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THE 2-CELL AS A PARTIALLY ORDERED SPACE

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In this paper we prove a Jordan Curve Theorem (Theorem 1) for certain two dimensional partially ordered spaces.

We use this result to give a new characterization of the closed 2-cell (Theorm 2).

By a *partially ordered space* we X mean a Hausdorff space X with a partial order which is closed when regarded as a subset of $X \times X$ ($X \times X$ has the product topology).

For $x \in X$ we set

$$L(x) = \{y \in X \mid y \leqq x\}$$
$$M(x) = \{y \in X \mid x \leqq y\}$$

and

$$\Gamma(x) = L(x) \cup M(x) .$$

If $A \subset X$ we let

$$L(A) = \bigcup \{L(x) \mid x \in A\} .$$

We define $M(A)$ and $\Gamma(A)$ analogously. We let L (resp. M) denote the set of minimal (resp. maximal) elements of X.

A *chain* is a totally ordered set. An *order arc* is a compact and connected chain. A separable and nondegenerate order arc is homeomorphic to $[0, 1]$. A *continuum* is a compact, connected, Hausdorff space. An *arc* is a continuum with exactly two noncutpoints. A *circle* is a continuum such that every pair of points separates it.

DEFINITION. If X is a partially ordered space and $A \subset X$ let

$$C(A) = L(A) \cap M(A) .$$

A subset A of X is *convex* if and only if $A = C(A)$.

L. Nachbin proved the following result ([4], p. 48).

LEMMA 1.1. (*Nachbin*). *A compact partially ordered space X has a basis of convex open sets.*

The following three lemmas appear in [5]. For completeness we sketch their proofs here.

LEMMA 1.2. *Let X be a compact partially ordered space such* *that L is closed. If for each $x \in X$ $L(x)$ has a unique minimal element $p(x)$ then the function $p: X \to L$ is a retraction.*

Proof. We need only show p is continuous. Let $(x_i$ be a net converging to x in X and let y be a cluster point of $p(x_i))$. Then $y \in L$ since L is closed. Since the partial order on X is closed $y \in L(x)$. Hence $y = p(x)$.

LEMMA 1.3. *Let X be a compact partially ordered space such that L is closed and for each $x \in X$ $L(x)$ is an order arc. Let 2^X denote the space closed subsets of X with the finite topology* [3]. *Then the function $f: X \to 2^X$ defined by $f(x) = L(x)$ is continuous.*

Proof. It is well known (Michael [3]) that 2^X is a compact Hausdorff space and that the family of closed and connected subsets of X is closed in 2^X. Let $(x_i$ be a net converging to x in X and let A be a cluster point of $L(x_i))$. Since the partial order on X is closed $A \subset L(x)$. Clearly $x \in A$ and A meets L since L is compact. Since A is connected and no proper connected subset of $L(x)$ contains both x and $L(x) \cap L$, $A = L(x)$.

LEMMA 1.4. *Let X be a compact partially ordered space such that L and M are closed and for each $x \in X$ $\Gamma(x)$ is an order arc. Then the projection $\pi: X \to M$ defined by letting $\pi(x) \in M(x) \cap M$ is continuous and open.*

Proof. By Lemma 1.2 we need only show that π is open. By Lemma 1.3 the function $f: M \to 2^X$ defined by letting $f(m) = L(m)$ is a homeomorphism onto $f(M) \subset 2^X$.

Let $x \in X$ and let U be a neighborhood of x. Then the pair $\langle U, X \rangle$ is a basic open neighbourhood of $L(m)$ in 2^X (Michael [3]). Hence

$$\pi(U) = f^{-1}(\langle U, X \rangle \cap f(M))$$

is a neighbourhood of $\pi(x)$ in M.

LEMMA 1.5. *If X is as in Lemma 1.4 then X is locally connected if and only if M is locally connected.*

Proof. By Lemma 1.2 M is a retract of X so M is locally connected if X is locally connected.

Suppose M is locally connected and let π be as in Lemma 1.4. Let $x \in X$ and let U be a neighborhood of x. By Lemma 1.1 we may suppose U is a convex open neighbourhood of x. By Lemma 1.4 $\pi(U)$

is an open neighbourhood of $\pi(x)$ in M. Since M is locally connected there exists a connected open set V in M such that $\pi(x) \in V \subset \pi(U)$. Then

$$\pi^{-1}(V) \cap U = L(V) \cap U$$

is a convex open neighbourhood of x. If $L(V) \cap U = A \cup B$ where A and B are nonvoid and open in X then $\pi(A)$ and $\pi(B)$ are open in M and $V = \pi(A) \cup \pi(B)$. Since V is connected $\pi(A) \cap \pi(B)$ is nonvoid. Let $z \in \pi(A) \cap \pi(B)$. Then $L(z) \cap U = \pi^{-1}(z) \cap U$ is a connected set such that $L(z) \cap U \subset A \cup B$ and $L(z) \cap U$ meets both A and B. Thus $A \cap B$ is nonvoid and $L(V) \cap U$ is connected.

LEMMA 1.6. *Let X be a compact partially ordered space such that L and M are closed and for each $x \in X$ $L(x)$ is an order arc. If M is locally connected then X is locally connected.*

Proof. Define a set Y by

$$Y = \{(m, x) \mid m \in M \text{ and } x \in L(m)\} .$$

Give Y the partial order $(m, x) \leqq^* (n, y)$ if and only if $m = n$ and $x \in L(y)$. Define $g: Y \to X$ by $g(m, x) = x$ and give Y the smallest topology $\mathscr{U}$ such that g is continuous with respect to $\mathscr{U}$.

For each open set V of M let

$$0_V = \{(m, x) \in Y \mid m \in V \text{ and } x \in L(m)\} .$$

Let $\mathscr{W}$ be the topology on Y generated by $\mathscr{U}$ and

$$\{0_V \mid V \text{ is an open subset of } M\} .$$

Then $\mathscr{W}$ is a Hausdorff topology. It follows from Alexander's Lemma (Kelly [7], p. 139) and Lemma 1.3 that $\mathscr{W}$ is a compact topology. Furthermore, the given partial order on Y is closed with respect to $\mathscr{W}$. The detailed proofs of the above statements appear in [5], Theorem 2.7.

With the above partial order and the topology $\mathscr{W}$ Y is a compact partially ordered space which satisfies the hypotheses of Lemma 1.4. The set of maximal elements of Y is homeomorphic to M. Hence, Y is locally connected by Lemma 1.5. Now, X is the continuous image of the compact, locally connected, Hausdorff space Y so X is locally connected.

LEMMA 1.7. *Let X be a compact partially ordered space such that M is a continuum and for each $x \in X$ $L(x)$ is an order arc. If F is a compact convex subset of X such that for each $m \in M$ $L(m) \cap F$*

is nonvoid then F is connected.

Proof. The relation R on $F \times M$ defined by setting $(x, m) \in R$ if and only if $x \leqq m$ is upper-semicontinuous [2]. It follows by a well-known result on upper-semicontinuous relations [2] that F is connected.

2. A jordan curve theorem. In this section we shall prove the following theorem:

THEOREM 1. *Let X be a compact partially ordered space such that*

(i) *M is an arc with endpoints 0 and 1,*

(ii) *L is closed,*

(iii) *$L(m)$ is a nondegenerate order arc for each m in M,*

(iv) *for each cutpoint m of M, $L(m)$ separates X into components P and Q such that either $\bar{P}$ or $\bar{Q}$ meets L. Let $B = L \cup M \cup L(0) \cup L(1)$. Then each circle in $X \backslash B$ separates X and no pair of points separates X.*

To prove Theorem 1 we shall use an approach somewhat similar to that used by Whyburn [6] in his proof of the Jordan Curve Theorem. We shall show that any circle in X may be approximated arbitrarily closely by a circle which is the union of a finite number of convex arcs. We shall then prove that a circle which is the union of a finite number of convex arcs separates X.

For the remainder of this section let X be as in Theorem 1. Let M have its natural order $\leqq$ with initial point 0. Then $a \leqq b$ in M if and only if a lies in every subcontinuum of M which contains both 0 and b. For $a, b \in M$ with $a \leqq b$ let $[a, b]$ denote the arc in M which is irreducible with respect to containing a and b. Let

$$[a, b[= [a, b] \backslash \{b\}$$

and let

$$]a, b] = [a, b] \backslash \{a\} .$$

For $m \in M$ let

$$P_m = L([0, m]) \backslash L(m)$$

and let

$$Q_m = L([m, 1]) \backslash L(m) .$$

LEMMA 2.1. *If $m \in M \backslash \{0, 1\}$ then P_m and Q_m are connected and P_m is separated from Q_m.*

LEMMA 2.2. *If L is not trivial then L is an arc.*

Proof. By Lemma 1.2 L is a retract of X. Hence L is connected. If L is not a point then by condition (iv) of Theorem 1 the only noncutpoints of L are in $L(0)$ and $L(1)$. Thus L is an arc.

LEMMA 2.3. *If $x, z \in M$ with $x < z$, $y \in P_z \cap Q_x$ and $w \in L(x) \cap L(z)$ then $w \in L(y)$.*

Proof. Let $m \in M \cap M(y)$. By Lemma 2.1 $x < m < z$. Then $(L(z) \cup L(x)) \cap M(w)$ is a connected set which meets both components of $X \backslash L(m)$. Hence $w \in L(m)$. Now $y \in Q_x$ and $w \in L(x)$ so $y \nleqq w$. Since $L(m)$ is a chain and $y, w \in L(m)$ $w \leqq y$.

DEFINITION. An arc C in X is said to have FT if C is the union of a finite number of convex arcs. If C is an arc with FT then for each m in M $L(m) \cap C$ consists of a finite number of components.

DEFINITION. Let C be an arc with FT and let $x \in X \backslash C$. Let D be a component of $C \cap L(x)$ such that D does not contain an endpoint of C. We say D is a *turnabout of C in $L(x)$* if and only if there exists a neighbourhood U of D in C and $m \in M(x) \cap M$ such that $U \subset L([0, m])$ or $U \subset L([m, 1])$. If D is a turnabout of C in $L(x)$ then for each $n \in M(x) \cap M$ either $U \subset L([0, n])$ or $U \subset L([n, 1])$.

LEMMA 2.4. *Let C be an arc with FT and let $m \in M$ such that the endpoints of C lie in $X \backslash L(m)$. The number of components of $C \cap L(m)$ which are not turnabouts of C in $L(m)$ is odd if and only if exactly one of the endpoints of C lies in P_m.*

Proof. Let A be a component of $C \cap L(m)$. Each sufficiently small neighbourhood of A in C meets both P_m and Q_m if and only if A is not a turnabout of C. Hence the number of times that C crosses $L(m)$ is odd if and only if the number of components of $C \cap L(m)$ which are not turnabouts of C in $L(m)$ is odd.

LEMMA 2.5. *If A is an arc in X with endpoints b and c then there exists a convex arc F with endpoints b and c such that $F \subset C(A)$.*

Proof. If there exists $y \in A$ with $b, c \in M(y)$ let x be maximal in $L(b) \cap L(c)$ and let

$$F = M(x) \cap (L(b) \cup L(c)) .$$

Then F is a convex arc with endpoints b and c such that $F \subset C(A)$.

Suppose, therefore, that there does not exist $y \in A$ with $b, c \in M(y)$. We may assume by Lemma 2.3 that if $m, n \in M$ with $b \in L(m)$ and $c \in L(n)$ then $m < n$. Let $r \in M(b) \cap M$ and let $s \in M(c) \cap M$ such that r is maximal in $M(b) \cap M$ and s is minimal in $M(c) \cap M$ (with respect to the total order on M). For each $x \in [r, s]$ let $g(x)$ be minimal in $L(x) \cap A$ and let

$$G = \{g(x) \mid x \in [r, s]\} .$$

For each $e \in [r, s]$ let f_e be minimal in $[r, e]$ such that $g(f_e) \in L(e)$ and let h_e be maximal in $[e, s]$ such that $g(h_e) \in L(e)$.

Let $e \in [r, s]$ such that $r < f_e$. Let $e_i)_{i \in I}$ and $d_i)_{i \in I}$ be two nets in $[r, f_e[$ which converge to f_e. Suppose the nets $g(e_i))$ and $g(d_i))$ converge to m and n respectively. Then $m, n \in L(f_e)$. Suppose $m < n$. By Lemma 1.1 there exist convex open neighbourhoods U and V of m and n respectively such that $L(U) \cap M(V)$ is void.

Pick $j \in I$ so that D, the arc in A which is irreducible with respect to containing $g(e_j)$ and m, is contained in U. For each i $m \in Q_{d_i} \cap Q_{e_i}$. Also $g(e_j) \in P_{f_e}$. By Lemma 1.3 there exists $k \in I$ such that $g(e_j) \in P_{d_k}$ and $g(d_k) \in V$. Then $L(d_k)$ separates D. Now

$$L(d_k) \cap D \subset L(d_k) \cap A \cap U \neq \emptyset .$$

If $z \in L(d_k) \cap A \cap U$ then $z < g(d_k)$ by the choice of U and V. This contradicts the choice of $g(d_k)$. Hence $m = n$. We denote m by m_e. If $t \in [f_e, h_e]$ then $g(t) = g(e)$ by Lemma 2.3. Similarly if $e \in [r, s]$ such that $h_e < s$ then

$$\overline{g([h_e, s])} \cap L(h_e)$$

consists of a single point. We denote this point by n_e.

If $e \in [r, s]$ such that $f_e = r$ we let $m_e = b$ and if $h_e = s$ we let $n_e = c$. For each $e \in [r, s]$ let p_e be maximal in $L(m_e) \cap L(n_e)$ and let

$$H = \{m_e, n_e, p_e \mid e \in [r, s]\} .$$

Since $C(H) \subset C(G)$ it follows by the above argument that H is closed.

We let $F = C(H)$. It is easy to check that F is closed. By Lemma 1.7 F is connected. It is obvious from the above arguments that the only noncutpoints of H are b and c. Thus F is a convex arc containing b and c. Also $F \subset C(A)$.

LEMMA 2.6. *Let A be an arc in X with endpoints b and c and let U be any neighbourhood of A. There exists an arc E with FT such that $E \subset U$ and the endpoints of E are b and c.*

Proof. For each $x \in A$ let $V(x)$ be a closed and connected neighbourhood of x in A such that $C(V(x)) \subset U$. Since A is compact there exists an integer n and $a_1, \cdots, a_n \in A$ with

$$A \subset \bigcup \{V(a_i) \mid i = 1, \cdots, n\} .$$

We may suppose n is the smallest such integer and that $V(a_i) \cap V(a_j)$ is nonvoid if and only if $|i - j| \leqq 1$.

The natural order on A with initial point b induces a total order on $V(a_i)$ for each $i = 1, \cdots, n$.

By Lemma 2.5 there exists for each $i = 1, \cdots, n$ a convex arc B_{2i-1} with the same endpoints as $V(a_i)$ such that

$$B_{2i-1} \subset C(V(a_i)) \subset U .$$

For each $i = 1, \cdots, n-1$ let B_{2i} be a convex arc whose initial point is the terminal point of $V(a_i)$ and whose terminal point is the initial point of $V(a_{i+1})$ such that

$$B_{2i} \subset C(V(a_i)) \subset U .$$

One can now construct by an induction argument an arc

$$E \subset \bigcup \{B_i \mid i = 1, \cdots, 2n - 1\} \subset U$$

such that E has FT and the endpoints of E are b and c.

LEMMA 2.7. *Let C be a convex arc in X and let $m \in M\backslash C$ such that $L(m) \cap C$ is a turnabout of C in $L(m)$. If z is maximal in $C \cap L(m)$ then one of the components of $C\backslash z$ is a chain.*

Proof. We may suppose that $C \subset L([0, m])$. Let w be maximal in $M(z) \cap C$ and let n be minimal in M such that $w \in L(n)$. Then $n \in [0, m]$.

If $C \not\subset L([0, n])$ let $c \in C\backslash L([0, n])$. By Lemma 2.3

$$L([0, m]) \subset L([0, n]) \cup M(z) .$$

Hence $c \in M(z)$. Since C is convex the component of $C\backslash z$ which contains c lies in $M(z)\backslash(M(w)\backslash w)$. This component of $C\backslash z$ is a chain since C is convex.

If $C \subset L(|0, n])$ and w is not an endpoint of C let F and G be the components of $C\backslash w$. The endpoints of C lie in P_n. Let $n_i)$ be a net in $[0, n[$ which converges to n. By Lemma 1.3 $L(n_i))$ converges to $L(n)$ in 2^X. Eventually, therefore, $L(n_i) \cap F$ and $L(n_i) \cap G$ are nonvoid. For each i $w \in Q_{n_i}$ hence

$$L(n_i) \cap C = (L(n_i) \cap F) \cup (L(n_i) \cap G)$$

is disconnected. This contradicts the assumption that C is convex.

Thus w is an endpoint of C and $(L(w) \cap M(z))\backslash z$ is a component of $C\backslash z$ which is a chain.

LEMMA 2.8. *Let $x \in X$ and let U be a convex connected neighbourhood of x. Let C be a convex arc in $X\backslash U$ such that C has no endpoints in $L(U)$ and C has a turnabout in $L(x)$. Then $L(U) \cap C$ is a chain and if $z \in U$ such that $L(z) \cap C$ is nonvoid then $L(z) \cap C$ is a turnabout of C in $L(z)$.*

Proof. Let $m \in M(x) \cap M$ and suppose $C \subset L([0, m])$. Let y be maximal in $L(x) \cap C$. By Lemma 2.7 there is a component T of $C\backslash y$ which is a chain. Then $T \subset M(y)$. Let t be the endpoint of C which is in T and let $p \in M(t) \cap M$.

Let $z \in U \cap P_m$ and let $n \in M(z) \cap M$. Just suppose $p \in [n, m]$. Then $z \in P_p$ and so $L(p)$ separates U. Let $a \in L(p) \cap U$. Since

$$L(t) \cap M(y) \subset C \quad \text{and} \quad C \cap U$$

is void, $a \nleq t$. Since $L(p)$ is a chain $t < a$. This contradicts the assumption that C does not have an endpoint in $L(U)$. Thus $p < n$. By Lemma 2.3 it follows that $U \cap P_m \subset L(t)$. This proves the lemma.

LEMMA 2.9. *If C is a circle with FT in X and $C \subset X\backslash M$, then C separates X.*

Proof. Let

$$A = \left\{ x \in X\backslash C \,\middle|\, \begin{array}{l} \text{the number of components of } C \cap L(x) \text{ which} \\ \text{are not turnabouts of } C \text{ in } L(x) \text{ is odd} \end{array} \right\}$$

and let

$$D = \left\{ x \in X\backslash C \,\middle|\, \begin{array}{l} \text{the number of components of } C \cap L(x) \text{ which} \\ \text{are not turnabouts of } C \text{ in } L(x) \text{ is even} \end{array} \right\} .$$

Then $X\backslash C = A \cup D$ and $A \cap D$ is void. We shall show first of all that A and D are open in X.

We may suppose that $C = A_1 \cup \cdots \cup A_q$ where each A_i is a convex arc such that if $A_i \cap A_j$ is nonvoid then either $A_i = A_j$ or $A_i \cap A_j$ consists of an endpoint of A_i and A_j.

Let $x \in A$ and let $m \in M(x) \cap M$. Let $C_1, \cdots, C_k$ be the set of components of $C \cap L(x)$. By Lemmas 1.1, 1.3 and 1.6 there exists a convex connected neighbourhood U of x such that

(i) $U \subset X\backslash C$,

(ii) if $p \in L(U)$ is an endpoint of A_i for some $i \in \{1, \cdots, q\}$ then $p \in L(x)$,

(iii) if $i \in \{1, \cdots, q\}$ such that A_i meets $L(U)$ then A_i meets $L(x)$.

We shall prove that $U \subset A$. For each $w \in U$ define a function f_w with domain the set of components of $L(w) \cap C$ and with range the set of components of $L(x) \cap C$ as follows: Let P be a component of $C \cap L(w)$. If P meets $L(x)$ let $f_w(P)$ be the unique component of $C \cap L(x)$ which meets P. If P does not meet $L(x)$ then $P \subset A_i$ for some unique $i \in \{1, \cdots, q\}$. Let $f_w(P)$ be the unique component of $C \cap L(x)$ which meets A_i. To prove that $U \subset A$ it will suffice to prove that for each $w \in U$ and each $i \in \{1, \cdots, k\}$ the number of elements of $f_w^{-1}(C_i)$ which are not turnabouts of C in $L(w)$ is odd and only if C_i is not a turnabout of C in $L(x)$.

Let y be maximal in C_1. We may suppose A_1 and A_2 each have exactly one endpoint in C_1 and that endpoint is y. We may also suppose $C_1 \subset A_1$.

Case 1. Suppose C_1 is a turnabout of C in $L(x)$. We may suppose $A_1 \cup A_2 \subset L([0, m])$.

Since A_1 has only one endpoint in $L(u)$ it follows that if $z \in U \cap P_m$ then $L(z)$ meets A_1.

Let n be minimal in M such that $y \in L(n)$ and let $n_i)$ be a net in $[0, n[$ which converges to n. For each i let $U_i = U \cap L([0, n_i])$. Since $L(n_i)$ separates U and U is convex and connected it follows that U_i is connected. By the choice of n and by Lemma 1.3

$$U \backslash L([n, 1]) = \bigcup U_i .$$

Let $V = U \backslash L([n, 1])$ then V is a convex connected open set such that $A_1 \subset X \backslash V$ and the endpoints of A_1 lie in $X \backslash L(V)$. By Lemma 2.8 for each $z \in V$ $L(z) \cap A_1$ is nonvoid and is not a turnabout of A_1 in $L(z)$. Similarly for each $x \in V$ $L(z) \cap A_2$ is nonvoid and is not a turnabout of A_2 in $L(z)$.

If $z \in U \backslash V$ then $L(z) \cap (A_1 \cup A_2)$ is either void or is a turnabout of C in $L(z)$.

Case 2. Suppose C_1 is not a turnabout of C in $L(x)$. We may suppose $A_2 \subset L([0, m])$ and $A_1 \subset L([m, 1])$.

If $z \in (U \cap P_m) \backslash M(y)$ then by the argument of Case 1 $L(z) \cap A_2$ is nonvoid and is not a turnabout of C in $L(z)$. Also $L(z) \cap A_1 \subset L(y) \backslash \{y\}$. If $L(z) \cap A_1$ is nonvoid it is a turnabout of C in $L(z)$.

If $z \in U \cap (L([m, 1]) \cup M(y))$ then $L(z) \cap (A_1 \cup A_2)$ is nonvoid and connected and is not a turnabout of C in $L(z)$.

Thus $U \subset A$ and A is open. Similarly D is open. Since C is not an arc there exists $m \in M$ such that C meets both P_m and Q_m. By

Lemma 2.4 there exists a component E of $L(m) \cap C$, such that E is not a turnabout of C in $L(m)$. Let $x, y \in L(m)\backslash C$ such that

$$E = M(x) \cap L(y) \cap C .$$

If $x \in A$ then $y \in D$ and if $x \in D$ then $y \in A$. Thus both A and D are nonvoid and so C separates X.

We are finally ready to prove Theorem 1.

Proof of Theorem 1. Let C be a circle in $X\backslash B$ and let $m \in M$ such that C meets both P_m and Q_m. Let a be maximal in $C \cap P_m$ and let b be maximal in $Q_m \cap C$. Let S and T be the two arcs in C which are irreducible with respect to containing a and b.

Let y be maximal in $C \cap L(m)$. We may suppose $y \in T$. Let x be minimal in $T \cap L(m)$. Let $n \in L(x)\backslash C$ such that

$$M(n) \cap L(x) \cap C = \{x\} .$$

Suppose that C does not separate X. Since $X\backslash C$ is connected and locally connected by Lemma 1.6 there exists a continuum D in $X\backslash C$ such that $m, n \in D$.

Let Z be the arc in T which is irreducible with respect to containing x and y. Let U and V be convex, open and connected neighbourhoods of a and b respectively such that the closure of $U \cup V$ does not meet $Z \cup L(m)$.

Let Z', S' and T' be arcs with FT which are obtained from Z, S and T respectively by the method of Lemma 2.6 so that

$$\begin{aligned} &Z' \subset X\backslash(D \cup S \cup U \cup V) \\ &S' \subset X\backslash(D \cup Z' \cup (L(x) \cap M(n))) \\ &T' \subset X\backslash D \end{aligned}$$

and

$$S' \cap (T \cup T') \subset U \cup V .$$

Let S'' be an arc in S' which is irreducible with respect to having one endpoint in $T' \cap U$ and the other in $T' \cap V$. Let T'' be an arc in T' such that $E = S'' \cup T''$ is a circle. Then E is a circle with FT in $X/(D \cup M)$.

Now, $T'' \cap L(m) \subset L(y) \cap M(x)$ and the number of components of $T'' \cap L(m)$ which are not turnabouts of T'' in $L(m)$ is odd by Lemma 2.4. Also,

$$S'' \cap L(m) \cap M(n) \subset (L(y)\backslash\{y\}) \cap (M(x)\backslash\{x\}) .$$

Let $p, q \in Z' \cap L(m)$ such that $p < q$ and

$$M(p) \cap L(q) \cap Z' = \{p, q\} .$$

Let R be the arc in Z' which is irreducible with respect to containing p and q. Then

$$\cdot P = R \cup (L(q) \cap M(p))$$

is a circle with FT in $X \backslash M$. Since $S'' \cap Z'$ is void

$$P \cap S'' \subset (L(q) \cap M(p)) \backslash \{p, q\} .$$

The endpoints of S'' lie in the same component of $X \backslash P$ as does m. Hence, by Lemma 2.4 and Lemma 2.9 the number of components of $S'' \cap L(q) \cap M(q)$ which are not turnabouts of S'' in $L(m)$ is even. It follows since Z' has FT and $S'' \cap Z'$ is void that the number of components of $S'' \cap L(m) \cap M(n)$ which are not turnabouts of S'' in $L(m)$ is even. Hence m and n lie in distinct components of $X \backslash E$. This is a contradiction since $E \cap D$ is void and D is a continuum which contains m and n. Thus C separates X.

To prove that no pair of points separates X it suffices to prove that if $m \in]0, 1]$ then $\bar{P}_m \cap L(m)$ is a nondegenerate arc. Let

$$m \in]0, 1] \subset M \quad \text{and let} \quad p \in L(m) \backslash m$$

such that $p \notin L(0)$. Let n be minimal in M such that $p \in L(n)$. Let $n_i)$ be a net in $[0, n[$ which converges to n. Br Lemma 1.3 the net $L(n_i))$ converges to $L(n)$. Hence $p \in \bar{P}_n \subset \bar{P}_m$.

3. Characterization of the 2-cell.

We prove that if X is as in Theorem 1 and also metric then X is homeomorphic to the closed 2-cell.

THEOREM 2. *If X is a compact metric partially ordered space such that*

(i) *M is an arc and L is closed,*

(ii) *$L(m)$ is a nondegenerate order arc for each $m \in M$,*

(iii) *for each cutpoint m of M $L(m)$ separates X into components P and Q such that either $\bar{P}$ or $\bar{Q}$ meets L,*

then X is homeomorphic to a closed 2-cell.

Proof. We shall use Bing's Characterization of the 2-sphere.

Clearly X is a continuum. By Lemma 1.6 X is locally connnected. We proved in Theorem 1 that no pair of points separates X.

Let D be the unit disc in the plane. Let B be as in Theorem 1. By Lemma 2.2, B is a simple closed curve. Let $f: S^1 \rightarrow B$ be a homeomorphism of the boundary S^1 of D onto the subset B of X.

Let Y be the adjunction space of X with D under the map f. We shall prove that Y is a 2-sphere. Since the boundary of X in Y is the simple closed curve B it will follow that X is a closed 2-cell.

It is clear that Y is a locally connected, metric continuum such that no pair of points of Y separates Y. It remains to show that every simple closed curve in Y separates Y.

Let C be a simple closed curve in Y. Let $y \in Y \backslash (X \cup C)$ and let U be an open disc containing y such that $\bar{U}$ is a closed disc in $Y \backslash X$. It is easy to define a closed partial order on $Y \backslash U$ so that $Y \backslash U$ satisfies all the hypotheses of Theorem 1. Then C is a simple closed curve in $Y \backslash U$ such that C does not meet the boundary of $Y \backslash U$. By Theorem 1, C separates $Y \backslash U$ and hence C separates Y. Thus Y is a 2-sphere and X is a closed 2-cell.

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