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# MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES

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**One of the important investigations in the theory of summability is that of finding necessary and sufficient conditions on an infinite matrix in order that the matrix should transform one (complex) sequence space into the same or another sequence space. In this note some such theorems are given.**

Let

$C_0 =$ the space of null sequences;

$C =$ the space of convergent sequences;

$\Gamma =$ the space of sequences $x = \{x_p\}$ such that $|x_p|^{1/p} \to 0$, as $p \to \infty$. The space $\Gamma$ can be regarded as the space of all integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$;

$\Gamma^* =$ the space of sequences $s = \{s_p\}$ such that the sequence $\{|s_p|^{1/p}\}$ is bounded. $\Gamma^*$ may also be considered as the space conjugate to $\Gamma$ regarded as the space of integral functions $f(z) = \sum_{p=1}^{\infty} x_p z^p$. Each continuous linear functional $U \in \Gamma^*$ is of the form $U(f) = \sum_{p=1}^{\infty} s_p z_p$.

Let $A = (a_{np})$, $(n, p = 1, 2, \cdots)$, be an infinite matrix of complex elements. The $A$ transform of $x = \{x_p\}$, $y = \{y_n\}$ is the sequence defined by the equations

$$y_n = \sum_{p=1}^{\infty} a_{np} x_p, \ (n = 1, 2, \cdots) . \tag{1}$$

Here $y = \{y_n\}$ and $x = \{x_p\}$ are complex sequences. Similarly, the $A$ transform of $s = \{s_p\}$, $t = \{t_n\}$ is the sequence defined by the equations

$$t_n = \sum_{p=1}^{\infty} a_{np} s_p, \ (n = 1, 2, \cdots) . \tag{2}$$

Here also $t = \{t_n\}$ and $s = \{s_p\}$ are both complex sequences.

The following theorems are true:

THEOREM I. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* $\Gamma$ *whenever* $\{x_p\}$ *belongs to* $C_0$, *it is necessary and sufficient that*

(I, 1) *the sequence* $\{\theta_n\}$ *is a null sequence, where*

$$\theta_n = \left(\sum_{p=1}^{\infty} |a_{np}|\right)^{1/n}, \ (n = 1, 2, \cdots) . \tag{3}$$

Theorem I holds even if $C_0$ is replaced by $C$.

THEOREM II. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* $\Gamma^*$ *whenever* $\{x_p\}$ *belongs to* $C$, *it is necessary and sufficient that*

(II, 1) *the sequence* $\{\theta_n\}$ *is bounded, where* $\theta_n, (n = 1, 2, \cdots)$, *are given by* (3).

THEOREM III. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* $C$ *whenever* $\{x_p\}$ *belongs to* $\Gamma$, *it is necessary and sufficient that*

(III, 1) $|a_{np}|^{1/p} \leqq M$ *independently of* $n, p$;

(III, 2) $\lim_{n\to\infty} a_{np} = a_p$ *exists for each fixed* $p$.

THEOREM IV. *Let* (2) *hold. In order that* $\{t_n\}$ *should belong to* $C$ *whenever* $\{s_p\}$ *belongs to* $\Gamma^*$, *it is necessary and sufficient that*

(IV, 1) *the sequence* $\{f_n(z)\}$ *of integral functions*

$$f_n(z) = \sum_{p=1}^{\infty} a_{np} z^p, \ (n = 1, 2, \cdots)\,, \tag{4}$$

*is uniformly bounded on every compact set* (*of the complex plane*);

(IV, 2) = (III, 2) $\lim_{n\to\infty} a_{np} = a_p$ *exists for each fixed* $p$.

THEOREM V. *Let* (1) *hold. In order that* $\{y_n\}$ *should belong to* $\Gamma^*$ *whenever* $\{x_p\}$ *belongs to* $\Gamma$, *it is necessary and sufficient that*

(V, 1) $|a_{np}|^{1/(n+p)} \leqq M$ *independently of* $n, p$.

THEOREM VI. *Let* (2) *hold. In order that* $\{t_n\}$ *should belong to* $\Gamma$ *whenever* $\{s_p\}$ *belongs to* $\Gamma^*$, *it is necessary and sufficient that*

(VI, 1) $|f_n(z)|^{1/n} \to 0$, *as* $n \to \infty$, *uniformly on every compact set* (*of the complex plane*), *where* $\{f_n(z)\}$ *is the sequence of integral functions* $f_n(z)$ *given by* (4).

THEOREM VII. *Let* (1) *hold with* $a_{ij} = 0$ *for* $i > j$. *In order that* $\{y_n\}$ *should belong to* $\Gamma$ *whenever* $\{x_p\}$ *belongs to* $\Gamma$, *it is necessary and sufficient that*

(VII, 1) $|a_{np}|^{1/p} \leqq M$ *independently of* $n, p$.

The matrix transformation of $\Gamma^*$ into $\Gamma^*$ was studied by Heller [6].

The sufficiency, in each case is a straightforward calculation. The necessity of any of the above conditions is proved by taking special sequences, and constructing sequences to contradict the given condition, or by using Functional Analysis. Indeed, to prove the necessity of (III, 1), let $U_n(x) = y_n = \sum_{p=1}^{\infty} a_{np} x_p, (n = 1, 2, \cdots)$, for each fixed $x = \{x_p\} \in \Gamma$. Then $\{U_n(x)\}$ represents a sequence of continuous linear functionals on $\Gamma$ ([4], Th. 4). Here $\{|a_{np}|^{1/p}\}$ is bounded for each fixed $n$. Since $\{y_n\} \in C$, it follows that $\overline{\lim}_{n\to\infty} |U_n(x)| < \infty$ for each fixed $x \in \Gamma$. Define for each $x \in \Gamma$, $|x|$ = upper bound $(|x_p|^{1/p}, p \geqq 1)$.

Then for $x, x' \in \Gamma$, $|x - x'|$ defines a metric or distance in $\Gamma$. With the metric, $\Gamma$ is a complete metric space. Therefore, by Theorem 11 of ([1], p.19), there is a closed sphere $S$ and a fixed number $M$ such that

$$(5) \qquad |U_n(x)| \leqq M \text{ for } x \in S \text{ and all } n \geqq 1 .$$

Take the sphere $S$ as $|x| \leqq d$. Set $x_p = (d/2)^p$ and $x_j = 0$ for all $j \neq p$ so that $|x| \leqq d/2$ and hence $x = \{x_p\} \in S \subset \Gamma$. Then, by (5), it at once follows that

$$|U_n(x)| = |a_{np}(d/2)^p| \leqq M .$$

That is, $|a_{np}|^{1/p} \leqq M^{1/p}(2/d) < 2m(M)/d$ where $m(M) = \max(1, M)$. This proves the necessity of (III, 1).

A similar proof applies to condition (VII, 1).

Finally, I thank Professor V. Ganapathy Iyer for his help and guidance. I also thank the referee for drawing my attention to the papers of Sheffer [10] and Zeller [14], and other useful comments. Conditions of Theorems V, VI and VII neither include nor are included in Sheffer's conditions. However, $\Gamma$ and $\Gamma^*$ are included in the spaces considered by Sheffer. Sheffer [10] and Zeller [14] also dealt with the spaces of all power series with a certain minimal radius of convergence.

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