theorem 1.18) and since $\{y_n\}$ is bounded there is a scalar $s > 1$ such that $y_n \in sV_b$ for all n. Hence $S_\sigma(y_n) \in sV$ for all $\sigma \in \Phi$ and $n \in \omega$. Then for each $\sigma \in \Phi$ we have $\sum_{k\in\sigma} a_k x_k \in S_\sigma(y_{N_\sigma}) + V \subset sV + V \subset sU$, proving that $\{\sum_{k\in\sigma} a_k x_k\}_{\sigma\in\Phi}$ is bounded. Since $\{x_k, f_k\}$ is boundedly complete there is a $y \in E$ such that $a_k = f_k(y)$ for each $k \in \omega$, and recalling the definition of a_k we have $\lim_{n\to\infty} f_k(y_n - y) = 0$ for each $k \in \omega$.

We will prove that y is the weak sequential limit of $\{y_n\}$. Suppose it is not. Then $u_n = y_n - y$ does not tend weakly to θ and so there is an $h \in E^*$, an $\varepsilon > 0$, and a subsequence $\{z_m\}$ of $\{u_n\}$ such that

(i) $h(z_m) > \varepsilon$ for each $m \in \omega$.

In accordance with Variation 1.20 let B be the equicontinuous subset of E^* corresponding to $\{h\}$. By the result of the first paragraph of proof $\lim_{m\to\infty} S_\sigma(z_m) = \theta$. Therefore for the neighborhood of zero $U = \varepsilon/5[B \cup \{h\}]_0$ there is a positive integer m_0 such that $R_{m_0}(z_1) \in U$ and then there is a $n_1 > m_0$ such that $S_{m_0}(z_{n_1}) \in U$. By induction there are increasing sequences $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ such that

(ii) $|g(R_{m_k}(z_{n_k}))| \leqq \varepsilon/5$ and $|g(S_{m_k}(z_{n_{k+1}}))| \leqq \varepsilon/5$ for all $g \in B \cup \{h\}$ and $k \in \omega$.

For each positive integer k define $w_k = \sum_{k=m_{k-1}+1}^{m_k} f_k(z_{n_k})x_k$. Since $z_{n_k} - w_k = S_{m_{k-1}}(z_{n_k}) + R_{m_k}(z_{n_k})$ we have, using (ii),

(iii) $|g(w_k) - g(z_{n_k})| \leqq 2\varepsilon/5$ for all $g \in B \cup \{h\}$ and $k \in \omega$.

Now using (i) and (iii) we have $|h(w_k)| \geqq |h(z_{n_k})| - |h(z_{n_k}) - h(w_k)| > \varepsilon - 2\varepsilon/5$. Thus

(iv) $|h(w_k)| > 3\varepsilon/5$ for each $k \in \omega$.

Let F be the subspace of l^1 of finitely nonzero sequences. If $t = \{t_k\} \in F$ then define $T(t) = \sum_{k=1}^{\infty} t_k w_k$. For $t \neq 0$ pick the scalars ε_k so that $\varepsilon_k t_k h(w_k) = |t_k h(w_k)|$; then, $|h(\sum_{k=1}^{\infty} \varepsilon_k t_k w_k)| = \sum_{k=1}^{\infty} |t_k| \cdot |h(w_k)| > 3\varepsilon/5 \sum_{k=1}^{\infty} |t_k|$. By the generalized Gurevic Theorem there must be a $g_t \in B$ such that $|g_t(\sum_{k=1}^{\infty} t_k w_k)| > 3\varepsilon/5 \cdot \sum_{k=1}^{\infty} |t_k|$.

(v) For each $t \in F$ there is a $g_t \in B$ such that $|g_t(T(t))| \geqq 3\varepsilon/5 \, ||t||$.

Define $H(t) = \sum_{k=1}^{\infty} t_k z_{n_k}$ for each $t \in F$. From (iii) we get $|g(H(t) - T(t))| \leqq 2\varepsilon/5 \, ||t||$ for all $g \in B$; hence

$$|g(H(t))| \geqq |g(T(t))| - \frac{2\varepsilon}{5} ||t||.$$

Now using (v) we have

(vi) for $t \in F$ there is a $g_t \in B$ such that $|g_t(H(t))| \geqq \varepsilon/5 \, ||t||$.

If Q is the Minkowski functional of B_0 then $|g(x)| \leqq Q(x)$ for each $x \in E$ and all $g \in B$. Thus in view of (vi) we have $|Q(H(t))| \geqq \varepsilon/5 \, ||t||$ for every $t \in F$. It now follows from Lemmas 2.2 and 2.3 that H is a topological isomorphism from F into E.

Let $\overline{H(F)}$ be the closure of $H(F)$ in the completion $\hat{E}$ of E. Then $\overline{H(F)}$ is topologically isomorphic to l^1, and hence is weakly ω-complete. Since the weakly Cauchy sequence $\{z_{n_k}\}$ is contained in $H(F)$, it is weakly convergent to an element z of $\hat{E}$. Therefore $\{u_n\}$ is weakly convergent to z and we have for each $k \in \omega$, $f_k(z) = \lim_{n\to\infty} f_k(u_n) = 0$ by the final result of the first paragraph of proof. This implies that $z = \theta$ and hence $\{u_n\}$ does converge weakly to θ, contradicting the original assumption that it does not.

(2) $\rightarrow$ (3). Lemma 2.10.

(3) $\rightarrow$ (1). If $\{x_k\}$ is not boundedly complete then there is a sequence $\{a_k\}$ such that $D = \{\sum_{k=1}^{\infty} a_k x_k\}_{n=1}^{\infty}$ is bounded but the series does not converge and hence is not Cauchy since E is ω-complete. Thus there is an equicontinuous subset A of E^* and increasing sequences $\{m_k\}$ and $\{n_k\}$ with $m_{k+1} > n_k$ such that $y_k \equiv \sum_{j=m_k}^{n_k} a_j x_j \notin A_0$ for all $k \in \omega$. Hence, for each positive integer k there is a $g_k \in A$ such that

(i) $|g_k(y_k)| > 1$.

Let F be the subspace of c_0 of finitely nonzero sequences. Define $T: F \rightarrow [y_k]$ as follows: $T(t) = \sum_{k \in \sigma} t_k y_k$ where σ is the set of subscripts on which t is not 0. We shall show that T is continuous. Since $\{x_k\}$ is unconditional and D is bounded, the set of unordered partial sums $\{\sum_{k \in \sigma} a_k x_k : \sigma \in \Phi\}$ is bounded (if V is a barrel then V_b is also a neighborhood of zero and hence there is a scalar s such that $\sum_{k=1}^{n} a_k x_k \in sV_b$ for all $n \in \omega$; it follows that $\sum_{k \in \sigma} a_k x_k \in sV$ for all

$\sigma \in \Phi$). Hence if B_0 is a neighborhood of zero (B is equicontinuous) there is a number M such that $|f\sum_{k\in\sigma} a_k x_k| < M$ for all $f \in B$ and $\sigma \in \Phi$. Let $\rho = \bigcup_{k\in\sigma}\{j: m_k \leqq j \leqq n_k\}$; then $|\sum_{k\in\sigma} f(y_k)| = |f\sum_{k\in\sigma} y_k| = |f\sum_{j\in\rho} a_j x_j| < M$. Therefore $\sum_{k\in\sigma}|f(y_k)| \leqq 4M$ for all $f \in B$ and $\sigma \in \Phi$. Now suppose $t \in F$ and $\|t\| < 1/4M$. If $f \in B$ then $|f(T(t))| = |\sum_{k\in\sigma} t_k f(y_k)| \leqq \sum_{k\in\sigma}|t_k| \cdot |f(y_k)| \leqq \|t\| \cdot \sum_{k\in\sigma}|f(y_k)| < 1$. Therefore $T(t) \in B_0$, and we have proved that T is continuous.

Let t be a nonzero member of F and pick j so that $|t_j| = \|t\|$. Now choose $\{\alpha_k\}$ so that $\alpha_k t_k g_j(y_k) = |t_k g_j(y_k)|$. Then $|g_j \sum_{k\in\sigma}\alpha_k t_k y_k| = |\sum_{k\in\sigma}\alpha_k t_k g_j(y_k)| = \sum_{k\in\sigma}|t_k| \cdot |g_j(y_k)| \geqq |t_j| \cdot |g_j(y_j)| > |t_j| = \|t\|$. If B is the equicontinuous set corresponding to A then by the Generalized Gurevic Theorem there must be an $h_t \in B$ such that

(ii) $|h_t T(t)| > \|t\|$.

If Q is the Minkowski functional of B_0 then $|h(x)| \leqq Q(x)$ for all $x \in E$ and $h \in B$. Therefore, $|Q(T(t))| \geqq \|t\|$ for each $t \in F$. By Lemma 2.3 T is one-to-one and open and so T is a topological isomorphism of F onto $[y_k]$. Thus their completions, c_0 and $\overline{[y_k]}$, are topologically isomorphic.

A converse to 2.9 can now be proved. If $\{x_k, f_k\}$ is an unconditional shrinking basis in a barrelled space E, then $\{f_k\}$ is a strongly unconditional, boundedly complete basis for E^* (2.1 and 1.5). Therefore if E^* is a barrelled space (in the strong topology) then by Theorem 2.11, E^* is weakly ω-complete. Thus we have proved the following.

THEOREM 2.12. *Let E be a barrelled space$_2$ with an adjoint E^* which is also barrelled, and suppose that E has an unconditional basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then E^* is weakly ω-complete.*

If the adjoint of a locally convex metrizable space is separable (in the strong topology) then E^* is a bound space and hence barrelled [10, p. 217, 22.17]. Thus as a corollary to 2.12 we have the following.

THEOREM 2.13. *Let E be a barrelled metrizable space with an unconditional basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then E^* is weakly ω-complete.*

We include a summary of the results on shrinking bases.

COROLLARY 2.14. *Let E be a locally convex space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. Consider the following six conditions.*

(1) *$\{f_k, \pi(x_k)\}$ is a Schauder basis for the dual space E^*.*

(2) *$\{f_k, \pi(x_k)\}$ is a boundedly complete Schauder basis for E^*.*

(3) $\{f_k, \pi(x_k)\}$ *is an unconditional Schauder basis for* E^*.

(4) E *is a non-*l^1 *space.*

(5) E^* *is separable.*

(6) E^* *is weakly* ω*-complete.*

Then (5) *implies* (4) (*even if* E *has no basis*) *and so each of* (4) *and* (5) *is a necessary condition for* (1). (1), (2), *and* (3) *are equivalent if* E *is barrelled* ((1) *and* (2) *are equivalent even if the basis* $\{x_k\}$ *is conditional*). *Each of* (4) *and* (5) *is sufficient for* (1) *if* E *is a* ω*-complete barrelled space.* (6) *implies* (1) *if* E *is a locally convex space such that weak* bounded sets are strongly bounded*; *and* (1) *implies* (6) *if* E *is both barrelled and metrizable. Thus all six statements are eqvivalent if* E *is a Fréchet space.*

Recall (Proposition 1.6) that a barrelled space$_2$ with a Schauder basis is reflexive if and only if the basis is both boundedly complete and shrinking. Also recall that a reflexive space is barrelled and its adjoint is reflexive and barrelled. Thus in view of 2.11 and 2.14 we have the following.

THEOREM 2.15. *Let* E *be a* ω*-complete barrelled space*$_2$ *with an unconditional Schauder basis. Consider the following conditions on* E *and* E^*.

(1) E *is weakly* ω*-complete.*

(2) E *is a non-*c_0 *space.*

(3) E *is a non-*l^1 *space.*

(4) E^* *is weakly* ω*-complete.*

(5) E^* *is a non-*c_0 *space.*

(6) E^* *is a non-*l^1 *space.*

If E *is reflexive then* (1), (2), *and* (3) *hold. If in addition* E^* *is* ω*-complete* (*as it will be if* E *is a bound space*) *then all six statements hold. Then conjunction of* (1) *and* (3) *imply that* E *is reflexive as does also the conjunction of* (2) *and* (3).

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