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THE ASYMPTOTIC BEHAVIOR OF THE KLEIN-GORDON EQUATION WITH EXTERNAL POTENTIAL, II

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Let $U_0(t)$ and $U(t)$ be the one-parameter groups governing the time development of solutions of the Klein-Gordon equation, $\Box\varphi = m^2\varphi$, and the perturbed equation, $\Box\varphi = m^2\varphi + V(\vec{x})\varphi$, respectively. In a previous work the author obtained sufficient conditions on the potential $V(\vec{x})$ which guaranteed the existence of the wave operators, $W_\pm := s - \lim U(-t)U_0(t)$ as $t \to \pm\infty$. Here it is shown that if, in addition, the associated (Schrödinger) wave operators, $W_\pm^S := s - \lim e^{i(m^2I+V-\Delta)t}e^{-i(m^2I-\Delta)t}$ as $t \to \infty$, are complete and the Invariance Theorem is valid then the $W_\pm$ are also complete and are isometries. Finally, these results are used to show that the scattering operator, $W_+^{-1}W_-$, is unitarily implemented in Fock space.

The similarity between the wave operators $W_\pm$ and $W_\pm^S$ observed in [1] as far as their existence theories are concerned, is clearly reaffirmed in their completeness theories. Indeed, the proof of the above results is based on the development of an explicit relationship between these wave operators. Connections of this sort were observed by Birman [3, p. 114, § 5] for abstract differential equations of the form $\varphi_{tt} + A\varphi = 0$. Sufficient conditions for such a relationship in this more general framework were obtained by Kato [4, §§ 9, 10] and used to study both potential and obstacle scattering for the wave equation [4, § 11].

In this investigation of the Klein-Gordon equation the argument will be directed so as to take best advantage of the above general results of Kato. However some generalizations will be necessary in order to establish the cited results on the Lorentz-invariant as well as the finite-energy solution spaces of the Klein-Gordon equation. Because a specific equation is being considered some simplification of Kato's arguments will also be possible.

1. Preliminaries. In this section the concepts discussed above are given precise definitions. Some related results which are directly used in the proofs of the main theorems are also included in summarized form.

Suppose Δ is the Laplacian in three dimensions and A^2 is the self-adjoint realization of $m^2I - \Delta$ on $L^2(E^3)$. Throughout this paper V is taken to be a real-valued function of three (space) variables and in

$L^p(E^3)$ for some $2 \leqq p \leqq \infty$.[1] With these hypotheses on V it is a fairly standard result that the perturbed operator, $A^2 + V$, is self-adjoint with $D(A^2 + V) = D(A^2) = D(\Delta)$. This self-adjoint realization of $A^2 + V$ will be denoted by B^2.

So that fractional powers of the above operators can be compared we ask that the perturbation satisfy a restriction on the size of its negative part:

(i) $\| V_- \|_q < M(q)$ for any $q \geqq 3/2$ (including ∞) where $M(q)$ is a constant depending only on q and m.

REMARK. More specifically $M(q) = \text{constant}.\, m^{(3-2q)/q}$ where the constant is that appearing in the Sobolev inequalities [6, p. 125]. The precise value of $M(q)$ is inessential in what follows. All that is needed is that the q-norm of V_- is sufficiently small for at least one $q \geqq 3/2$.

PROPOSITION 1.1. *For perturbations V, as above, satisfying condition* (i), *the self-adjoint operators A^θ, B^θ satisfy*

$$m^\theta \| \varphi \| \leqq \| A^\theta \varphi \| \leqq C_1^\theta \| B^\theta \varphi \| \leqq C_2^\theta \| A^\theta \varphi \| \tag{1}$$

for all $\varphi \in D(B^\theta) = D(A^\theta)$ and all $0 \leqq \theta \leqq 1$. In addition

$$C_2^{-\theta} \| A^{-\theta} \varphi \| \leqq C_1^{-\theta} \| B^{-\theta} \varphi \| \leqq \| A^{-\theta} \varphi \| \leqq m^{-\theta} \| \varphi \| \tag{2}$$

for all $\varphi \in L^2(E^3)$ and all $0 \leqq \theta \leqq 1$. C_1 and C_2 are constants depending on V, m, p and q.

Proof. [1, Lemma 2.4, Th. 2.5].

In order to discuss the solution spaces of the $K - G$ equation we shall first write it in its equivalent vector-valued form

$$i \frac{d}{dt} \begin{pmatrix} \varphi \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \dot{\varphi} \end{pmatrix} \tag{3}$$

which has as its formal solution

$$\begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{pmatrix} = U_0(t) \begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0) \end{pmatrix} = \begin{pmatrix} \cos At & A^{-1} \sin At \\ -A \sin At & \cos At \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0) \end{pmatrix} \tag{4}$$

where $\varphi(0), \dot{\varphi}(0)$ are the Cauchy data at $t = 0$. Indeed, it is a fairly well known fact that equation (4) rigorously defines the solution of the $K - G$ equation on $H(A, \alpha)$ (defined below) in the sense that $t \to U_0(t)$ is a one-parameter group of unitary transformations on $H(A, \alpha)$ with infinitesmal generator $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$. The solution spaces $H(A, \alpha)$

[1] $\| \ \|_p$ will denote the usual norm in $L^p(E^3)$. However, for notational convenience $\| \ \|_2$ will be replaced by $\| \ \|$ and the associated inner product will be written as $(,)$.

are described in the following.

DEFINITION. For each $\alpha \in \boldsymbol{R}$, the complex Hilbert space $H(A, \alpha)$ is the completion of $D(A^\alpha) \oplus D(A^{\alpha-1})$ with respect to the inner product

$$(\Phi, \Psi)_{A,\alpha} = \left(\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}, \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}\right)_{A,\alpha}$$
$$= (A^\alpha\varphi_1, A^\alpha\psi_1) + (A^{\alpha-1}\alpha_2, A^{\alpha-1}\psi_2) .$$

As a direct sum $H(A, \alpha)$ will be written as $D[A^\alpha] \oplus D[A^{\alpha-1}]$.

REMARK. Our primary interest is in the finite energy $(H(A, 1))$ and the Lorentz-invariant $(H(A, \frac{1}{2}))$ solution spaces of the $K-G$ equation. We shall handle both simultaneously by proving the main results on $H(A, \theta)$ for all $0 \leqq \theta \leqq 1$. For θ in this range it can be checked that the above completion is only required in the second summand of $H(A, \theta)$. In fact, except for the norm, $D[A^{\theta-1}]$ is isomorphic to the Sobolev space $W^{\theta-1,2}(E^3)$ and hence contains non-$L^2(E^3)$ elements.

Condition (i) insures that B^2, like A^2, is a nonnegative (self-adjoint) operator. For this reason the above discussion can be repeated with A replaced by B to obtain the dynamical propagators $U(t)$ on the solution spaces, $H(B, \theta)$, of the perturbed $K-G$ equation. The following observation, which is a direct consequence of Proposition 1.1, will be convenient in the next section.

PROPOSITION 1.2. *With the hypothesis of Proposition* 1.1 $H(A, \theta)$ *and* $H(B, \theta)$ *are isomorphic as linear spaces for each* $0 \leqq \theta \leqq 1$ *and the norms satisfy*

$$K_1 \|\cdot\|_{A,\theta} \leqq \|\cdot\|_{B,\theta} \leqq K_2 \|\cdot\|_{A,\theta} \tag{5}$$

where K_1 *and* K_2 *are constants depending on* C_1 *and* C_2. *It follows that* $U_0(t)\colon H(B, \theta) \to H(B, \theta)$ *and* $U(t)\colon H(A, \theta) \to H(A, \theta)$ *are uniformly bounded.*

The above result allows us to form products of the finite-time propagators even though they were defined on a priori different spaces and hence define the wave operators.

DEFINITION. The (free-to-physical) wave operators $W_\pm$ are given by

$$W_\pm = s - \lim_{t\to\pm\infty} U(-t)U_0(t)$$

whenever this strong limit exists on all of $H(A, \theta)$.

REMARK. The existence of the strong limit is demanded on all of $H(A, \theta)$ because the generator of $U_0(t), \begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$, is spectrally absolutely continuous (c.f. Lemma 2.2 to follow). For notational convenience the θ-dependence of $W_\pm$ is deleted since the conditions obtained are valid for all $0 \leqq \theta \leqq 1$.

If one further restriction is made on V,

(ii) $V \in L^p(E^3)$ for any $2 \leqq p < 3$,

then the following existence theorem can be proved [1, Th. 4.1].

THEOREM 1.3. *If V is real-valued and satisfies conditions* (i) *and* (ii) *then $W_\pm$ exist on $H(A, \theta)$ for each $0 \leqq \theta \leqq 1$.*

2. **Main results.** In this section the isometric nature and the completeness of $W_\pm: H(A, \theta) \to H(B, \theta)$ will be established for perturbations which satisfy the additional conditions

(iii) $W_\pm^S = s - \lim_{t \to \pm\infty} e^{iB^2t} e^{-iA^2t}$ are complete;

(iv) $W_\pm^S = s - \lim_{t \to \pm\infty} e^{i\varphi(B^2)t} e^{-i\varphi(A^2)t}$ for φ as in Invariance Theorem.[2]

The method of proof will be to establish a relationship between $W_\pm$ and $W_\pm^S$ by using the ideas concerning identification operators proved by Kato [4, §§ 9, 10]. Indeed the proof will be directed so as to take best advantage of these general results of Kato.

We begin by considering the transformation $\Gamma(A, \theta): H(A, \theta) \to L^2(E^3) \oplus L^2(E^3)$ formally defined by the equation

$$\Gamma(A, \theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} A^\theta & iA^{\theta-1} \\ A^\theta & -iA^{\theta-1} \end{pmatrix} .$$

This transformation, which is the analog of one considered by Birman [3, p. 114, § 5] and Kato [4, p. 335, 8.9], will provide us with a unitary operator which "diagonalizes" $U_0(t)$ in an operationally convenient way.

LEMMA 2.1. *For each $0 \leqq \theta \leqq 1$,*

$$\Gamma(A, \theta): D(A^\theta) \oplus D(A^{\theta-1}) (\subset H(A, \theta)) \to L^2(E^3) \oplus L^2(E^3)$$

defined above has a unique unitary extension

$$\tilde{\Gamma}(A, \theta): H(A, \theta) \to L^2(E^3) \oplus L^3(E^3) .$$

In addition

[2] The strongest version of condition (iv) required is with $\varphi(\lambda) = \lambda^{\theta/2}, 0 \leqq \theta \leqq 1$. This is not an operationally weaker condition, however, since the full Invariance Theorem [5, p. 544-7] must be used to determine conditions on V for it to occur.

$$\tilde{\Gamma}(A, \theta) U_0(t) \tilde{\Gamma}(A, \theta)^{-1} = e^{-iAt} \oplus e^{iAt} . \tag{6}$$

Proof. For $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(A^\theta) \oplus D(A^{\theta-1})$, a straight-forward computation using the defining equation gives $\| \Gamma(A, \theta)\Phi \| = \| \Phi \|_{A,\theta}$.[3] Furthermore

$$\Gamma(A, \theta)(D(A^\theta) \oplus D(A^{\theta-1})) = R(A^\theta) \oplus R(A^\theta) = L^2(E^3) \oplus L^2(E^3) .$$

Thus the isometry $\Gamma(A, \theta)$ has a unique extension to one with domain the $H(A, \theta)$-closure of $D(A^\theta) \oplus D(A^{\theta-1})$ (i.e., all of $H(A, \theta)$) and range $L^2(E^3) \oplus L^2(E^3)$. This unitary extension is

$$\tilde{\Gamma}(A, \theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} A^\theta & i\widetilde{A^{\theta-1}} \\ A^\theta & -i\widetilde{A^{\theta-1}} \end{pmatrix} \quad \text{where} \quad \widetilde{A^{\theta-1}}: D[A^{\theta-1}] \to L^2(E^3)$$

is the unitary transformation defined by $\widetilde{A^{\theta-1}}\varphi = A^{\theta-1}\varphi$ for all $\varphi \in L^2(E^3) \subset D[A^{\theta-1}]$. A simple algebraic computation shows that

$$\tilde{\Gamma}(A, \theta) U_0(t) = \{e^{-iAt} \oplus e^{iAt}\} \tilde{\Gamma}(A, \theta)$$

on a suitable dense set from which the relation (6) follows by continuity.

Before applying the above to the problem at hand we shall obtain a more precise description of the absolutely continuous part of the generator of $U_0(t)$ $\left(\text{i.e., of } \begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix} \text{ on } H(A, \theta)\right)$ since it is at the basis of the completeness problem for $W_\pm$. In particular we shall relate the subspace of absolute continuity of $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$ to that of A by means of an adaption to the present situation of a result of Kato [4, p. 355, Lemma 8.1].

LEMMA 2.2. *Let $P_{A,\theta}$ and Q_A denote (the projections in $H(A, \theta)$ and $L^2(E^3)$ onto) the subspaces of absolute continuity of $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$ and A respectively then the following conditions are equivalent*:

(a) $\Phi \in P_{A,\theta}$;

(b) $\tilde{\Gamma}(A, \theta)\Phi \in Q_A \oplus Q_A$;

(c) $A^\theta \varphi_1 \in Q_A$ *and* $\widetilde{A^{\theta-1}}\varphi_2 \in Q_A$.

Proof. Since Q_A is a closed linear subspace of $L^2(E^3)$ [5, p. 516,

[3] The norm $(\|\cdot\|^2 + \|\cdot\|^2)^{1/2}$ in $L^2(E^3) \oplus L^2(E^3)$ is also denoted by $\|\cdot\|$ since there is no possibility of confusion.

Th. 1.5] (b) and (c) are clearly equivalent. Suppose $E(\lambda)$ and $e(\lambda)$ are the spectral projections for A and $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$ respectively. Then equation (6) is equivalent to $\tilde{\Gamma}(A, \theta)e(S)\tilde{\Gamma}(A, \theta)^{-1} = \{E(S) \oplus -E(S)\}$ for all Borel sets $S \supset \boldsymbol{R}$. Thus

$$\| e(S)\Phi \|_{A,\theta} = \| \{E(S) \oplus -E(S)\}\tilde{\Gamma}(A, \theta)\Phi \|$$

from which the equivalence of (a) and (b) is immediate.

REMARK 1. Because $m^2 I - \Delta$ is spectrally absolutely continuous, A and hence $\begin{pmatrix} 0 & iI \\ -iA^2 & 0 \end{pmatrix}$, is likewise. This motivates the definition of $W_\pm$ in the previous section.

REMARK 2. Clearly if condition (i) is satisfied (so that B^θ is a nonnegative self-adjoint operator for each $0 \leqq \theta \leqq 1$), the above two results can be proved with A replaced by B. In general, however, B will not be spectrally absolutely continuous so that $P_{B,\theta} \neq I$.

Returning to the main problems we now indicate how the above may be used to provide a connection between the quasi-relativistic wave operators $W_\pm$ and the nonrelativistic wave operators $W_\pm^S$. This will be accomplished by comparing each to the wave operator

$$W'_\pm := s - \lim_{t \to \pm\infty} U(-t)\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta)U_0(t) .$$

The requirement that the identification operator [4, p. 343, 1.2 and p. 346, Definition 3.1] $\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta) \in B(H(A, \theta), H(B, \theta))$ is satisfied, since $\tilde{\Gamma}(A, \theta)$ and $\tilde{\Gamma}(B, \theta)$ are unitary.

THEOREM 2.3. *If the perturbation V satisfies conditions* (i) *and* (iv), *then*

(a) $W'_\pm$ *exist if and only if* $W_\pm^S$ *exist;*

(b) $W'_\pm$ *are complete if and only if* $W_\pm^S$ *are complete.*

Proof. Relation (6) for A, and the corresponding one for B can be used to obtain

$$\begin{aligned}&\tilde{\Gamma}(B, \theta)U(-t)\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta)U_0(t)\tilde{\Gamma}(A, \theta)^{-1} \\ &\quad = \{e^{iBt}e^{-iAt} \oplus e^{-iBt}e^{iAt}\} .\end{aligned}$$

Because the Γ-operators are bounded with bounded inverse, standard results on strong limits can be used on the above equation to give

$$\begin{aligned}\tilde{\Gamma}(B, \theta)W'_\pm\tilde{\Gamma}(A, \theta)^{-1} &= s - \lim_{t \to \pm\infty} \{e^{iBt}e^{-iAt} \oplus e^{-iBt}e^{iAt}\} \\ &= W_\pm^S \oplus W_\mp^S .\end{aligned} \tag{7}$$

The last equality follows from the invariance condition (iv). This establishes part (a). Similarly, part (b) follows from (7) and the equivalence of the first two statements in Lemma 2.2.

REMARK. The existence and completeness of $W_{\pm}^{S}$ are equivalent to the same questions for the more familiar wave operators,

$$s-\lim_{t\to\pm\infty} e^{i(V-\Delta)t}e^{-i(-\Delta)t},$$

since the associated prewave operators are identical. In particular, the existence of the latter is assured for potentials which satisfy condition (ii) [5, p. 534-5]; the completeness follows if $V\in L^1(E^3)\cap L^2(E^3)$ [5, p. 546, Example 4.10]. The proof of the completeness shows that condition (iii) and (iv) are closely related. It is interesting to distinguish them, however, since the latter is used for other purposes (e.g., in equation (7) and in a more essential manner in Lemma 2.5 to follow).

All that remains then is to show that $W_{\pm}=W'_{\pm}$. This will require condition (i), (iv) and the existence of $W_{\pm}^{S}$ (e.g., condition (ii)) in an explicit way. We now state this as a theorem, the proof of which is rather lengthy, and as a result, will proceed as a sequence of lemmas.

THEOREM 2.4. *If V satisfies conditions* (i), (ii) *and* (iv) *then $W_{\pm}=W'_{\pm}$ in the sense that the existence of one implies the existence of the other and their equality.*

Proof. A straightforward application of Theorem 4.2 of [4] shows that sufficient conditions for the equality of $W_{\pm}$ and $W'_{\pm}$ are

(a) $\tilde{\Gamma}(B,\theta)^{-1}\tilde{\Gamma}(A,\theta)$ and $I\in B(H(A,\theta),H(B,\theta))$, and

(b) $s-\lim_{t\to\pm\infty}(\tilde{\Gamma}(B,\theta)^{-1}\tilde{\Gamma}(A,\theta)-I)U_0(t)=0$ on $H(A,\theta)$.

The first part of (a) has already been noticed to be true if condition (i) is satisfied. The second part follows from Proposition 1.2 which likewise requires condition (i). In addition $U_0(t)\colon H(A,\theta)\to H(B,\theta)$ is uniformly bounded by K_2 (c.f. Proposition 1.2). Thus it suffices to establish (b) on a dense subset of $H(A,\theta)$; say $D(A)\oplus L^2(E^3)$. For $\Phi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\in D(A)\oplus L^2(E^3)$,

$$\begin{aligned}(8)\qquad &(\tilde{\Gamma}(B,\theta)^{-1}\tilde{\Gamma}(A,\theta)-I)U_0(t)\Phi\\ &=\left[\frac{1}{2}\begin{pmatrix}B^{-\theta} & B^{-\theta}\\ -iB^{1-\theta} & iB^{1-\theta}\end{pmatrix}\begin{pmatrix}A^{\theta} & iA^{\theta-1}\\ A^{\theta} & -iA^{\theta-1}\end{pmatrix}-\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\right]U_0(t)\Phi\\ &=\begin{pmatrix}B^{-\theta}A^{\theta}-I & 0\\ 0 & B^{1-\theta}A^{\theta-1}-I\end{pmatrix}\begin{pmatrix}\varphi_0(t)\\ \varphi_0(t)\end{pmatrix}\end{aligned}$$

where $\varphi_0(t)$ is the solution of the $K - G$ equation with Cauchy data φ_1, φ_2 at $t = 0$, and $\dot{\varphi}_0(t)$ is its time derivative. Thus

$$\begin{aligned} &\| (\tilde{\Gamma}(B, \theta)^{-1}\tilde{\Gamma}(A, \theta) - I)U_0(t)\Phi \|^2_{B,\theta} \\ &\quad = \| B^\theta(B^{-\theta}A^\theta - I)\varphi_0(t) \|^2 + \| B^{\theta-1}(B^{1-\theta}A^{\theta-1} - I)\dot{\varphi}_0(t) \|^2 \\ &\quad = \| (A_\theta - B^\theta)\varphi_0(t) \|^2 + \| (A^{\theta-1} - B^{\theta-1})\dot{\varphi}_0(t) \|^2 . \end{aligned}$$

The last equation follows from Proposition 1.1 (i.e., $D(A) = D(B) \subset D(A^\theta) = D(B^\theta)$) and the fact that $D(A) \oplus L^2(E^3)$ is invariant under $U_0(t)$. Thus (b) is implied by $\| (A^\theta - B^\theta)\varphi_0(t) \|$ and $\| (A^{\theta-1} - B^{\theta-1})\dot{\varphi}_0(t) \| \to 0$ as $t \to \pm\infty$.

We now reduce the conditions, step-by-step, to one which is much more amenable [4, p. 361, Condition 10.2 and Th. 10.5]. Let

$$\Phi_\pm = \frac{1}{2}\begin{pmatrix} (W^S_\pm + W^S_\mp), & i(W^S_\pm - W^S_\mp) \\ -i(W^S_\pm - W^S_\mp), & (W^S_\pm + W^S_\mp) \end{pmatrix}\Phi$$

and $\begin{pmatrix} \varphi_\pm(t) \\ \dot{\varphi}_\pm(t) \end{pmatrix} = U(t)\Phi_\pm$.

LEMMA 2.5. *Under the hypothesis of Theorem 2.4,* $\| A^\theta\varphi_0(t) - B^\theta\varphi_\pm(t) \|$ *and* $\| A^{\theta-1}\dot{\varphi}_0(t) - B^{\theta-1}\dot{\varphi}_\pm(t) \|$ *tend to zero as* $t \to \pm\infty$.

Proof. As previously observed the hypothesis implies the existence of $W^S_\pm$ which, by the invariance condition equals $s - \lim_{t\to\pm\infty} e^{iB^\theta t}e^{-iA^\theta t}$ for each $\theta \geqq 0$ (in particular for $0 \leqq \theta \leqq 1$). Now

$$\begin{aligned} A^\theta\varphi_0(t) &= A^\theta(\cos At\varphi_1 + A^{-1}\sin At\varphi_2) \\ &= \frac{1}{2}e^{-iAt}(A^\theta\varphi_1 + iA^{\theta-1}\varphi_2) + \frac{1}{2}e^{iAt}(A^\theta\varphi_1 - iA^{\theta-1}\varphi_2) . \end{aligned}$$

But the existence of $W^S_\pm$ implies that $s - \lim_{t\to\pm\infty}(e^{-iAt} - e^{-iBt}W^S_\pm) = 0$ and $W^S_\pm A^\theta = B^\theta W^S_\pm$ (using the invariance condition and the fact that $Q_A = I$). It is clear then that

$$\begin{aligned} \Big\| A^\theta\varphi_0(t) - \Big\{ &\frac{1}{2}e^{-iBt}(B^\theta W^S_\pm\varphi_1 + iB^{\theta-1}W^S_\pm\varphi_2) \\ &+ \frac{1}{2}e^{iBt}(B^\theta W^S_\mp\varphi_1 - iB^{\theta-1}W^S_\mp\varphi_2)\Big\}\Big\| \end{aligned}$$

tends to zero as $t \to \pm\infty$. A straightforward algebraic computation shows that the term in braces is $B^\theta\varphi_\pm(t)$. This establishes the first part of the lemma and the second part can be proved similarly.

By writing

$$\begin{aligned}\|(A^{\theta}-B^{\theta})\varphi_0(t)\| &= \|A^{\theta}\varphi_0(t)-B^{\theta}\varphi_{\pm}(t)+B^{\theta}\varphi_{\pm}(t)-B^{\theta}\varphi_0(t)\| \\ &\leqq \|A^{\theta}\varphi_0(t)-B^{\theta}\varphi_{\pm}(t)\|+\|B^{\theta}(\varphi_0(t)-\varphi_{\pm}(t))\|\ ,\end{aligned}$$

it is clear that $\|(A^{\theta}-B^{\theta})\varphi_0(t)\|\to 0$ as $t\to\pm\infty$ if $\|B^{\theta}(\varphi_0(t)-\varphi_{\pm}(t))\|\to 0$ as $t\to\pm\infty$. Similarly $\|(A^{\theta-1}-B^{\theta-1})\dot\varphi_0(t)\|\to 0$ as $t\to\pm\infty$ if $\|B^{\theta-1}(\dot\varphi_0(t)-\dot\varphi_{\pm}(t))\|\to 0$ as $t\to\pm\infty$.

LEMMA 2.6. *Under the hypothesis of Theorem* 2.4,

$$\|B^{\theta}(\varphi_0(t)-\varphi_{\pm}(t))\| \quad \textit{and} \quad \|B^{\theta-1}(\dot\varphi_0(t)-\dot\varphi_{\pm}(t))\|\to 0$$

as $t\to\pm\infty$ *if* $\|B(\varphi_0(t)-\varphi_{\pm}(t))\|\to 0$ *as* $t\to\pm\infty$.

Proof. Since $\Phi\in D(A)\oplus L^2(E^3)$, $\varphi_0(t)$ and $\varphi_{\pm}(t)\in D(A)=D(B)$ [8, p. 614, Th. 2.1]. But $\|B^{\theta}\psi\|=\|B^{\theta-1}B\psi\|\leqq(mC_1^{-1})^{\theta-1}\|B\psi\|$ for all $\psi\in D(B)$ by Proposition 1.1, which establishes the first part. The second part follows directly from the existence of $W_{\pm}^{S}$ and (iv). To see this write

$$\begin{aligned}\dot\varphi_0(t) &= -A\sin At\varphi_1+\cos At\varphi_2 \\ &= -\frac{i}{2}e^{-iAt}(A\varphi_1+i\varphi_2)+\frac{i}{2}e^{iAt}(A\varphi_1-i\varphi_2)\ ,\end{aligned} \tag{10}$$

and

$$\dot\varphi_{\pm}(t) = -\frac{i}{2}e^{-iBt}W_{\pm}^{S}(A\varphi_1+i\varphi_2)+\frac{i}{2}e^{iBt}W_{\mp}^{S}(A\varphi_1-i\varphi_2)\ . \tag{11}$$

Thus $\|\dot\varphi_0(t)-\dot\varphi_{\pm}(t)\|\to 0$ as $t\to\pm\infty$ if $s-\lim(e^{-iAt}-e^{-iBt}W_{\pm}^{S})=0$ as $t\to\pm\infty$ which follows from conditions (ii) and (iv). The proof is completed by again observing that

$$\|B^{\theta-1}(\dot\varphi_0(t)-\dot\varphi_{\pm}(t))\|\leqq(mC_1^{-1})^{\theta-1}\|\dot\varphi_0(t)-\dot\varphi_{\pm}(t)\|\ .$$

LEMMA 2.7. $\|B(\varphi_0(t)-\varphi_{\pm}(t))\|\to 0$ *as* $t\to\pm\infty$ *if* $\|Ve^{-iBt}W_{\pm}^{S}\psi\|\to 0$ *as* $t\to\pm\infty$ *for all* $\psi\in D(A^2)$.

Proof. This is essentially condition (*e*) of Theorem 10.5 of [4]. A careful examination of the proof shows that it suffices to have $s-\lim_{t\to\pm\infty}Ve^{-iBt}=0$ on $\{W_{\pm}^{S}\psi;\psi\in D(A^2)\}$ rather than on all of $D(B^2)\cap Q_B$. Condition (iv) is used in the present formulation but in a rather inessential way.

LEMMA 2.8. *Under the hypotheses of Theorem* 2.4,

$$\|V(e^{-iBt}W_{\pm}^{S}-e^{-iAt})\psi\|\to 0$$

as $t \to \pm\infty$ *for all* $\psi \in D(A^2)$.

Proof. Since $\psi \in D(A^2)$, $e^{-iAt}\psi$ and $e^{-iBt}W_\pm^S\psi \in D(A^2) = D(B^2)$ [8, p. 614, Th. 2.1]. Now

$$\| V(e^{-iBt}W_\pm^S - e^{-iAt})\psi \| \leqq \| V \|_p \| (e^{-iBt}W_\pm^S - e^{-iAt})\psi \|_q$$

where $q = 2p(p-2)^{-1}$. The last term is estimated using inequalities of the Sobolev type [6, p. 125] to obtain

$$\| (e^{-iBt}W_\pm^S - e^{-iAt})\psi \|_q \leqq \text{constant} \| (-\Delta)(e^{-iBt}W_\pm^S - e^{-iAt})\psi \|^\gamma \cdot \| (e^{-iBt}W_\pm^S - e^{-iAt})\psi \|^{1-\gamma} \tag{12}$$

where $\gamma = 3(2p)^{-1}$. The result will now follow if it can be shown that the first term on the right in (12) is uniformly bounded in t and the second tends to zero as $t \to \pm\infty$. The second requirement follows from the existence of $W_\pm^S$ and the invariance condition provided $\gamma < 1$ or $p > 3/2$ which is guaranteed by the hypothesis. Turning to the second requirement,

$$\begin{aligned} \| (-\Delta)(e^{-iBt}W_\pm^S - e^{-iAt})\psi \| &\leqq \| A^2(e^{-iBt}W_\pm^S - e^{-iAt})\psi \| \\ &\leqq \| A^2 e^{-iBt}W_\pm^S\psi \| + \| A^2\psi \| . \end{aligned}$$

To show that the first term on the right of the above inequality is bounded recall [1, Th. 2.1] that if $V \in L^p(E^3)$ for any $p \geqq 2$, there exist constants $a < 1$ and b, such that for $\chi \in D(A^2)$,

$$\| B^2\chi - A^2\chi \| = \| V\chi \| \leqq a \| A^2\chi \| + b \| \chi \| .$$

Hence

$$\| A^2\chi \| \leqq (1-a)^{-1}(\| B^2\chi \| + b \| \chi \|) . \tag{13}$$

Applying (13) to the above and using well-known properties of $W_\pm^S$ one obtains

$$\begin{aligned} \| A^2 e^{-iBt}W_\pm^S\psi \| &\leqq (1-a)^{-1}(\| B^2 e^{-iBt}W_\pm^S\psi \| + b \| e^{-iBt}W_\pm^S\psi \|) \\ &\leqq (1-a)^{-1}(\| A^2\psi \| + b \| \psi \|) \end{aligned}$$

which proves the lemma.

Clearly, the above result reduces the proof of Theorem 2.4 to showing that $\| Ve^{-iAt}\psi \| \to 0$ as $t \to \pm\infty$ for all $\psi \in D(A^2)$.

LEMMA 2.9. *If* $V \in L^p(E^3)$ *for any* $2 \leqq p < \infty$, *then* $\| Ve^{-iAt}\psi \| \to 0$ *as* $t \to \pm\infty$ *for all* $\psi \in D(A^2)$.

Proof. We first show that it suffices to prove the result on a core of A^2 (i.e., a set $\mathscr{C} \subset D(A^2)$ such that for each $\psi \in D(A^2)$, there exists a sequence $\{\psi_n\} \subset \mathscr{C}$ such that $\| A^2(\psi - \psi_n) \| + \| \psi - \psi_n \| \to 0$ as

$n \to \infty$). If ψ and ψ_n are as above then the observation follows from

$$\begin{aligned} \| Ve^{-iAt}\psi \| &= \| Ve^{-iAt}(\psi - \psi_n + \psi_n) \| \\ &\leqq \| Ve^{-iAt}(\psi - \psi_n) \| + \| Ve^{-iAt}\psi_n \| \\ &\leqq \| A^2 e^{-iAt}(\psi - \psi_n) \| + \| Ve^{-iAt}\psi_n \| \\ &\leqq \| A^2(\psi - \psi_n) \| + \| Ve^{-iAt}\psi_n \| \, . \end{aligned}$$

Of course, the above computation requires $V \in L^p(E^3)$ for any $p \geqq 2$ so that $\| V\chi \| \leqq \| A^2\chi \|$ for all $\chi \in D(A^2)$.

In particular take $\mathscr{C} = \mathfrak{F}C_c^\infty(E^3)$ (i.e., the image under Fourier transformation of $C_c^\infty(E^3)$). $\mathscr{C}$ is a core for A^2 if and only if $C_c^\infty(E^3)$ is a core for $M_{k^2+m^2}$ [5, p. 300]. The latter condition is true since $M_{k^2+m^2}$ maps $C_c^\infty(E^3)$ onto $C_c^\infty(E^3)$ [5, p. 166, 5.19]. All that remains then is to show that $\| Ve^{-iAt}\psi \| \to 0$ as $t \to \pm\infty$ for all $\psi \in \mathscr{C}$. Now

$$\| Ve^{-iAt}\psi \| \leqq \| V \|_p \, \| e^{-iAt}\psi \|_q$$

where $q = 2p(p-2)^{-1}$. But $\| e^{-iAt}\psi \|_r = 0(|t|^{-3\{(1/2)-(1/r)\}})$ as $|t| \to \infty$ for each $2 \leqq r \leqq \infty$ and each $\psi \in \mathscr{C}$ by a variant of Proposition 4.2 of [1] which is a direct consequence of a result of Segal [7, p. 95, Lemma 3]. Thus the decay is established if $q > 2$ or $2 \leqq p < \infty$.

The above results can be used in a fairly obvious manner to prove the result indicated at the beginning of this section; namely,

THEOREM 2.10. *If conditions* (i)-(iv) *are satisfied then the* $W_\pm$ *are complete.*

REMARK. A careful examination of the above proofs shows that condition (ii) is used only to show that $W_\pm^S$ exist. Thus the above theorem is valid if condition (ii) is replaced by the weaker condition

(ii)′ $W_\pm^S$ exist.

Indeed the same change gives an alternate formulation of the existence Theorem 1.3. This result is more appealing from the viewpoint of the similarly of $W_\pm^S$ and $W_\pm$ but the proof requires the very restrictive condition (iv). It is interesting however, that condition (i) is present in both versions.

One further result which follows from the above is the isometric nature of the $W_\pm$. More specifically,

THEOREM 2.11. *If conditions* (i), (ii)′ *and* (iv) *are satisfied then for each* $0 \leqq \theta \leqq 1$, $W_\pm$: $H(A, \theta) \to H(B, \theta)$ *are isometries.*

Proof. Theorems 2.3 and 2.4 give

$$W_{\pm} = \tilde{\Gamma}(B, \theta)^{-1}\{W_{\pm}^{S} \oplus W_{\mp}^{S}\}\tilde{\Gamma}(A, \theta)\ , \tag{14}$$

from which the result immediately follows since the Γ-operators are unitary and the $W_{\pm}^{S}$ are isometries.

3. **Application.** In this section the preceding results will be used to show that the scattering operator, $S = W_{+}^{-1}W_{-}$, is unitarily implementable in the free representation of the quantized Klein-Gordon field with mass m. We shall introduce only the most basic concepts here and direct the reader to [2] and the references therein for a more detailed and systematic discussion.

The unique, relativistically invariant, classical dynamical system associated with the $K - G$ field in three space consists of the real Hilbert space $H_r(A, \frac{1}{2})$ (the real part of $H(A, \frac{1}{2})$) and the nondegenerate, skew-symmetric bilinear form $\mathrm{Re}(J\cdot, \cdot)_{A,1/2}$ where $J = \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix}$. A transformation on $H_r(A, \frac{1}{2})$ which preserves the above form is called symplectic. It is well-known that the symplectic transformations form a group. By means of a straightforward algebraic computation [e.g., 2, p. 391, Lemma 3.4], it can be shown that both $U_0(t)$ and $U(t)$, and hence the prewave operators $W(t)$, are symplectic. In addition, it is not difficult to show that strong limits of symplectic operators are likewise symplectic. Thus $W_{\pm}$ and S are symplectic in the above sense.

A quantization of the above classical $K - G$ field is basically a map $\Phi \rightarrow Q(\Phi)$ from $H_r(A, \frac{1}{2})$ into unitary operators on a complex Hilbert space $\mathscr{H}$ which satisfy the Weyl (exponentiated) form of the commutation relations. The most familiar of these, and the one with which we shall deal, is called the Fock-Cook quantization. It will be denoted by Q_0 on $\mathscr{H}_0$. If $T: H_r(A, \frac{1}{2}) \rightarrow H_r(A, \frac{1}{2})$ is symplectic then $\Phi \rightarrow Q_0(T\Phi)$ is another quantization. If it is unitarily equivalent to the Fock-Cook quantization, T is said to be unitarily implementable (in the free representation of the $K - G$ field with mass m). This situation occurs if and ond only if T, as an operator on $H_r(A, \frac{1}{2})$, is bounded with bounded (everywhere defined) inverse such that $T^*T - I$ is Hilbert-Schmidt [2, p. 388, Corollary 2.3].

THEOREM 3.1. *S is unitarily implementable in the free representation of the $K - G$ field with mass m if conditions* (i)-(iv) *are satisfied.*

Proof. Since $W_{\pm}$ are complete, $D(W_{+}^{-1}) = R(W_{+}) = R(W_{-}) = P_{B,1/2}$, and hence S is well defined on $H(A, \frac{1}{2})$. In addition, since $R(W_{+}^{-1}) = D(W_{+}) = H(A, \frac{1}{2})$, the image of $H(A, \frac{1}{2})$ under S is all of $H(A, \frac{1}{2})$. Furthermore, the isometric nature of $W_{\pm}: H(A, \frac{1}{2}) \rightarrow H(B, \frac{1}{2})$ implies that $S: H(A, \frac{1}{2}) \rightarrow H(A, \frac{1}{2})$ is an isometry, and hence unitary. Thus

$S\colon H_r(A, \frac{1}{2}) \rightarrow H_r(A, \frac{1}{2})$ is orthogonal and the required conditions for unitary implementability are satisfied trivially.

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