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MAPPINGS AND REALCOMPACT SPACES

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The problem of preserving realcompactness under perfect and closed maps is studied. The main result is that realcompactness is preserved under closed maps if the range is a normal, weak *cb*, k-space. Generalizing a result of Frolík, we show that realcompactness is preserved under perfect maps if the range is weak *cb*. Moreover, the problem of preserving realcompactness under perfect maps may be reduced to the following: When does the absolute of X being realcompact imply that X is realcompact? Likewise the problem of preserving topological completeness under perfect maps may be reduced to an analogous question. The following special case is also proved. If ϕ is a closed map from X onto a weak *cb*, q-space Y, then X is realcompact implies that Y is realcompact.

A function is a k-covering map if any compact set in the image space is contained in the image of a compact set in the domain. We show that all closed maps whose domain X is topologically complete are k-covering maps. Next a representation is obtained for a closed image Y of a topologically complete, G_δ-space. Namely $Y = Y_a \cup (\cup Y_i)$ where $\phi^{-1}(y)$ is compact for all $y \in Y_a$ and each Y_i is discrete in Y. This generalizes a theorem of K. Morita and is similar to a theorem of Arhangel'skiĭ which replaces topologically complete with point-paracompact.

If ϕ is an open-closed mapping from a realcompact space X onto a k-space Y, then $\partial\phi^{-1}(y)$ is compact for every $y \in Y$. It then follows from a result of Isiwata's that Y is also realcompact. Now let ϕ be a WZ-map of X onto a realcompact space Y. We show that X is realcompact if and only if $\mathrm{cl}_{\nu X}\,\phi^{-1}(y) = \phi^{-1}(y)$. This generalizes Isiwata's result that X is realcompact if $\phi^{-1}(y)$ is a C^*-embedded, realcompact subset of X.

The reader is referred to [6] for the basic ideas of rings of continuous functions. The following characterization of realcompactness will be used in this paper. A completely regular, Hausdorff space X is *realcompact* if and only if for each p in $\beta X - X$ there exists an f in $C(\beta X)$ such that $f(p) = 0$ and $f(x) > 0$ if x is in X. A map will be used to designate a continuous onto function. A map is said to be *closed* (*open*) if the image of each closed (open) subset of the domain is closed (open) in the range. A map is called a *Z-map* if the image of each zero-set in X is closed in Y. If the inverse image of each compact set in the range is compact, then the map is said to be *compact*. *Perfect maps* are those which are closed and compact.

A map is *minimal* if the image of every proper closed subset of the domain is a proper subset of the range. It is well known [17, 5] that if f is a compact mapping of X onto Y, then there exists a closed subspace X_0 of X such that $f_0 = f \mid X_0$ is a minimal map onto Y. Let ϕ map X onto Y and let Φ denote the continuous extension of ϕ from βX onto βY. A map is a *WZ-map* if $\mathrm{cl}_{\beta X}\, \phi^{-1}(y) = \Phi^{-1}(y)$ for every y in Y. It is shown in [10] that all Z-maps are WZ-maps. A subspace S is said to be *regular closed* if $S = \mathrm{cl}\,\mathrm{int}\, S$. Similarly, a subspace S is said to be *regular open* if $S = \mathrm{int}\,\mathrm{cl}\, S$. A space X is a G_δ-space if it is a G_δ subset of βX.

1. **Realcompactness and perfect mappings.** Realcompactness is not preserved under perfect maps. An example will be presented at the end of this chapter. However, Frolìk has proved that realcompactness is preserved under perfect maps if the range is normal and countably paracompact. In this section it will be shown that this can be weakened by only requiring that the range space is weak *cb*. A space X is a *weak cb-space* [13] if each locally bounded, lower semicontinuous function on X is bounded above by a continuous function. The following theorem is proved in [13, 3.1, p. 237].

THEOREM 1.1. *The following statements are equivalent for any topological space X.*

(a) *X is weak cb.*

(b) *Given a positive normal lower semicontinuous function g on X, there exists f in $C(X)$ such that $0 < f(x) \leqq g(x)$ for each x in X.*

(c) *Given a decreasing sequence $\{F_n\}$ of regular closed sets with empty intersection, there exists a sequence $\{Z_n\}$ of zero-sets with empty intersection such that $Z_n \supset F_n$ for each n.*

A space X is called *almost realcompact* [5, p. 128] if for every maximal open filter $\mathscr{A}$ of X with $\bigcap \overline{\mathscr{A}} = \varnothing$, there exists a countable subfamily $\{A_i\}$ of $\mathscr{A}$ such that $\bigcap \bar{A}_i = \varnothing$. Every realcompact space is almost realcompact. The next theorem generalizes Frolík's result that every normal, countably paracompact, almost realcompact space is realcompact.

THEOREM 1.2. *Let X be a completely regular, Hausdorff space. If X is almost realcompact and weak cb, then X is realcompact.*

Proof. Let $\mathscr{B}$ be a free zero-ultrafilter. Set $\mathscr{A} = \{U\colon\ U$ is open and there exists a Z in $\mathscr{B}$ such that $Z \subset U\}$. Let $\mathscr{A}'$ be an open ultrafilter containing $\mathscr{A}$. By regularity $\bigcap \overline{\mathscr{A}'} = \varnothing$. There exists a countable subcollection $\{A_i\}$ of $\mathscr{A}'$ such that $\bigcap \bar{A}_i = \varnothing$. Set

$N_n = \bigcap_{i=1}^{n} A_i$. Then $\{N_n\}$ is a decreasing sequence of open sets and $\bigcap \bar{N}_n = \varnothing$. Since X is weak *cb*, there exists a sequence of zero sets $\{Z_n\}$ such that $Z_n \supset \bar{N}_n$ and $\bigcap Z_n = \varnothing$. Now $\bar{N}_n$ meets all elements of $\mathscr{B}$, so Z_n meets all elements of B. Thus, for each positive integer n, $Z_n \in \mathscr{B}$, and, hence, X is realcompact.

Frolík has proved the following theorem [5, 8, p. 134].

THEOREM 1.3. *Let f be a perfect map of X onto Y. If X is almost realcompact, then Y is almost realcompact. If Y is almost realcompact and X is regular, then X is almost realcompact.*

The next corollary follows from Theorems 1.2 and 1.3.

COROLLARY 1.4. *Let f be a perfect map of a realcompact space X onto a completely regular, Hausdorff space Y. If Y is weak cb, then Y is realcompact.*

Associated with each regular Hausdorff space X is a completely regular Hausdorff space $\tilde{X}$, called the absolute of X, with the following properties. The absolute $\tilde{X}$ is extremally disconnected and there exists a minimal perfect map of $\tilde{X}$ onto X. Further, if Y is extremally disconnected and is the preimage of a perfect minimal map onto X, then Y is homeomorphic to $\tilde{X}$. The next theorem is proved in [17, p. 308].

THEOREM 1.5. *If f is a perfect map from a regular Hausdorff space X onto a regular Hausdorff space Y and g is a perfect map from an extremally disconnected space E onto a closed subset of Y, then there exists a perfect map h from E onto a closed subset of X such that $g = fh$.*

THEOREM 1.6. *Let f be a perfect map of X onto a regular Hausdorff space Y. Then if X is realcompact, the absolute of Y is also realcompact.*

Proof. This follows from 1.5 and the fact [6, 10.16, p. 148] that the perfect preimages of realcompact spaces are realcompact.

Thus the problem of preserving realcompactness under perfect maps may be reduced to the following: Under what conditions does $\tilde{X}$ realcompact imply X is realcompact? Every extremally disconnected space is weak *cb*, so by Theorems 1.2 and 1.3, we have the following:

THEOREM 1.7. *Let X be a regular Hausdorff space. Then X is* *almost realcompact if and only if $\tilde{X}$ is realcompact.*

There exists a nonrealcompact space Y such that Y is the union of two closed realcompact subsets Y_1 and Y_2 [16]. Let X be the topological sum of Y_1 and Y_2. Clearly, X is realcompact. Let f be the canonical map of X onto Y. Then f is a perfect map of a realcompact space onto a space that is not realcompact. This example is due to R. L. Blair. Note also that X is almost realcompact and not realcompact.

2. Realcompactness and closed mappings. Throughout the remainder of this paper, X and Y will denote completely regular, Hausdorff spaces. As we have seen in the preceding section, realcompactness is not preserved under closed mappings. The following special case has been proved by Isiwata [10, 7.5, p. 477]. If ϕ is a closed mapping from a locally compact, countably paracompact, normal space X onto Y, then Y is realcompact when X is realcompact. This theorem will be generalized in this section.

A space X is a *k-space* if a set is closed when its intersection with any compact set is closed. Locally compact spaces, first countable spaces, p-spaces in the sense of Arhangel'skii [1] and G_δ-spaces [2, 12, p. 563] are k-spaces.

If $\phi\colon X \to Y$ and $f \in C(X)$, let

$$f^i(y) = \inf \{f(x)\colon x \in \phi^{-1}(y)\}$$

and

$$f^s(y) = \sup \{f(x)\colon x \in \phi^{-1}(y)\} .$$

It has been shown [13, 11, p. 235] that if ϕ is a minimal perfect map, then f^s is normal upper semicontinuous. In an analogous manner, it can be shown that f^i is normal lower semicontinuous if ϕ is a minimal perfect map. If ϕ is a continuous function of X onto Y, Φ will denote the continuous extension of ϕ from βX onto βY.

LEMMA 2.1. *Let X be a topologically complete space and ϕ a closed map of X onto the space Y. If C is a compact subset of $\beta X - X$ and $\Phi(C) \subset Y$, then $\Phi(C)$ is finite.*

Proof. Let $C \subset \beta X - X$ and $\Phi(C) \subset Y$. Suppose that $\Phi(C)$ is not finite. Then $\Phi(C)$ contains an infinite set I such that I is discrete in the relative topology. For each $y \in I$, pick $x_y \in \Phi^{-1}(y) \cap C$. The set $S = \{x_y\colon y \in I\}$ is infinite, hence it must have an accumulation point in C, say p. The space X is topologically complete, so there exists a locally finite cozero cover $\mathscr{U}$ of X such that if $U \in \mathscr{U}$ then $p \notin \mathrm{cl}_{\beta X} U$ [18, 2.6, p. 172].

Select $x_1 \in \phi^{-1}(I)$ and $U_1 \in \mathscr{U}$ such that $x_1 \in U_1$. Suppose that $x_1, \cdots, x_{n-1}$ and $U_1, \cdots, U_{n-1}$ have been selected such that $x_i \in U_i \in \mathscr{U}$ and if $i \neq j$, then $\phi(x_i) \neq \phi(x_j)$ and $U_i \neq U_j$. Set $y(i) = \phi(x_i)$. Now $p \in \mathrm{cl}_{\beta X} (S \backslash \{x_{y(i)}: i \leqq n-1\})$ and $p \notin \mathrm{cl}_{\beta X} \bigcup_{i=1}^{n-1} U_i$. Thus there exists an open set N containing p such that $N \cap (\bigcup_{i=1}^{n-1} U_i) = \emptyset$. Pick

$$x_{y(n)} \in N \cap S \backslash \{x_{y(i)}: i \leqq n-1\} .$$

Next select $x_n \in \phi^{-1}(y(n)) \cap N$, and $U_n \in \mathscr{U}$ such that $x_n \in U_n$. Note that if $i < n$, then $y_i \neq y_n$ and $U_i \neq U_n$. Now $\{\{x_i\}\}_i$ is a locally finite collection, so $\{x_i\}_i$ is a closed subset of X. Since ϕ is a closed map $\{y_i\}_{i=1}^{\infty}$ must be a closed subset of Y. But $\{y_i\}$ is an infinite discrete subset of the compact set $\Phi(C)$. Thus we have reached a contradiction.

LEMMA 2.2. *Let X be a topologically complete space and ϕ a closed map of X onto the space Y. If C is a compact subset of $\beta X - X$, then all compact subsets of $\Phi(C) \cap Y$ are finite.*

Proof. This follows from 2.1.

LEMMA 2.3. *Let X be a topologically complete space and ϕ a closed map of X onto a k-space Y. If C is a compact subset of $\beta X - X$, then $\Phi(C) \cap Y$ is closed and discrete in the relative topology.*

Proof. $\Phi(C)$ is a compact subset of βY, so $\Phi(C) \cap Y$ is a closed subspace of Y, and hence is a k-space. Let $F \subset \Phi(C) \cap Y$ and K be any compact subset of $\Phi(C) \cap Y$. By 2.2 K is finite and so $F \cap K$ is closed. Since $\Phi(C) \cap Y$ is a k-space, F must be closed. Hence $\Phi(C) \cap Y$ is discrete.

THEOREM 2.4. *Let ϕ be a closed map of X onto a normal, weak cb, k-space Y. Then X realcompact implies that Y is realcompact.*

Proof. There exists a closed subspace X_0 of βX such that $\Phi_0 = \Phi \mid X_0$ is a minimal perfect map onto βY. Pick $q \in \beta Y - Y$ and $p \in \Phi_0^{-1}(q)$. Since p is not in X there exists a nonnegative function f in $C(\beta X)$ such that $f(p) = 0$ and $f(x) > 0$ if $x \in X$. Define $f^i(y) = \inf\{f(x): x \in \Phi_0^{-1}(y)\}$. Then f^i is a normal lower semicontinuous function on βY and $Z(f^i) = \Phi(Z(f))$. Set $Y_0 = \Phi(Z(f)) \cap Y$. Since $Z(f)$ is a compact subset of $\beta X - X$, Y_0 is discrete and closed in Y. Pick $x_y \in \phi^{-1}(y)$. Then $F = \{x_y: y \in Y_0\}$ is a closed and discrete subset of X. Hence $\phi \mid F$ is a homeomorphism and so Y_0 is realcompact. If

$q \notin \mathrm{cl}_{\beta Y} Y_0$, then there exists a $g \in C(\beta Y)$ such that $g(y) = 1$ if $y \in Y_0$ and $g(q) = 0$. Now suppose $q \in \mathrm{cl}_{\beta Y} Y_0$. Since Y is normal, Y_0 is C^*-embedded in Y and hence $\beta Y_0 = \mathrm{cl}_{\beta Y} Y_0$. By the realcompactness of Y_0, there exists a nonnegative $g_0 \in C^*(\mathrm{cl}_{\beta Y} Y_0)$ such that g_0 is positive on Y_0 and $g_0(q) = 0$. Let $g \in C(\beta Y)$ such that $g \mid \mathrm{cl}_{\beta Y} Y_0 = g_0$ and $g \geqq 0$. The function $f^i + g$ is a normal lower semicontinuous function on βY that is positive on Y and $f^i + g(q) = 0$. Since Y is weak *cb*, there exists a $h \in C^*(Y)$ such that $0 < h \leqq f^i + g$. Let h^* be the continuous extension of h from βY into the reals. Since normal lower semicontinuous functions are determined on dense subsets, we have that $0 \leqq h^*(q) \leqq f^i + g(q) = 0$. Thus Y is realcompact.

Note that the full strength of 2.3 was not used in the proof of the theorem. All that was needed was that if Z is a zero set and $Z \subset \beta X - X$, then $\Phi(Z) \cap Y$ is discrete. This particular result follows from a lemma of Arhangel'skiĭ [3, 1.2, p. 202].

Let Y be a k-space, ν a point-finite covering of X, and let ϕ: $X \to Y$ be a closed mapping of X onto Y. Then $N = \{y \in Y$: no finite $\nu' \subset \nu$ covers $\phi^{-1}(y)\}$ is discrete in Y.

If $Z \subset \beta X - X$, let f be a nonnegative element of $C^*(\beta X)$ such that Z is the zero set of f and $f \leqq 1/2$. Set

$$U_n = \{x \in X: 1/n + 2 < f(x) < 1/n\} .$$

Clearly $\nu = \{U_n\}$ is a point-infite covering of X and $Y \cap \Phi(Z) = N$. However, Corollary 2.3 does not follow from this, since X is Lindelöf if and only if for every compact subset of $\beta X - X$ there is a zero-set Z such that $C \subset Z$ and $Z \cap X$ is empty.

By a usual method found in [10] and [14] we obtain the following lemma.

LEMMA 2.5. *If ϕ is a map of a space X onto Y and $\partial\phi^{-1}(y)$ is compact for every $y \in Y$, then there exists a closed subspace X_0 such that $\phi_0 = \phi \mid X_0$ has the property that inverses of points are compact and $\phi_0(X_0) = Y_0$.*

COROLLARY 2.6. *If ϕ is a closed map of a realcompact space X onto a weak cb-space Y, and the $\partial\phi^{-1}(y)$ is compact for each $y \in Y$, then Y is realcompact.*

Another special case of realcompactness being preserved under closed maps will be proved in the next section (3.5).

3. Topologically complete spaces. The following is a problem in [6, 8E.1, p. 126]. For any subset S of a realcompact space X, if

$f|S$ is bounded for all f in $C(X)$, then cl S is compact. As a generalization of this, we have the following.

LEMMA 3.1. *If S is a subset of a topologically complete space X, and $f|S$ is bounded for all f in $C(X)$, then* cl S *is compact.*

Proof. Suppose $\mathrm{cl}_X S$ is not compact. Then there exists a point p in $\mathrm{cl}_{\beta X} S\backslash X$ and locally finite partition of unity Φ such that $p \notin \mathrm{cl}_{\beta X}$ (cozero (ϕ)) for each $\phi \in \Phi$. Let $\Phi' = \{\phi \in \Phi$: there exists a $x \in S$ such that $\phi(x) \neq 0\}$. Let ϕ^β denote the Stone extension of ϕ. If $\phi \in \Phi$, then $\phi^\beta(p) = 0$, since $\phi^\beta(p) > 0$ implies that $p \in \mathrm{cl}_{\beta X}$ cozero (ϕ). Now if Φ' is finite, then $f = \sum_{\Phi'} \phi^\beta \in C(\beta X)$, $f(x) = 1$ if $x \in S$, and $f(p) = 0$. Since $x \in \mathrm{cl}_{\beta X} S$, this is impossible and so Φ' must be infinite. Let $\{\phi_n\} \subset \Phi'$ such that $n \neq m$ implies that $\phi_n \neq \phi_m$. Let x_n be an element of S such that $\phi_n(x_n) \neq 0$ and set $f_n = n\phi_n/\phi_n(x_n)$. Now $\sum f_n \in C(X)$ since the cozero sets of $\{f_n\}$ are locally finite, but $\sum f_n$ is not bounded on S. This concludes the proof.

Michael has defined a point $y \in Y$ to be a *q-point* if it has a sequence of neighborhoods $\{N_i\}$ such that if $y_i \in N_i$ and the y_i are all distinct, then $y_1, y_2, \cdots$ has an accumulation point in Y. If every y in Y is a q-point, then Y is called a q-space. The q-spaces include the first countable spaces, the locally compact spaces and the p-spaces in the sense of Arhangel'skiĭ [1]. Michael has proved the next theorem [14, 2.1, p. 173].

THEOREM 3.2. *Let $f: X \rightarrow Y$ be continuous, closed and onto, where X is T_1. If $y \in Y$ is a q-point, then every continuous, real-valued function on X is bounded on $\partial f^{-1}(y)$.*

The next corollary follows from 3.1 and 3.2.

COROLLARY 3.3. *If f is a continuous, closed map of a topologically complete space X onto a q-space Y, then $\partial f^{-1}(y)$ is compact for each $y \in Y$.*

A function f is a *k-covering map* if any compact set in the image space is contained in the image of a compact set in the domain. Note that $f: X \rightarrow Y$ is a k-covering mapping if and only if the induced mapping f_c^* from $C(Y)$ to $C(X)$ endowed with their respective compact-open topologies is a homeomorphism into [1, p. 154]. As a generalization of Corollary 1.2 of [14], we have the following:

THEOREM 3.4. *Let X be topologically complete and $f: X \rightarrow Y$ continuous, closed and onto. Then f is a k-covering map.*

Proof. Let C be a compact subset of Y and consider $f_0 = f | f^{-1}(C)$. Since f_0 is a closed continuous map onto C, by 3.3, $\partial f_0^{-1}(y)$ is compact for each y in C. It is well known that if f has the property that inverses of points are compact and f is closed, then f is a compact map. So from 2.5, there exists a closed set F in $f^{-1}(C)$ such that $f_0 | F$ is a closed and compact map from F onto C. Thus F is compact and $f(F) = C$.

Since all realcompact spaces are topologically complete, the next corollary follows from 2.6 and 3.3.

COROLLARY 3.5. *If ϕ is a closed continuous map of a realcompact space X onto a weak cb, q-space, then Y is realcompact.*

THEOREM 3.6. *Let f be a perfect map of X onto Y. If Y is topologically complete, then X is topologically complete.*

Proof. This follows from [6, 15.11, p. 224] and [6, p. 147].

COROLLARY 3.7. *Let f be a perfect map of X onto Y. If X is topologically complete, then $\tilde{Y}$ is topologically complete.*

Proof. This follows from 3.6 and 1.5.

Thus the problem of preserving topological completeness under perfect maps reduces to finding conditions under which $\tilde{X}$ topologically complete implies that X is. The author has been unable to find any such conditions. Note again that the example in § 1 is also an example of a space that is not topologically complete and is the perfect image of a realcompact space. Further any counterexample when the range is weak *cb* would involve a space of measurable cardinal. As in the case of realcompactness, if f is a closed map of a topologically complete space X onto a q-space Y, there exists a topologically complete space X_0 such that Y is the perfect image of X_0.

The next theorem is similar to a theorem of Arhangel'skiĭ [3, 1.1, p. 202] in that it replaces point-paracompact with topologically complete.

THEOREM 3.8. *If X is a topologically complete, G_δ-space and ϕ is a closed map of X onto Y, then $Y = \bigcup_{i=0}^{\infty} Y_i$, where each Y_i is closed and discrete if $i \geqq 1$ and $\phi^{-1}(y)$ is compact for all $y \in Y_0$.*

Proof. If X is a G_δ-space, then $\beta X - X = \bigcup C_i$ where each C_i is compact. Set $Y_0 = \{y: \phi^{-1}(y)$ is compact$\}$ and if $i \geqq 1$, let $Y_i =$

$\Phi(C_i) \cap Y$. Since G_δ-spaces are k-spaces, it follows by 2.3 that Y_i is closed and discrete. If $y \in Y \backslash Y_0$, then $y \in \Phi(\beta X \backslash X) \cap Y = \bigcup_{i=0}^{\infty} Y_i$. Thus $Y = \bigcup_{i=0}^{\infty} Y_i$.

The next theorem is a slight generalization of a theorem of Isiwata's [10, 7.3, p. 476] in that it replaces realcompactness by topologically complete.

THEOREM 3.9. *Let ϕ be a closed mapping from a locally compact, topologically complete space X onto Y. Then Y is locally compact if and only if $\partial\phi^{-1}(y)$ is compact for every $y \in Y$.*

Proof. This follows from 2.5 and 3.3.

4. Realcompactness and *WZ*-maps. The following lemma is due to Isiwata [10, 6.1, p. 467].

LEMMA 4.1. *If $\phi: X \rightarrow Y$ is an open WZ-mapping, y is not isolated, $\phi^{-1}(y)$ is not compact, and if there is a function f in $C(\beta X)$ such that $0 \leqq f \leqq 1, f > 0$ on X and $f(x) = 0$ for some x in $\Phi^{-1}(y) \backslash \Phi^{-1}(y)$, then $Z_{\beta Y}(f^i)$ is a neighborhood in βY of y.*

Using this lemma Isiwata proves the following theorem. Hence we shall give a different proof that uses the technique employed in 2.4.

THEOREM 4.2. *If ϕ is an open and closed mapping from a realcompact space X onto a space Y such that $\partial\phi^{-1}(y)$ is compact for every $y \in Y$, then Y is also realcompact.*

Proof. Pick q in $\beta Y - Y$ and p in $\Phi^{-1}(q)$. Then there exists a $f \in C(\beta X)$ such that $f(x) > 0$ if $x \in X$ and $f(q) = 0$. Since ϕ is open and closed, Φ is open [10, 4.4, p. 464] which implies that $f^i \in C(\beta Y)$ [10, 4.1, p. 463]. Now if $f^i(y) = 0$, then $\phi^{-1}(y)$ is not compact, $\operatorname{int} \phi^{-1}(y) \neq \emptyset$, and hence y is isolated. Since $Z(f^i)$ is both open and closed, it is C-embedded in Y. The subspace $Z(f^i)$ is realcompact since it is discrete and homeomorphic to a closed subset of X. These last two properties imply that there exists a nonnegative $g \in C(\beta Y)$ such that $g(q) = 0$ and $g(y) > 0$ if $y \in Z(f^i)$. Now the function $g + f^i$ is positive on Y and $g + f^i(q) = 0$. It follows that Y is realcompact.

THEOREM 4.3. *If ϕ is an open-closed mapping from a realcompact space X onto a k-space Y, then $\phi^{-1}(y)$ is compact for all nonisolated points $y \in Y$.*

Proof. Assume that $\phi^{-1}(y)$ is not compact and y is not isolated. Pick $p \in \Phi^{-1}(y)\backslash X$. Since X is realcompact there exists a $f \in C(\beta X)$ such that f is positive on X, $0 \leqq f \leqq 1$, and $f(p) = 0$. By 4.1, $Z_Y(f^i)$ is a neighborhood of y. But by 2.3 $Z_Y(f^i)$ is discrete. Hence y is isolated, which is a contradiction.

COROLLARY 4.4. *If ϕ is an open-closed mapping from a realcompact space X onto a k-space Y, then Y is also realcompact.*

Proof. This follows from Theorems 4.2 and 4.3.

Arhangel'skii mentions in [1] that he has shown if X is a completely regular, Hausdorff, k-space which admits a complete uniform structure and ϕ is an open closed map of X onto Y, then either $\phi^{-1}(y)$ is compact or y is isolated for each y in Y. The author does not know if he uses the fact that X is a k-space or only that Y is a k-space.

Next, we consider preimages of realcompact spaces. Perfect preimages of realcompact spaces are realcompact. In [10, 5.3, p. 466], Isiwata proves the next theorem.

THEOREM 4.5. *Let $\phi: X \rightarrow Y$ be a Z-mapping and let $\phi^{-1}(y)$, $y \in Y$, be a C^*-embedded, realcompact subset of X. If Y is realcompact, then so is X.*

Unfortunately, the condition that $\phi^{-1}(y)$ be C^*-embedded is not a necessary condition if X is realcompact. Indeed, all closed subsets will be C^*-embedded if and only if X is normal. The following theorem generalizes 4.4.

THEOREM 4.6. *Let ϕ be a WZ-map of X onto a realcompact space Y. Then X is realcompact if and only if $\mathrm{cl}_{\nu X}\, \phi^{-1}(y) = \phi^{-1}(y)$ for each $y \in Y$.*

Proof. The sufficiency is clear. Assume that $\mathrm{cl}_{\nu X}\, \phi^{-1}(y) = \phi^{-1}(y)$ for each $y \in Y$ and let $\phi_0 = \Phi \mid \nu X$. Note that $\phi_0(\nu X) = Y$ since Y is realcompact [6, 8.7, p. 118] and $\Phi^{-1}(y) = \mathrm{cl}_{\beta X}\, \phi^{-1}(y)$ implies that $\phi_0^{-1}(y) = \mathrm{cl}_{\nu X}\, \phi^{-1}(y)$. Thus $\nu X = \bigcup_{y \in Y} \phi_0^{-1}(y) = \bigcup_{y \in Y} \phi^{-1}(y) = X$, and X is realcompact.

LEMMA 4.7. *If ϕ is a z-map of X onto Y and $\phi^{-1}(y)$ is C^*-embedded, then $\phi^{-1}(y)$ is C-embedded.*

Proof. Let Z be a zero-set in X which is disjoint from $\phi^{-1}(y)$.

Then $\phi(Z)$ is closed in Y and $y \notin \phi(Z)$. So there exists a $f \in C(Y)$ such that $f(y) = 0$ and $f(\phi(Z)) = \{1\}$. Now $f\phi \in C(X)$ and $f\phi$ is 0 on $\phi^{-1}(y)$ and 1 on Z. It follows that $\phi^{-1}(y)$ is C-embedded [6, 1.18, p. 19].

LEMMA 4.8. *Let ϕ map X onto Y and $y \in Y$. If $\phi^{-1}(y)$ is realcompact and C-embedded, then* $\mathrm{cl}_{\nu X}\, \phi^{-1}(y) = \phi^{-1}(y)$.

Proof. Since $\phi^{-1}(y)$ is C-embedded in X, it is C-embedded in νX. Hence,

$$\mathrm{cl}_{\nu X}\, \phi^{-1}(y) = \nu(\phi^{-1}(y)) = \phi^{-1}(y) .$$

Thus we see that 4.5 is actually a generalization of 4.4.

If $F \subset X$, F is said to be *Z-embedded* in X if for every zero-set Z in F, there exists a zero-set Z' in X such that $Z' \cap F = Z$. If F is Lindelöf and $F \subset X$, then F is Z-embedded in X [8]. Further, if F is Z-embedded and completely separated from every zero-set disjoint from it, then it is C-embedded [7]. Thus we have the following.

COROLLARY 4.9. *Let ϕ be a Z-mapping of X onto Y. If $\phi^{-1}(y)$ is Lindelöf for each $y \in Y$, and Y is realcompact, then X is realcompact.*

If X is locally compact, a G_δ-space, a p-space, or if every point is a G_δ, then every element of X has a compact zero-set containing it. We conclude this section with the following theorem.

THEOREM 4.10. *Let ϕ be a compact mapping from X onto Y and let every element of Y have a compact zero-set containing it. Then Y realcompact implies that X is realcompact.*

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