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# FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS II

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**In a previous paper one of the authors considered groups $G$ with r. b. $n$ (representation bound $n$) and $n < p^2$ for some prime $p$. Here we continue this study. We first offer a new proof of the fact that if $n = p - 1$ then $G$ has a normal Sylow $p$-subgroup. Next we show that if $n = p^{3/2}$ then $p^2 \nmid |G/O_p(G)|$. Finally we consider $n = 2p - 1$ and with the help of the modular theory we obtain a fairly precise description of the structure of $G$. In particular we show that its composition factors are either p-solvable or isomorphic to $PSL(2, p)$, $PSL(2, p - 1)$ for $p$ a Fermat prime or $PSL(2, p + 1)$ for $p$ a Mersenne prime.**

Now the irreducible characters of $PSL(2, \mathrm{p})$ have degrees (see [10] p. 128) $1, p, p \pm 1$, $(p \pm 1)/2$ for $p$ odd and those of $PSL(2, 2^a)$ have degrees (see [10] p. 134) $1, 2^a, 2^a \pm 1$. Thus for $p > 2$ the linear groups of the preceding paragraph do in fact have r.b. $(2p - 1)$.

The notation here is standard. In addition, if $\chi$ is a character of $G$ we let det $\chi$ denote the linear character which is the determinant of the representation associated with $\chi$. Also $n_p(G)$ denotes the number of Sylow $p$-subgroups of $G$.

LEMMA 1. *Let $G$ be a group with* r.b.$n$. *and let $N \neq G$* be a *subgroup. Suppose $G = \bigcup_{i=0}^{t} Nx_iN$ is the $(N, N)$-double coset decomposition of $G$ with $x_0 = 1$. Set $a_i = |Nx_iN|/|N| = [N: N \cap N^{x_i}]$. Then $n \geqq (a_1 + a_2 + \cdots + a_t)/t$.*

*Proof.* Let $\theta = (1_N)^G$ be the character of the permutation representation of $G$ on the cosets of $N$. Then $\theta(1) = [G: N]$, $[\theta, 1_G] = 1$ and $\|\theta\|^2 = 1 + t$. Since $[\theta, 1_G] = 1$ we can write

$$\theta = 1_G + b_1\chi_1 + \cdots + b_s\chi_s$$

where the $\chi_i$ are distinct nonprincipal irreducible characters of $G$. Thus since $G$ has r.b.$n$ we have

$$\begin{aligned} 1 + nt &= 1 + n(\|\theta\|^2 - 1) = 1 + n(b_1^2 + \cdots + b_s^2) \\ &\geqq 1 + n(b_1 + \cdots + b_s) \geqq 1 + b_1\chi_1(1) + \cdots + b_s\chi_s(1) \\ &= \theta(1) = 1 + (a_1 + a_2 + \cdots + a_t) \end{aligned}$$

and the result follows.

LEMMA 2. *Let $G$ be a group with* r.b.$n$.
( i ) Let $N \neq G$ *be a subgroup. Then*

$$n \geqq \min \{[N: N \cap N^x] \mid x \in G - N\} .$$

(ii) *Let $\pi$ be a set of primes and let $H$ be a maximal $\pi$-subgroup of $G$. Then either $H \triangle G$ or* $n \geqq \min \{[H: H \cap H^x] \mid x \in G - N(H)\}$.

*Proof.* (i) follows immediately from Lemma 1. Now let $H$ be as in (ii) and suppose $H$ is not normal in $G$. Set $N = N(H) \neq G$. Since $H$ is a maximal $\pi$-subgroup it follows that $H = O_\pi(N)$. Thus if $x \in G$ then $H^x = O_\pi(N^x)$ so $H \cap N^x = H \cap H^x$ and

$$[N: N \cap N^x] \geqq [H: H \cap N^x] = [H: H \cap H^x] .$$

Thus the result follows from (i).

Applying Lemma 2(ii) with $\pi = \{p\}$ and $H$ a Sylow $p$-subgroup of $G$ yields

THEOREM 3. *Let $p$ be a prime and let $G$ be a group with* r.b.$(p-1)$. *Then* $n_p(G) = 1$.

This result was originally proved in [7] (Theorem E) in a much more complicated way.

LEMMA 4. *Let $G$ have* r.b.$(p^2-1)$ *and let $Q_1$ and $Q_2$ be $p$-subgroups of $G$ with $\langle Q_1, Q_2\rangle$ not a $p$-group. Then $n_p(C(Q_1) \cap C(Q_2)) = 1$. If further the Sylow $p$-subgroups of $G$ are abelian, then*

$$n_p(N(Q_1) \cap N(Q_2)) = 1$$

*Proof.* Set $W = \langle Q_1, Q_2\rangle$. Since $W$ is not a $p$-group we see that $n_p(W) > 1$. We assume now that $n_p(C) > 1$ where $C = C(Q_1) \cap C(Q_2) = C(W)$ and derive a contradiction. Set $Z = W \cap C$ so that $Z$ is central in $W$ and $C$ and let $\bar{W} = W/Z, \bar{C} = C/Z$. Since $Z$ is central we have easily $n_p(\bar{W}) > 1$, $n_p(\bar{C}) > 1$ and $(WC)/Z = \bar{W} \times \bar{C}$. By Theorem 3 both $\bar{W}$ and $\bar{C}$ have irreducible characters of degree $\geqq p$ and hence $\bar{W} \times \bar{C}$ has an irreducible character of degree $\geqq p^2$. This is a contradiction since $G$ has r.b.$(p^2-1)$ and this property is inherited by subgroups and quotient groups. If the Sylow $p$-subgroups of $G$ are abelian then any $p$-group normalizing $Q_i$ centralizes it. Thus the second result follows from the first.

THEOREM 5. *Let $p$ be a prime and let $G$ be a group with* r.b.$p^{3/2}$. *Then* $p^2 \nmid |G/O_p(G)|$.

*Proof.* If $p = 2$ then $G$ has r.b.2 and the result follows from Theorem C of [7]. Thus we can assume that $p \geqq 3$ and clearly also that $O_p(G) = \langle 1 \rangle$. Since $p^2 - p - 1 \geqq [p^{3/2}]$ for $p \geqq 3$, Proposition 1.3 of [6] implies that a Sylow $p$-subgroup $P$ of $G$ is abelian. We assume that $|P| \geqq p^2$ and derive a contradiction. Set $n = p^{3/2}$.

Let $N = N(P)$ so that $N \neq G$. By Lemma 2(i) there exists $w \in G - N$ with $n \geqq [N: N \cap N^w] \geqq [P: P \cap P^w]$. Set $Q = P \cap P^w$ so since $p^2 > n$ and $w \notin N$ we see that $[P: Q] = p$ and hence $Q \neq \langle 1 \rangle$. Let $M = N(Q)$. Since $P \triangle N$, $P^w \triangle N^w$ we have $Q \triangle (N \cap N^w)$. Also $Q \triangle P$ and $P \nsubseteq N^w$. Hence $M \cap N \supseteqq \langle P, N \cap N^w \rangle$ so $[N: N \cap M] \leqq n/p = p^{1/2}$.

We now make the following crucial observation. If $[M: M \cap M^x] < p^2$ for some $x \notin M$ then $Q$ and $Q^x$ commute elementwise and $x \in MNM$. To see this suppose that $Q$ and $Q^x$ do not commute. Then since the Sylow $p$-subgroups of $G$ are abelian, $\langle Q, Q^x \rangle$ is not a $p$-group. By Lemma 4, $n_p(M \cap M^x) = 1$ so if $U = O_p(M \cap M^x)$ then $U$ is also a Sylow $p$-subgroup of $M \cap M^x$. Now $p^2 \nmid [M: M \cap M^x]$ and $Q \nsubseteq M^x$ clearly so $QU$ is a Sylow $p$-subgroup of $M$. Since $N_M(QU) \supseteqq \langle Q, M \cap M^x \rangle$ we have $[M: N_M(QU)] < p$ and hence by Sylow's theorem $QU \triangle M$ and $n_p(M) = 1$. This is a contradiction since $Q = P \cap P^w$ and $P, P^w \subseteqq M$. Thus $Q$ and $Q^x$ commute and since $Q \neq Q^x$ and $[P: Q] = p$ it follows that $QQ^x = P^{y^{-1}}$ is a Sylow $p$-subgroup of $G$. Thus $Q$, $Q^y$ and $Q^{xy}$ are all contained in $P$. By Burnside's lemma these three groups are conjugate in $N$. Thus $Q^y = Q^h$, $Q^{xy} = Q^k$ for some $h, k \in N$. This yields $yh^{-1} \in M$, $xyk^{-1} \in M$ so

$$x = (xyk^{-1})kh^{-1}(yh^{-1})^{-1} \in MNM .$$

Since $Q \neq \langle 1 \rangle$ we have $M \neq G$. Let $G = \bigcup_{i=0}^{t} Mx_iM$ be the $(M, M)$-double coset decomposition of $G$ with $x_0 = 1$. Set $a_i = |Mx_iM|/|M|$ and suppose that there are precisely $r$ such $i \neq 0$ with $a_i < p^2$ and $s$ with $a_i \geqq p^2$. Then by Lemma 1, $p^{3/2} = n \geqq (r + p^2 s)/(r + s)$. Clearly $r \neq 0$ here so

$$p^{3/2} \geqq (r + p^2 s)/(r + s) > p^2/(1 + r/s) .$$

If $s = 0$ then by the preceding paragraph $Q$ commutes with all its conjugates. This implies that $\langle Q^x \mid x \in G \rangle$ is a nontrivial normal $p$-subgroup of $G$, a contradiction. Thus $s \geqq 1$. Also if $a_i < p^2$ then $Mx_iM \subseteqq MNM$ by the above. Since we have seen that $[N: N \cap M] \leqq p^{1/2}$ we have $r \leqq p^{1/2} - 1$ since the double coset $M$ itself is not counted. Thus $r/s \leqq p^{1/2} - 1$ and

$$p^{3/2} > p^2/(1 + r/s) \geqq p^{3/2}$$

a contradiction and the result follows.

We now turn to the main result of this paper.

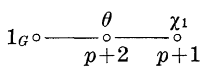
THEOREM 6. *Let $p$ be a prime and let $G$ be a nonabelian simple group with* r.b.$(2p-1)$. *Then $p > 2$ and we have one of the following.*

(i) *$G$ is a $p'$-group.*
(ii) $G \cong PSL(2, p)$ for $p > 3$.
(iii) *$G \cong PSL(2, p-1)$ for $p$ a Fermat prime, $p > 3$.*
(iv) *$G \cong PSL(2, p+1)$ for $p$ a Mersenne prime.*

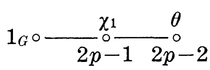
*Proof.* Since groups with r.b.3 are solvable (Corollary 6.5 of [8]) we have $p > 2$. By Theorem 5 since $p^{3/2} \geqq 2p-1$ for $p > 2$ we have $p^2 \nmid |G|$. If $p \nmid |G|$ then $G$ satisfies (i) above. Thus we can assume now that $G$ has a Sylow $p$-subgroup $P$ of order $p$. Let $B_1(p)$ denote the principal $p$-block of $G$. We will use freely the structure of $B_1(p)$ and its associated tree as described in [1] and [2]. Since $B_1(p)$ contains a nonprincipal irreducible character $\chi$ and $\chi(1) < 2p$, Lemma 1 of [3] implies that $P$ is self centralizing. Let $N = N(P)$ and let $e = |N/P|$ so that $e \mid (p-1)$. By Burnside's transfer theorem $e > 1$. We assume now that $G \not\cong PSL(2, p)$ and $G \not\cong PSL(2, p-1)$ for $p$ a Fermat prime.

*Step* 1. Let $\theta$ denote an exceptional character in $B_1(p)$. Then the tree of $B_1(p)$ must be one of the following.

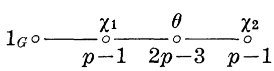
(1) $e = 2$



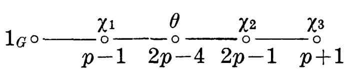
(2) $e = 2$



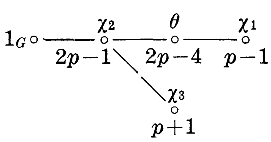
(3) $e = 3$



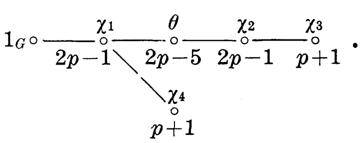
(4) $e = 4$



(5) $e = 4$



(6) $e = 5$



Here the degree of the character is written below the character designation.

Since $G$ is nonabelian and simple it follows that every nonprincipal irreducible representation of $G$ is faithful. Thus by Theorem 2 of [5] the degrees of the nonprincipal ordinary irreducible characters of $G$ are all $\geqq p-1$ and by Theorem 1 of [4] the degrees of the nonprincipal irreducible Brauer characters of $G$ are $\geqq 2(p-1)/3$. Now $G$ has r.b.$(2p-1)$ and the degrees of the ordinary irreducible characters of $B_1(p)$ satisfy $\chi(1) \equiv \pm 1(p)$ if $\chi$ is nonexceptional and $\theta(1) \equiv \pm e(p)$. This yields easily $\chi(1) = p-1, p+1, 2p-1$ or $\chi = 1_G$ and $\theta(1) = 2p - e, p + e$ or $\theta(1) = e$ if $e = p - 1$.

Suppose first that $p = 3$. Since $e \mid (p-1)$ we have $e = 2$ and the tree is a line with three vertices. Now the center degree is maximal and the principal character must occur so we have clearly

$$1_G \circ\!\text{——}\!\overset{\chi_1}{\underset{5}{\circ}}\!\text{——}\!\overset{\chi_2}{\underset{4}{\circ}}\,.$$

This is either tree (1) or tree (2) according to which of $\chi_1$ of $\chi_2$ we consider exceptional.

Now assume that $p \geqq 5$. Let $\chi \in B_1(p)$ with $\chi(1) = p - 1$ and suppose that $\chi$ is modular reducible. We can denote this latter fact graphically by

$$\overset{\chi_1}{\circ}\!\underset{r}{\text{——}}\!\overset{\chi}{\circ}\!\underset{s}{\text{——}}\!\overset{\chi_2}{\circ}$$

where $r$ and $s$ are degrees of modular constituents. Since $r + s \leqq p - 1$ we cannot have both $r, s \geqq 2(p-1)/3$. This implies that one of the modular constituents is the principal character and the other has degree $p - 2$.

Now let $\chi \in B_1(p)$ with $\chi(1) = p + 1$ and suppose $\chi$ is modular reducible. Say we have

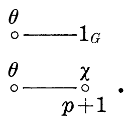
$$\overset{\chi_1}{\circ}\!\underset{r}{\text{——}}\!\overset{\chi}{\circ}\!\underset{s}{\text{——}}\!\overset{\chi_2}{\circ}\,.$$

Here by the alternating nature of the tree $\chi_1, \chi_2 \neq 1_G$ so $r, s \geqq 2(p-1)/3$. Since $r + s \leqq p + 1$ this yields $4(p-1)/3 \leqq p + 1$ so $p = 5$ or $7$. In fact either $p = 5, r = s = 3$ or $p = 7, r = s = 4$. We consider these in turn. Let $p = 5$. Since $e \mid (p-1)$ we have $e = 2$ or $4$. Certainly $e \neq 2$ here since $B_1(p)$ contains $e + 1$ ordinary irreducible characters including $1_G$ and hence $e = 4$. Thus the degrees of the nonprincipal ordinary irreducible characters of $B_1(p)$ are $4, 6$ or $9$ and the nonprincipal Brauer characters have degrees $\geqq 3$. The former fact implies that $\chi_1$ and $\chi_2$ are modular reducible so the tree having five vertices is a straight line. This yields easily

$$1_G \circ\!\text{——}\!\overset{\chi_1}{\underset{4}{\circ}}\!\text{——}\!\overset{\chi}{\underset{6}{\circ}}\!\text{——}\!\overset{\chi_2}{\underset{9}{\circ}}\!\text{——}\!\overset{\chi_3}{\underset{6}{\circ}}$$

and we obtain tree (4). Now let $p = 7$ so that $e = 2, 3$ or $6$. Clearly neither $\chi_1$ nor $\chi_2$ can be modular irreducible so the tree has at least five vertices and hence $e = 6$. Thus the degrees of the nonprincipal ordinary irreducible characters of $B_1(p)$ are 6, 8 or 13 and the non-principal Brauer characters have degrees $\geqq 4$. This latter fact implies that neither $\chi_1$ nor $\chi_2$ can have degree 6 and hence $\chi_1(1) = \chi_2(1) = 13$. Now at least one of $\chi_1$ or $\chi_2$, say $\chi_1$, is not adjacent to $1_G$. Hence all characters adjacent to $\chi_1$ have degree $p + 1$ and as we have seen these are either modular irreducible or have constituents of degree 4. This shows that all modular constituents of $\chi_1$ have degree divisible by 4, a contradiction since $\chi_1(1) = 13$. Thus this case does not occur. It now clearly suffices to assume for the remainder of this step that all $\chi \in B_1(p)$ with $\chi(1) = p + 1$ are modular irreducible.

Let $\theta$ denote an exceptional character of $B_1(p)$ and we consider the possible branches leaving the vertex associated with $\theta$. Suppose first that $\theta(1) \equiv +e(p)$. The above implies easily that we can only have

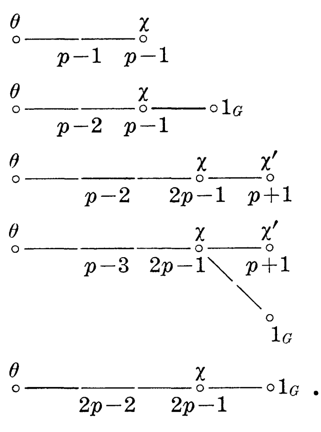



Now the first branch must occur precisely once and let the second branch occur $a$ times. Since the tree has $e$ edges we have

$$1 + a = e$$
$$1 + a(p + 1) = \theta(1)\ .$$

Now $e \geqq 2$ so $a \geqq 1$ and hence $\theta(1) > p$. Thus $\theta(1) = p + e$ and we obtain $a = 1, e = 2$ and this is tree (1).

Now let $\theta(1) \equiv -e(p)$ so that $\theta(1) = 2p - e$. Using the above information and the alternating nature of the tree we see easily that the only possible branches leaving the vertex associated with $\theta$ are




If the last branch occurs then since $2p - e = \theta(1) \geqq 2p - 2$ we have $e = 2$ and this is tree (2). Thus we can assume that only the first four branches occur say with multiplicities $a, b, c, d$ respectively. Since there are precisely $e$ edges in the tree we have

$$a + 2b + 2c + 3d = e$$
$$a(p-1) + b(p-2) + c(p-2) + d(p-3) = \theta(1) = 2p - e\ .$$

Adding these two and dividing by $p$ yields

$$a + b + c + d = 2\ .$$

In addition the vertex of $1_G$ occurs precisely once so $b + d = 1$. Thus $a + c = 1$ and there are four possibilities which are easily seen to be trees (3), (4), (5) and (6).

*Step* 2. Let $N = N(P)$. We consider the restriction of the ordinary irreducible characters of $G$ to $N$.

Now $N = PE$ is a Frobenius group of order $pe$ with $E = \langle x \rangle$ cyclic of order $e$. $N$ has precisely $e$ linear characters, namely those of $N/P \cong E$, and the remaining irreducible characters have degree $e$. Let $\Delta$ denote any sum of irreducible characters of $N$ of degree $e$. Clearly $\Delta_E = \Delta(1)/e \cdot \rho_E$ where $\rho_E$ is the regular character of $E$. This yields easily

$$\Delta(x) = 0$$
$$(1) \qquad \det \Delta(x) = \begin{cases} 1 & \text{if } e \text{ is odd} \\ (-1)^{\Delta(1)/e} & \text{if } e \text{ is even}\ . \end{cases}$$

If $e$ is even let $\delta$ denote the linear character of $N$ given by $\delta(x) = -1$.

Let $\psi$ be an ordinary irreducible character of $G$ with $\psi \notin B_1(p)$. Since $P$ is self centralizing it follows that $B_1(p)$ is the unique $p$-block of positive defect and hence $\psi$ belongs to a block of defect 0. Thus $p \mid \psi(1)$. Since $G$ has r.b.$(2p-1)$ this yields $\psi(1) = p$ and clearly $\psi_P = \rho_P$. Thus $\psi_N = \lambda + \Delta$ where $\lambda$ is linear and $\Delta(1) = p - 1$. Now $G$ is simple so the linear character $\det \psi$ must be principal and hence $1 = \det \psi(x) = \lambda(x) \det \Delta(x)$. This yields by the above

$$(2) \qquad \psi_N = \begin{cases} 1_N + \Delta & \text{if } e \text{ is odd} \\ 1_N + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is even} \\ \delta + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is odd}\ . \end{cases}$$

Now let $\chi \in B_1(p)$ and let $m(\chi)$ denote the number of linear characters counting multiplicities which occur in $\chi_N$. Obviously $m(\chi)$ is the multiplicity of $1_P$ in $\chi_P$. Suppose first that $\chi_i, \chi_j$ are nonexceptional ordinary irreducible characters of $B_1(p)$ which are adjacent in the tree. Then $\chi_i + \chi_j = \Phi$ is a projective character and since $\Phi_P = \Phi(1)/p \cdot \rho_P$ we have easily

$$m(\chi_i) + m(\chi_j) = [\chi_i(1) + \chi_j(1)]/p\ .$$

Now suppose that $\chi_i$ is adjacent to the vertex of the exceptional characters $\{\theta_k\}$. Clearly $m(\theta_r) = m(\theta_s)$ for all $r, s$ so since $\chi_i + \sum_{k=1}^{(p-1)/e} \theta_k = \Phi$ we have with $\theta = \theta_1$

$$m(\chi_i) + (p-1)/e \cdot m(\theta) = [\chi_i(1) + (p-1)/e \cdot \theta(1)]/p.$$

Using the above two equations, the fact that the tree of $B_1(p)$ is connected and the obvious fact that $m(1_G) = 1$ we obtain easily for irreducible $\chi$

$$m(\chi) = \begin{cases} k-1 & \text{if } \chi(1) = kp - 1 \\ k & \text{if } \chi(1) = kp \pm e \\ k+1 & \text{if } \chi(1) = kp + 1\,. \end{cases} \tag{3}$$

There is of course additional information available, for example the fact that $\det \chi(x) = 1$ and the position of $\chi$ in the tree, which further limits the structure of $\chi_N$.

*Step* 3. If tree (1) occurs in step 1, then $p$ is a Mersenne prime and $G \cong PSL(2, p+1)$.

By assumption $e = 2$ and the tree of $B_1(p)$ has the form

$$1_G \circ\!\overset{1_G}{\text{————}}\!\underset{p+2}{\overset{\theta_i}{\circ}}\!\overset{\zeta}{\text{————}}\!\underset{p+1}{\overset{\chi}{\circ}}$$

where $\theta_i$ represents the $(p-1)/2$ exceptional characters with $\theta = \theta_1$. Let $\{\psi_j \mid j = 1, 2, \cdots, k\}$ denote the set of irreducible characters of $G$ not in $B_1(p)$.

Let $\alpha$ denote the unique nonprincipal linear character of $N$. By (1) and (2) we have

$$\begin{aligned} &\psi_{jN} = 1_N + \Delta'_j\,, \quad \psi_j(x) = 1 \quad \text{for } p \equiv 1 \ (4) \\ &\psi_{jN} = \alpha + \Delta'_j\,, \quad \psi_j(x) = -1 \quad \text{for } p \equiv 3 \ (4)\,. \end{aligned} \tag{4}$$

By (3), $\theta_{iN} = \lambda + \Delta_i$ and since the $\theta_i$ are all algebraically conjugate $\lambda$ is the same for all $i$. Thus $\det \theta_i(x) = 1$ and equation (1) yield

$$\begin{aligned} &\theta_{iN} = \alpha + \Delta_i\,, \quad \theta_i(x) = -1 \quad \text{for } p \equiv 1 \ (4) \\ &\theta_{iN} = 1_N + \Delta_i\,, \quad \theta_i(x) = 1 \quad \text{for } p \equiv 3 \ (4)\,. \end{aligned} \tag{5}$$

Now $\chi_N = a1_N + b\alpha + \Delta$ with $a + b = 2$ by (3) and since $x$ is a $p'$-element $a - b = \chi(x) = \zeta(x) = \theta(x) - 1$. Thus (5) yields

$$\begin{aligned} &\chi_N = 2\alpha + \Delta\,, \quad \chi(x) = -2 \quad \text{for } p \equiv 1 \ (4) \\ &\chi_N = 1_N + \alpha + \Delta\,, \quad \chi(x) = 0 \quad \text{for } p = 3 \ (4)\,. \end{aligned} \tag{6}$$

Equations (4), (5) and (6) and Frobenius reciprocity now yield

$$(7)\qquad \begin{aligned} &(1_N)^G = 1_G + \sum_1^k \psi_j\,, && \alpha^G = 2\chi + \sum_1^{(p-1)/2} \theta_i \quad \text{for } p \equiv 1 \ (4) \\ &(1_N)^G = 1_G + \chi + \sum_1^{(p-1)/2} \theta_i\,, && \alpha^G = \chi + \sum_1^k \psi_j \quad \text{for } p \equiv 3 \ (4)\,. \end{aligned}$$

Thus since $(1_N)^G(1) = \alpha^G(1) = [G\colon N]$ and $|N| = 2p$ we obtain easily

$$(8)\qquad \begin{aligned} &|G| = p(p^2 + 5p + 2)\,, \quad k = (p+5)/2 \quad \text{for } p \equiv 1 \ (4) \\ &|G| = p(p+1)(p+2)\,, \quad k = (p+1)/2 \quad \text{for } p \equiv 3 \ (4)\,. \end{aligned}$$

Now using $|C(x)| = \sum \eta(x)\overline{\eta(x)}$, where $\eta$ runs over all ordinary irreducible characters of $G$, along with equations (4), (5), (6) and (8) we have

$$(9)\qquad \begin{aligned} &|C(x)| = p + 7 \quad \text{for } p \equiv 1 \ (4) \\ &|C(x)| = p + 1 \quad \text{for } p \equiv 3 \ (4)\,. \end{aligned}$$

Since $(p+7) \nmid p(p^2 + 5p + 2)$ the case $p \equiv 1 \ (4)$ is eliminated. Thus $p \equiv 3 \ (4)$.

Set $S = C(x)$ so that $|S| = p + 1$ and $[G\colon S] = p(p+2)$. We consider $(1_S)^G$. Since this character is rational and the $\theta_i$ are algebraically conjugate we have

$$(1_S)^G = 1_G + a \sum_1^{(p-1)/2} \theta_i + b\chi + \sum_1^k c_j\psi_j\,.$$

Set $c = \sum_1^k c_j$. By considering degrees we have

$$p(p+2) = 1 + a(p+2)(p-1)/2 + b(p+1) + cp$$

and evaluating at $x$ yields

$$0 < (1_S)^G(x) = 1 + a(p-1)/2 - c$$

by (4), (5), (6) and the fact that $x \in S$. Certainly $a \leqq 2$. Also $b \leqq \chi(1)/2 = (p+1)/2$ by Frobenius reciprocity and the fact that $\chi(x) = 0$. Thus $a = 0$ yields a contradiction. If $a = 1$ then $b \equiv 0 \ (p)$ so $b = 0$ and $c = (p+3)/2 > 1 + a(p-1)/2$ again a contradiction. Thus $a = 2$ and we have easily

$$(10)\qquad (1_S)^G = 1_G + 2 \sum_1^{(p-1)/2} \theta_i + \chi$$

so $(1_S)^G(x) = p$ by (5) and (6). By definition of induced character and the fact that $S = C(x)$ this implies that $S$ contains precisely $p$ distinct conjugates of $x$. Since $|S| = p + 1$ this shows that $S$ is an elementary abelian 2-group and therefore that $S$ is a Sylow 2-subgroup of $G$ and $p$ is a Mersenne prime. By Burnside's lemma the nonidentity elements of $S$ are all conjugate in $N(S)$ so $N(S) > S$.

Set $H = N(S) > S$. Then $(1_H)^G$ is a national constituent of $(1_S)^G$ and $(1_H)^G(1) \leqq p(p+2)/3$. Thus by (10) we have easily

$$(1_H)^G = 1_G + \chi\ . \tag{11}$$

Therefore $G$ is a doubly transitive permutation group on the set $\Omega$ where $H = G_\infty$ for some point $\infty \in \Omega$. By (10) $\chi_S$ contains $1_S$ with multiplicity one so (11) implies that $S$ has two orbits on $\Omega$. Hence since $|S| = |\Omega| - 1$, $S$ is in fact a regular normal subgroup of $G_\infty$. Now $|H| = p(p+1)$ so if $\tilde{P}$ is a Sylow $p$-subgroup of $H$, then since $\tilde{P}$ is self centralizing and $|\tilde{P}| = |S| - 1$ we see that $G$ is sharply 3-transitive.

With the structure of $H$ as given above we can clearly identify $\Omega$ with $GF(p+1) \cup \{\infty\}$ in such a way that $S$ is the set of translations $\left\{\begin{pmatrix} z \\ z+r \end{pmatrix} \middle| r \in GF(p+1)\right\}$ and $\tilde{P}$ is the set $\left\{\begin{pmatrix} z \\ sz \end{pmatrix} \middle| s \in GF(p+1),\ s \neq 0\right\}$. Let $\tilde{x} \in G$ with $\tilde{x} = (0\ \infty)(1)\cdots$. Then $\tilde{x}$ has order 2 and normalizes $\tilde{P} = G_{\infty 0}$ so $\tilde{x}$ acts in a dihedral manner on $\tilde{P}$. If $\tilde{x} = \begin{pmatrix} z \\ f(z) \end{pmatrix}$ then for all $s \in GF(p+1)$, $s \neq 0$

$$\begin{pmatrix} z \\ sz \end{pmatrix}\begin{pmatrix} z \\ f(z) \end{pmatrix} = \begin{pmatrix} z \\ f(z) \end{pmatrix}\begin{pmatrix} z \\ s^{-1}z \end{pmatrix}$$

so $f(sz) = s^{-1}f(z)$. Setting $z = 1$ yields $f(s) = s^{-1}$. Thus $\tilde{x} = \begin{pmatrix} z \\ 1/z \end{pmatrix}$ and since $G = \langle H, \tilde{x}\rangle$ we have clearly $G \subsetneqq PSL(2, p+1)$. By (8) we have in fact $G \cong PSL(2, p+1)$ and this step follows.

*Step* 4. Completion of the proof.

We now consider the remaining trees in turn. Let $\{\psi_j \mid j = 1, 2, \cdots, k\}$ denote the set of ordinary irreducible characters of $G$ not in $B_1(p)$.

Suppose first that we have tree (2). If $p = 3$ this is the same as tree (1) so we assume that $p > 3$. From

$$\begin{array}{ccccc} & 1_G & \chi & \zeta & \theta_i \\ 1_G\circ & \text{———} & \circ & \text{———} & \circ \\ & & 2p-1 & & 2p-2 \end{array}$$

and (3) and det $\chi(x) = 1$ we have $\chi_N = 1_N + \Delta$. Let $\alpha$ be the unique nonprincipal linear character of $N$ so that we have by (3) $\theta_{iN} = a1_N + b\alpha + \Delta_i$ with $a + b = 2$. Since $x$ is a $p'$-element $a - b = \theta_i(x) = \zeta(x) = \chi(x) - 1 = 0$ so $\theta_{iN} = 1_N + \alpha + \Delta_i$. Now by (2) all the $\psi_j$ occur in either $(1_N)^G$ or $\alpha^G$ depending on the parity of $(p-1)/2$. Since $(1_N)^G(1) = \alpha^G(1)$, the above and Frobenius reciprocity imply that the $\psi_j$ occur in $\alpha^G$ and hence

$$(1_N)^G = 1_G + \chi + \sum_{1}^{(p-1)/2} \theta_i\ .$$

Since $|N| = 2p$ this yields $|G| = 2p(p^2 + 1)$. Now $\theta_i(1) \mid |G|$ so $(p-1) \mid (p^2+1)$ and this is easily seen to be a contradiction for $p > 3$.

Now consider tree (3)

$$1_G \circ \overset{1_G}{\text{———}} \underset{p-1}{\overset{\chi_1}{\circ}} \overset{\zeta_1}{\text{———}} \underset{2p-3}{\overset{\theta_2}{\circ}} \overset{\zeta_2}{\text{———}} \underset{p-1}{\overset{\chi_2}{\circ}} .$$

By (3), $\chi_{iN} = \Delta_i$ and $\chi_i(x) = 0$ for $i = 1, 2$. This implies that $\zeta_1(x) = -1$, $\zeta_2(x) = 0$ so $\theta_i(x) = -1$. Now by (3), $\theta_{iN} = a1_N + b\alpha + c\alpha^2 + \Delta'_i$ where $\alpha$ is a nonprincipal linear character and $a + b + c = 2$. Since $\theta_i(x) = -1$ we have easily $\theta_{iN} = \alpha + \alpha^2 + \Delta'_i$. Applying Frobenius reciprocity to the above and (2) we have

$$(1_N)^G = 1_G + \sum_1^k \psi_j , \qquad \alpha^G = \sum_1^{(p-1)/3} \theta_i$$

and this yields easily

$$|G| = p(p-1)(2p-3) , \qquad k = (2p-5)/3 .$$

Using $|C(x)| = \sum \bar{\eta}(x)\eta(x)$ along with the above and (2) we obtain $|C(x)| = p - 1$. Now clearly $x$ is a real element so $|C^*(x)| = 2(p-1)$ where $C^*(x) = \{g \in G \mid x^g = x \text{ or } x^{-1}\}$. Since $2(p-1)$ does not divide $|G|$ as given above, it follows that this tree does not occur.

Suppose tree (4) or (5) occurred. Since $\chi_1(1) = p - 1$ and $\det \chi_1(x) = 1$, (1) and (3) imply that $(p-1)/4$ is even. Hence by (2), $\psi_{jN} = 1_N + \Delta_j$. Now there are four linear characters of $N$ and at most two occur in $\theta_N$ so choose $\alpha \neq 1_N$ such that $\alpha$ does not occur in $\theta_N$. Thus $\alpha$ can occur only in $\chi_{2N}$ or $\chi_{3N}$ with multiplicity at most two. Hence

$$[G\colon N] = \alpha^G(1) \leqq 2\chi_2(1) + 2\chi_3(1) = 6p .$$

Now choose $\beta$ so that $\beta$ occurs in $\theta_N$. Then

$$[G\colon N] = \beta^G(1) \geqq \sum_1^{(p-1)/4} \theta_i(1) = (2p-4)(p-1)/4 .$$

Since $(p-1)/4$ is even and $(p-1)/4 \neq 2$ we have $(p-1)/4 \geqq 4$, $p \geqq 17$ and

$$6p \geqq \alpha^G(1) = \beta^G(1) \geqq 4(2p-4) ,$$

a contradiction.

Finally consider tree (6). By (1), (2) and (3) we have easily $\chi_{1N} = 1_N + \Delta$, $\chi_{2N} = 1_N \Delta'$, and $\psi_{jN} = 1_N + \Delta_j$. Now since $e = 5$ and $m(\theta) = 2$ we can choose a linear character $\alpha$ of $N$ with $\alpha \neq 1_N$ and such that $\alpha$ does not occur in $\theta_N$. Hence by the above and the fact that $m(\chi_3) = m(\chi_4) = 2$ we have $\alpha^G = a\chi_3 + b\chi_4$ with $a, b \leqq 2$. Thus since

$$[G : N] = \alpha^G(1) = a\chi_3(1) + b\chi_4(1) = (a+b)(p+1)$$

and $[G:N] \equiv 1$ $(p)$ we have $[G:N] = p+1$. Now choose $\beta$ so that $\beta$ occurs in $\theta_N$. Then

$$p + 1 = [G:N] = \beta^G(1) \geqq \sum_1^{(p-1)/5} \theta_i(1) = (2p-5)(p-1)/5\ ,$$

a contradiction since $5 \mid (p-1)$ implies that $p \geqq 11$. This therefore completes the proof of the theorem.

Finally we consider the remaining groups with r.b.$(2p-1)$.

THEOREM 7. *Let $p$ be a prime and let $G$ be a group with* r.b.$(2p-1)$. *Then we have one of the following.*

( i ) *$G$ has a normal abelian Sylow $p$-subgroup.*
( ii ) *$G$ is solvable and has $p$-length* 1.
(iii) *$G/Z(G) \cong PSL(2, p)$ or $PGL(2, p)$ for $p > 3$.*
( iv) *$G/Z(G) \cong PSL(2, p-1)$ for $p$ a Fermat prime, $p > 3$.*
( v ) *$G/Z(G) \cong PSL(2, p+1)$ for $p$ a Mersenne prime.*
( vi) $G/Z(G) \cong \mathrm{Sym}\,(4)$ for $p = 2$.

*Proof.* If $p = 2$ then $G$ has r.b.3. Thus by Corollary 6.5 of [8], $G$ satisfies (ii) or (vi) above. Now let $p > 2$. Since $2p - 1 \leqq p^{3/2}$, Theorem 5 implies that $p^2 \nmid |G/O_p(G)|$. With this additional fact it is easy to see that the proof of the main theorem of [6] applies also to groups with r.b.$(2p-1)$ with $p > 2$ yielding the same conclusion. (The $p > 2$ assumption is used crucially in the last paragraph of the proof of Proposition 3.1 of [6].) Thus either $G$ satisfies (i) or (ii) above or $G = P_1 \times G_1$ where $P_1$ is an abelian $p$-group and $p^2 \nmid |G_1|$. Clearly $G_1$ has r.b.$(2p-1)$ and if $G_1$ satisfies any of the above then so does $G$. Therefore it suffices to assume that $G = G_1$ or equivalently that $p^2 \nmid |G|$. We assume now that $G$ does not satisfy (i). This of course implies that $p \mid |G|$.

Let $K = O_{p'}(G)$ and let $H/K$ be a minimal normal subgroup of $G/K$. Then $p \mid |H/K|$ and since $p^2 \nmid |G/K|$ this implies that $H/K$ is the unique minimal normal subgroup. Now $H/K$ is a product of isomorphic simple groups and $p^2 \nmid |H/K|$ so $H/K$ is simple. If $|H/K| = p$ then $G$ is $p$-solvable of $p$-length 1. Thus since $G$ does not have a normal Sylow $p$-subgroup, Proposition 2.3 of [6] implies that $G$ is solvable and $G$ satisfies (ii). Hence it suffices to assume that $\bar{H} = H/K$ is a nonabelian simple group. It is convenient to first consider the possibility $p \geqq 5$.

Since $\bar{H}$ is the unique minimal normal subgroup of $\bar{G} = G/K$ we have $C_{\bar{G}}(\bar{H}) = \langle 1 \rangle$ and thus $\bar{G} \subsetneq \mathrm{Aut}\,\bar{H}$. Suppose $\bar{T}$ is a subgroup of $\bar{H}$ with $1 < [\bar{H}\colon \bar{T}] < 2p$. Since $\bar{H}$ is simple and $p \mid |\bar{H}|$ we cannot have $[\bar{H}\colon \bar{T}] < p$. Thus $p \leqq [\bar{H}\colon \bar{T}] < 2p$ and $\bar{T}$ is maximal in $\bar{H}$ and hence self normalizing. If $\bar{T}$ were abelian it would follow easily that

$\bar{T}$ is a T.I. set and then $\bar{H}$ is a simple Frobenius group, a contradiction. Thus $\bar{T}$ is nonabelian.

Let $\psi$ be an irreducible character of $K$ and let $\chi$ be an irreducible constituent of $\psi^H$. If $e = [\chi_K, \psi]_K$ then $\chi(1) = et\psi(1)$ where $t = [H: T]$ and $T$ is the inertial group of $\psi$ in $H$. Suppose $T < H$ and set $\bar{T} = T/K$. Since $\chi(1) < 2p$ we have $t < 2p$ and thus by the remarks of the preceding paragraph $t \geqq p$ and $\bar{T}$ is nonabelian. Thus we have $2p > \chi(1) = et\psi(1) \geqq ep\psi(1)$ so $e = \psi(1) = 1$. Now there exists an irreducible character $\eta$ of $T$ with $\eta^H = \chi$ and $\eta_K = e\psi = \psi$. Since $\bar{T}$ is nonabelian we can choose a nonlinear irreducible character $\beta$ of $T$ containing $K$ in its kernel. Thus since $\eta$ is linear, $\eta_0 = \eta\beta$ is also an irreducible character of $T$. Let $\chi_0$ be an irreducible constituent of $\eta_0^H$. Then $\beta(1)\psi = \eta_{0K}$ occurs in $\chi_{0K}$ and therefore $[\chi_{0K}, \psi] \geqq \beta(1) > 1$. The above reasoning applied to $\chi_0$ now yields a contradiction. Thus $H = T$ and $H$ fixes all irreducible characters of $K$. By Brauer's lemma, $H$ fixes all conjugacy classes of $K$. Let $P$ be a Sylow $p$-subgroup of $H$. Then $P$ fixes each class of $K$ and since $K$ is a $p'$-group, $P$ centralizes $K$. Thus if $C = C_H(K)$ then $KC > K$ and since $H/K$ is simple we have $H = KC$. Now $C/(C \cap K) \cong \bar{H}$ and $Z(C) \supseteqq C \cap K$ so $Z(C) = C \cap K$.

Let $D$ denote the last term in the derived series of $C$. Then clearly $D = D'$, $D/Z(D) \cong \bar{H}$ and $Z = Z(D) = D \cap K$. Thus $Z$ is a homomorphic image of the Schur multiplier of $\bar{H}$. By Theorem 6, $\bar{H} \cong PSL(2, p)$, $PSL(2, p-1)$ for $p$ a Fermat prime or $PSL(2, p+1)$ for $p$ a Mersenne prime. We have by assumption $p \geqq 5$. Also for $p = 5$, $PSL(2, p) \cong PSL(2, p-1)$ and we will view this group as $PSL(2, p)$. By [10] (Satz IX, p. 119) either $Z = \langle 1 \rangle$ or $\bar{H} \cong PSL(2, p)$, $D \cong SL(2, p)$ and $|Z| = 2$.

We show now that $K$ is central. Suppose first that $Z = \langle 1 \rangle$ so that $H \cong D \times K$. Let $\chi$ be a fixed irreducible character of $D$ with $\chi(1) = p$ and let $\lambda$ be an irreducible character of $K$. Then $\chi\lambda$ is an irreducible character of $H$ so $2p > \chi(1)\lambda(1) = p\lambda(1)$ and $\lambda(1) = 1$. Thus $K$ is abelian and central in $H$. If $K$ is not central in $G$, then some linear character $\lambda$ of $K$ is not fixed by $G$. This implies easily that if $\theta$ is a constituent of $(\chi\lambda)^G$ then $\theta(1) \geqq 2\chi(1) = 2$p, a contradiction. Thus $K$ is central in $G$ in this case. Now let $Z \neq \langle 1 \rangle$ so that $|Z| = 2$ and $D \cong SL(2, p)$. We have an epimorphism $D \times K \to DK = H$ where the kernel is the third subgroup $W$ of order 2 in the group generated by the copies of $Z$ in $D$ and $K$. Let $\lambda$ be an irreducible character of $K$. Since $Z$ is central in $K$ and $|Z| = 2$ it is easy to see from the character table of $SL(2, p)$ ([10], p. 128) that there exists an irreducible character $\chi$ of $D$ with $\chi(1) \geqq p$ and with $W$ in the kernel of $\chi\lambda$, an irreducible character of $D \times K$. Thus $\chi\lambda$ is a character of $H$. The preceding argument now shows first that $K$ is abelian and then that $K$ is central. We have therefore shown that $G/Z(G) \cong \bar{G}$ and it remains to identify $\bar{G}$.

Now $\bar{G} \subsetneqq \operatorname{Aut} \bar{H}$ and $\bar{H}$ is a 2-dimensional projective group so the possibilities for $\bar{G}$ are given by Satz 1 of [9]. Suppose first that $\bar{H} \cong PSL(2, p)$. Then either $\bar{G} \cong PSL(2, p)$ or $\bar{G} \cong PGL(2, p)$ and we have (iii). Note the fact that $PGL(2, p)$ has $r$.b.$(2p - 1)$ can be seen from the character table on page 136 of [10]. We consider the remaining two cases. Thus $\bar{H} \cong PSL(2, s)$ with $2^n = s = p \pm 1$ and $\bar{G}/\bar{H}$ is isomorphic to a subgroup of the Galois group of $GF(2^n)/GF(2)$, a cyclic group of order $n$. Suppose $\bar{G} > \bar{H}$ and let $t \in \bar{G}$ correspond to a nontrivial field automorphism $x \longrightarrow x^j$. Then in the notation of page 134 of [10], but replacing upper case by lower case letters, we have $t^{-1}at = a^j \neq a$. Since $s > 4$ by our assumption for $p = 5$ it follows easily that $a^j \neq a^{-1}$ so $a^j$ is not conjugate to $a$ in $\bar{H}$. From the character table of $\bar{H}$ we now see easily that $t$ moves some irreducible character of $\bar{H}$ of degree $s + 1$ and thus $\bar{G}$ has an irreducible character of degree at least $2(s + 1) \geqq 2p$, a contradiction. Hence $\bar{G} = \bar{H}$ and $G$ satisfies (iv) or (v). This completes the proof of the theorem for $p \geqq 5$.

Finally let $p = 3$. Since $\bar{H}$ is a nonabelian simple group with r.b.$(2p - 1)$, $\bar{H} \cong PSL(2, 4) \cong PSL(2, 5)$ by Theorem 6. Certainly $G$ is not 5-solvable and $G$ has r.b.$(2 \cdot 5 - 1)$. Thus by the prime 5 case already proved, $G/Z(G) \cong PSL(2, 5)$ or $PGL(2, 5)$. Since the latter group has an irreducible character of degree $6 > 2p - 1$ we have $G/Z(G) \cong PSL(2, 5) \cong PSL(2, p + 1)$ and $G$ satisfies (v). Thus the result follows.

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