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LOCALLY COMPACT TOPOLOGIES ON A GROUP AND THE CORRESPONDING CONTINUOUS IRREDUCIBLE REPRESENTATIONS

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It is shown that two different topologies on a group G both of which make it into a locally compact group, usually give rise to different continuous irreducible unitary representations. To be more precise: If the continuous irreducible unitary representations of G coincide for the two topologies, then these topologies are the same in the following cases: The topologies are comparable; There exists a normal subgroup of G, open and σ-compact in one of the topologies.

In the case of an abelian locally compact group G, complete information about its structure is contained in its dual object $\hat{G}$, which is, again, an abelian locally compact group. If G is an arbitrary locally compact group, its dual $\hat{G}$ is a topological space only—the space of equivalence classes of continuous irreducible unitary representations of G, furnished with an intricate topology. There is the natural question of how much information about G is contained in this space $\hat{G}$. This paper is an attempt to answer a very restricted problem in this context, namely the following: does the 'functor' $\wedge$ distinguish between two different topologies ρ and σ on an abstract group, each of which makes it into a locally compact group? To be more precise, we shall show (see §9 below) that in a large class of cases the following is true: If the irreducible representations of G continuous with respect to ρ coincide with those continuous with respect to σ, then $\rho = \sigma$. A special result of this sort has been obtained in [2], and we list it here for reference as

PROPOSITION 1. *If every irreducible representation of the locally compact group G is continuous, then G is discrete.*

The paper is divided into eight more sections, of which the first contain some simple lemmas about locally compact groups, mainly listed for reference, while the later ones treat special cases of pairs of locally compact topologies on a group with the same continuous irreducible representations. A survey of the results obtained is given in §9.

I owe to Professor K. Ross the remark that slightly stronger, analogous results, due to I. Kaplansky are known for some time in the Abelian case[1].

[1] (cf. [**6**, Corollary 24]; [**10**], [**15**]).

2. Generalities on locally compact topologies on a group. The simple lemmas of this section will permit us below to reduce questions of weak equivalence or equality of topologies on a group to the same questions on subgroups and quotient groups. For the following three subsections, G denotes a locally compact group.

2.1. The identity component G_0 of G is the intersection of all open subgroups of G, and G is totally disconnected, if and only if there exists a basis at e of open compact subgroups ([10, §7] or [14, p. 54 ff.]). From this it follows immediately that any subgroup of G and any quotient of G by a closed normal subgroup is totally disconnected, if G is. We recall that if G is *compact* and totally disconnected, the open and *normal* subgroups form a basis at e.

LEMMA 2.2. *The following statements are equivalent:*

(i) *G is σ-compact.*

(ii) *G is a Lindelöf space.*

(iii) *If V is a neighbourhood of the identity, G can be covered by countably many left (right) translates of V.*

(iv) *For every closed subgroup U of G, U and G/U are σ-compact.*

(v) *There is a closed subgroup U of G such that U and G/U are σ-compact.*

(vi) *G/G_0 is σ-compact.*

Proof. (i) and (ii) are equivalent for every locally compact space. The following are obvious: (ii) $\rightarrow$ (iii) $\rightarrow$ (i) $\rightarrow$ (iv) $\rightarrow$ (v) and (iv) $\rightarrow$ (vi). Assume (v), and let $(K_n)_{n\in N}$ be a compact countable cover of G/U. Let $(M_\alpha)_{\alpha\in A}$ be an open, relatively compact cover of G. Then finitely many of the open sets $M_\alpha U$ cover K_n, i.e., K_n is of the form $K'_n U$ with K'_n compact in G. If $(K''_m)_{m\in N}$ is a countable compact cover of U, then $(K'_n K''_m)_{n,m\in N}$ is a countable compact cover of G. Hence (v) $\rightarrow$ (i). As the identity component G_0 of G is compactly generated, it is σ-compact, whence (vi) $\rightarrow$ (v).

LEMMA 2.3. *Let $E(G)$ denote the set of all continuous positive definite functions on G with supremum equal to* 1. *Then the topology of G is the coarsest of all topologies with respect to which the elements of $E(G)$ are continuous.*

A proof can be found in [5]. Its idea is to check that for any symmetric neighborhood V of the identity, a suitable multiple of the convolution square of the characteristic function of V is an element of $E(G)$ with support in V^2.

NOTATION 2.4. Henceforth, G is a group, and ρ and τ are two topologies on G with respect to each of which G is a locally compact group. We denote the locally groups arising this way by G_ρ and G_τ, respectively. Their identity components, considered as abstract groups, are $G_{\rho,0}$ and $G_{\tau,0}$, respectively. If U is a subgroup of G, U_ρ denotes the topological group obtained by inducing ρ on U. If U is a (normal) subgroup of G, G_ρ/U denotes the topological space (topological group) which is the quotient of the topological group G_ρ by U. We write $G_\rho \simeq G_\tau(G_\rho/U \simeq G_\tau/U)$, if G_ρ and $G_\tau(G_\rho/U$ and $G_\tau/U)$ are equal as topological groups (topological spaces).

LEMMA 2.5. *Assume G_ρ and G_τ are both σ-compact groups. Then, if there exists a Hausdorff topology on G coarser than both ρ and τ, $G_\rho \simeq G_\tau$.*

A proof based on the Baire category theorem can be found in [16, p. 58].

LEMMA 2.6. *Again let G_ρ and G_τ be locally compact groups with the same underlying group G such that there exists a Hausdorff topology coarser than both ρ and τ. Let U be a subgroup of G, closed in G_ρ and G_τ such that $U_\rho \simeq U_\tau$ and $G_\rho/U \simeq G_\tau/U$. If U is σ-compact, then $G_\rho \simeq G_\tau$.*

Proof. Let V be an open symmetric neighborhood of e in G_ρ. Then there is a symmetric neighborhood V' of e in G_τ such that $VU = V'U$ and $V \cap U = V' \cap U$. U can be covered by countably many left translates $u_n V \cap U$, $u_n \in U$, $n \in N$ (2.2). Let $V'_n = V' \cup \{u_1, u_1^{-1}\} \cup \{u_2, u_2^{-1}\} \cup \cdots \cup \{u_n, u_n^{-1}\}$. Then $\tilde{V}'_n := \bigcup_m (V'_n)^m$ is an open σ-compact subgroup of G_τ, hence so is $\tilde{V}' := \bigcup_n \tilde{V}'_n$. Let us show that $V \subset \tilde{V}'$. If $v \in V$, there exists $v' \in V'$ and $u \in U$ such that $v = v'u$. Let $u \in u_n V \cap U$. Then $v \in (V'_n)^3 \subset \tilde{V}'$. It is obvious that the group $\tilde{V}$ constructed analogously from V is contained in $\tilde{V}'$, and even that the two groups $\tilde{V}'$ and $\tilde{V}$ coincide. As they are both σ-compact in their respective topologies, they are equal as topological groups (2.5). Hence V and V' are open both in G_ρ and G_τ.

3. Weak equivalence and first properties. Throughout this section we resume the notation 2.4 and add the following: If A is a C^*-algebra, $C(A)$ is the convex cone of positive linear functionals on A with norm not exceeding 1. $E(A)$ is the subset of $C(A)$ of states, i.e., of functionals of norm 1. $P(A)$ is the set of pure states. If T is an irreducible representation of A, $P_T(A)$ denotes the set of (pure) states whose canonically associated representation is unitarily equivalent to T.

$C(A)$, $E(A)$, $P_T(A)$ are furnished with the weak*-topology inherited from A'. The topological spaces so obtained are again denoted $C(A)$, $E(A)$ etc. $C(A)$ is a compact space.

If A is the C^*-algebra $A(G)$ of a locally compact group G, we sometimes write $C(G), \cdots, P_T(G)$ instead and identify these spaces in the usual manner with spaces of continuous positive definite functions on G (cf. [4, §13]). If we want to distinguish the topology ρ of G, we also write $C(G_\rho)$ etc. By a representation of a group, we always mean a unitary representation T in some Hilbert space $\mathscr{H}_T$.

DEFINITION 3.1. The locally compact groups G_ρ, G_τ (see 2.4 above) are called *weakly equivalent*, denoted $G_\rho \sim G_\tau$, if every irreducible representation of G continuous with respect to ρ is also continuous with respect to τ and vice versa.

In view of the canonical correspondence between continuous irreducible representations of a locally compact group G and continuous irreducible positive definite functions on G, we can restate this definition thus: $G_\rho \sim G_\tau$ if and only if $P(G_\rho) = P(G_\tau)$. It is obvious that $\sim$ is an equivalence relation.

LEMMA 3.2. *Let $G_\tau \sim G_\rho$.*

(i) *If U is subgroup of G, open in G_τ, then U is closed in G_ρ, and every continuous irreducible representation of U_τ is continuous with respect to ρ.*

(ii) *If N is a normal subgroup of G closed in G_ρ, then N is also closed in G_τ, and $G_\rho/N \sim G_\tau/N$.*

(iii) *There is a Hausdorff topology on G coarser than both ρ and τ.*

Proof. The C^*-algebra $A(U_\tau)$ of U_τ is a subalgebra of $A(G_\tau)$, and its irreducible representations are the restrictions of irreducible representations of $A(G_\tau)$ ([4, §2.10]). That is, the continuous irreducible representations of U_τ are the restrictions of irreducible continuous representations of G_τ. They are, therefore, continuous on U_ρ.

This argument also shows that the continuous irreducible positive definite functions of U_τ are the restrictions to U of those of G_τ. Consider the convex set $E_U(G_\tau)$ of continuous positive definite functions on G_τ which equal 1 on U. $E_U(G_\tau)$ is obviously compact in the weak*-topology of $A(G_\tau)'$. It contains the characteristic function χ_U of U. By the theorem of Krein-Milman, it is the closed convex hull of its extreme points $P_U(G_\tau)$. Let $f = f_1 + f_2$ be an element of $P_U(G_\tau)$, where the f_i are continuous positive definite on G_τ. Then the restrictions to U of the f_i are proportional, and as f is extreme in $E_U(G_\tau)$, the f_i are proportional. Hence f is an irreducible positive definite function on G_τ, and it follows that the elements of $P_U(G_\tau)$ are ρ-continuous. As $\chi_U \in E_U(G_\tau)$, U is the intersection of the sets $\{s \in G \mid f(s) = 1, f \in P_U(G_\tau)\}$, which are closed in G_ρ. Hence U is closed in G_ρ, whence (i). Statement (ii) says that the continuous irreducible representations of G_ρ or G_τ which contain N in their kernel are coincident, and this is obvious. The coarsest topology σ such that all ρ-continuous irreducible representations of G are σ-continuous meets the requirements of (iii).

COROLLARY. *Let ρ, τ be topologies on G which make it into a locally compact group; and let U be a subgroup of G. If U is open both in G_τ and G_ρ, then*

(i) *$G_\tau \simeq G_\rho$ if and only if $U_\tau \simeq U_\rho$ and*

(ii) *$G_\tau \sim G_\rho$ if and only if $U_\tau \sim U_\rho$.*

4. Weak equivalence with a compact group. We start with some lemmas on compact convex sets. In the first two cases we assume we are given a convex compact subset C of some locally convex vector space together with a complex vector space A of affine-linear continuous functions on C which separates the points of C.

With these data, we define a metric d_A on C by putting $d_A(x, y) = \sup\{|a(x) - a(y)|;\ a \in A, |a| \leqq 1\}$, $|a|$ being the sup-norm of $a \in A$.

DEFINITION. Let K be a subset of C and $\hat{K}$ its closed convex hull. We say that K is *A-uniform*, if the set of finite convex-linear combinations of extreme points of $\hat{K}$ is dense in $\hat{K}$ with respect to the topology defined by d_A.

LEMMA 4.1. *Let P be the set of extreme points of C. If P is the union of countably many disjoint, A-uniform sets $K_n, n \in N$, then C is A-uniform.*

Proof. Let $x \in C$. According to Choquet's theorem ([1, p. 57] or [3, Lemma 28]), there is a probability measure μ on C with barycenter x and concentrated on P. Let χ_n, $n = 1, 2, \cdots$, be the characteristic function of K_n and put $\mu_n = \mu\chi_n$. Then $\sum |\mu_n| = 1$, $|\mu_n|^{-1}\mu_n$ is a probability measure on C, which has a barycenter $x_n \in \hat{K}_n$ [1]. (We may assume that all the μ_n are $\neq 0$, omitting the others.)

Let an $\varepsilon > 0$ be given, $\varepsilon < 1/2$, say.

As K_n is compact and contained in P, it coincides with the set of extreme points of $\hat{K}_n$, hence there is a convex-linear combination y_n of points of K_n such that $d_A(x_n, y_n) < \varepsilon$.

Let n' be such that $1 \geqq m := \sum_{n=1}^{n'} |\mu_n| > 1 - \varepsilon$ and put $y = m^{-1} \sum_{n=1}^{n'} |\mu_n| y_n$. Then y is a finite convex-linear combination of points of P, and for all $a \in A$ with $|a| \leqq 1$ we have

$$\begin{aligned} |a(x) - a(y)| &= \left| \mu(a) - m^{-1} \sum_{n=1}^{n'} |\mu_n| a(y_n) \right| \\ &\leqq \left| \sum_{n=1}^{n'} \mu_n(a) - m^{-1} \sum_{n=1}^{n'} |\mu_n| a(y_n) \right| + \left| \sum_{n=n'+1}^{\infty} \mu_n(a) \right| \\ &\leqq \sum_{n=1}^{n'} |\mu_n| |a(x_n) - m^{-1} a(y_n)| + \varepsilon \leqq \varepsilon + m^{-1}\varepsilon + \varepsilon \leqq 4\varepsilon \, . \end{aligned}$$

That is, $d_A(x, y) \leqq 4\varepsilon$, and the lemma is proven.

4.2. We now return to the notation of 3. Let A be a C^*-algebra and view it in the obvious manner as a vector space of continuous affine-linear functions on $C(A)$. Let T be an irreducible representation of A in some Hilbert space $\mathscr{H}_T$ such that $T(a)$ is a compact operator for all $a \in A$. Then $T(A) = \mathscr{L}\mathscr{C}(\mathscr{H}_T)$, the algebra of all compact operators of $\mathscr{H}_T$ ([4, §4]).

LEMMA. (i) *The closed convex subcone $C_T(A)$ of $C(A)$ spanned by* 0 *and $P_T(A)$ is A-uniform.*

(ii) *If T is finite-dimensional, then $P_T(A)$ is compact, and every element of $C_T(A)$ is a finite linear combination of pure states in $P_T(A)$.*

Proof. (cf. [4, §4]). Let $x \in C_T(A)$. As a pointwise limit of linear functionals vanishing on ker T, x also vanishes on ker T and can be viewed as a positive linear functional on $A/\ker T \simeq \mathscr{L}\mathscr{C}(\mathscr{H}_T)$. Hence it is of the form $x: a \to \sum \lambda_n (T(a)\xi_i \mid \xi_i)$, where $\lambda_i > 0$, $\sum_{i=1}^{\infty} \lambda_i \leqq 1$ and $\{\xi_i\}$ is an orthonormal set of vectors in $\mathscr{H}_T$. Hence x lies in the convex compact cone spanned by 0 and the at-most-countably-many elements $a \to (T(a)\xi_i \mid \xi_i)$ of $P_T(A)$. This cone is A-uniform according to the preceding lemma, i.e., x is in the convex and d_A-closed hull of $P_T(A) \cap \{0\}$. (ii) is now obvious.

4.3. Now let A be the C^*-algebra of the locally compact group G. $C(A)$ is identified as usual with the set of continuous positive definite functions on G with $|x|_\infty = x(e) \leqq 1$.

LEMMA. $d_A(x, y) \geqq |x - y|_\infty$ *for* $x, y \in C(A) \subset L^\infty(G)$. *That is, the topology defined by d_A on $C(A)$ is finer than the topology of uniform convergence on G.*

Proof. $d_A(x, y) = \sup\{(x - y \mid a); a \in A, |a| \leqq 1\}$

$$= \sup \{(x - y \mid \Phi); \Phi \in L^1(G), |\Phi| \leqq 1\}$$
$$\geqq \sup \{(x - y \mid \Phi); \Phi \in L^1(G), |\Phi|_1 \leqq 1\}$$
$$= \sup \{|x(s) - y(s)|; s \in G\} = |x - y|_\infty ,$$

since $L^1(G)$ is dense in A and since the norm $|\Phi|$ of an element Φ of $L^1(G)$ in A does not exceed its norm $|\Phi|_1$ in $L^1(G)$.

LEMMA 4.4. *Let $G_\rho \sim G_\tau$ and suppose G_ρ compact. Let U be a subgroup of G which is a closed G_δ in G_ρ. Then a positive definite function, constant on U, is ρ-continuous if and only if it is τ-continuous.*

Proof. Let ${}_U\hat{G}_\rho$ denote the set of all equivalence classes $\hat{T}$ of continuous irreducible representations of G_ρ such that at least one of the positive definite functions associated to $\hat{T}$ is constant on U, and for every such $\hat{T}$ select a representative T contained in the left regular representation of G_ρ ([11, p. 164]). Furthermore, for every T so obtained select a (continuous) function $\Phi_T \in \mathscr{H}_T \subset L^2(G_\rho)$ of 2-norm 1, such that the corresponding positive definite function x_T: $s \rightarrow (T(s)\phi_T \mid \phi_T)$ is constant on U; i.e., such that ϕ_T is constant on right cosets Ut, $t \in G$. As all the $\mathscr{H}_T$'s are finite-dimensional, it is clear that the dimension of their Hilbert space span in $L^2(G_\rho)$ is equal to the dimension of the span of the ϕ_T in $L^2(G_\rho)$, which again is the cardinality of ${}_U\hat{G}_\rho$[2]. We want to show that this cardinality is countable. Let V_n, $n \in N$, be a sequence of compact neighborhoods of the identity such that $U = \cap UV_n$. Such a sequence exists since U is a compact G_δ. For every $n \in N$, there is a finite number m_n of elements s_n^m of G such that the right translates $UV_ns_n^m$, $m = 1, \cdots m_n$, cover G. Let χ_n^m denote the countably many characteristic functions of the sets $UV_ns_n^m$. If the inner products $(\Phi_T \mid \chi_n^m)$ all vanish for the function Φ_T, then $\Phi_T = 0$. Indeed, if $\Phi_T(s) = 0$ say $\Phi_T(s) > 0$, then $\Phi_T > 0$ in one of the sets $UV_ns_n^m$, and $(\Phi_T \mid \chi_n^m) > 0$. That is, we have shown that ${}_U\hat{G}_\rho$ is countable, or else, that there are at most countably many $\hat{T}$'s in $\hat{G}_\rho$ with one of the associated positive definite functions constant on U. As $G_\rho \sim G_\tau$, the same holds for G_τ.

Now let $C'_U(G_\rho)$ denote the set of all ρ-continuous positive definite functions on G, constant on U and of norm not exceeding 1. $C'_U(G_\rho)$ is a convex cone, compact in the weak*-topology of $A(G_\rho)'$. It is clearly contained in the compact convex cone $C_U(G_\rho)$ spanned by 0 and the sets $P_T(G_\rho)$, $\hat{T} \in {}_U\hat{G}_\rho = {}_U\hat{G}_\tau$. Lemma 4.2 tells that these sets are $A(G_\rho)$-uniform; according to Lemma 4.1 and the preceding argument, $C_U(G_\rho)$

[2] This is strictly true only if ${}_U\hat{G}_\rho$ is infinite. If it is finite, there is nothing to prove.

is $A(G_\rho)$-uniform. With Lemma 4.3 we can state: every ρ-continuous positive definite function x on G, constant on U, is a uniform limit on G of convex-linear combinations of irreducible ρ-continuous positive definite functions on G. But the latter are also τ-continuous, hence so is x. Interchanging ρ and τ in this argument, we see that, indeed, if x is τ-continuous and constant on U, it is also ρ-continuous.

COROLLARY. *If $G_\tau \sim G_\rho$ and G_ρ is compact and metrizable, then $G_\rho \simeq G_\tau$. In fact, ρ is the coarsest of all topologies on G with respect to which the elements of $P(G_\rho)$ are continuous.*

For then $\{e\}$ is a G_δ, and Lemma 2.3 yields the result.

PROPOSITION 4.5. *If $G_\tau \sim G_\rho$, G_ρ is compact, and U is an open subgroup of G_τ, then there exists a smallest open subgroup $\check{U}$ of G_ρ containing U.*

Proof. Let $\mathscr{U}$ be the set of all open subgroups of G_ρ containing U, ordered by inclusion. Let $(U_\alpha)_{\alpha \in A}$ be a chain of distinct elements in $\mathscr{U}$. Then A is at most countable. Indeed, the mapping $\alpha \longrightarrow$ (index of U_α in G) is one-to-one from A into N. Let $U' = \cap U_\alpha$. Then U' is a compact G_δ, open in G_τ, as it contains U. The characteristic function of U' is constant on U, positive definite, and τ-continuous. Hence it is ρ-continuous (4.4), and U' is ρ-open. We have shown that $\mathscr{U}$ is inductively ordered and hence has a smallest element, $\check{U}$.

5. Weakly equivalent topologies on G, one of which is the topology of a Lie group.

PROPOSITION. *If G_ρ is a Lie group and $G_\tau \sim G_\rho$, then $G_\tau \simeq G_\rho$.*

Proof. As $G_\rho/G_{\rho,0}$ is discrete and weakly equivalent to $G_\tau/G_{\rho,0}$, Proposition 1 yields that $G_{\rho,0}$ is open in G_τ also. Once we know that $G_{\rho,0,\rho} \simeq G_{\rho,0,\tau}$, the proposition is established (cf. 2.5). That is, we may assume without restriction that G_ρ is connected (cf. Corollary 3.2).

$G_{\tau,0}$ is a closed subgroup of G_ρ, hence σ-compact with respect to both ρ and τ (Lemma 2.2 iv)). In view of Lemmas 3.2, 2.5, and 2.6, we need only show that $G_\tau/G_{\tau,0} \simeq G_\rho/G_{\tau,0}$. That is, we may assume additionally that G_τ is totally disconnected.

Let U be a compact open subgroup of G_τ. According to Lemma 3.2 (i), U is a closed subgroup of G_ρ, hence U_ρ is a Lie subgroup of G_ρ, and, therefore, according to Lemma 2.5, discrete in both ρ and τ. Hence G_τ is discrete, and Proposition 1 yields $G_\rho \simeq G_\tau$.

6. Weakly equivalent totally disconnected topologies on G.

LEMMA 6.1. *Let $G_\tau \sim G_\rho$, τ finer than ρ, and ρ totally disconnected. Then $G_\tau \simeq G_\rho$.*

Proof. There is a compact open subgroup of G_ρ, open also in G_τ as $\tau \supseteqq \rho$, so Corollary 3.2 tells that it is no restriction to assume that G_ρ is compact.

G_τ is totally disconnected, too, and hence has a basis at e of open compact subgroups. Let U be one of them. There exists a smallest open subgroup $\check{U}$ of G_ρ containing U (Proposition 4.5). Now G_ρ has a basis at e of open normal subgroups, $\mathscr{N}$. We have $\check{U} \subset UN$ for all $N \in \mathscr{N}$, hence $\check{U} \subset \cap NU = U$: U is also open in G_ρ, and $G_\tau \simeq G_\rho$.

LEMMA 6.2. *Let $G_\tau \sim G_\rho$, $\tau \supseteqq \rho$, and G_τ totally disconnected. Then $G_\rho \simeq G_\tau$.*

Proof. There is an open subgroup of G_ρ, open also in G_τ, which is almost connected. According to Corollary 3.2 it is no restriction to assume that G_ρ is almost connected. We want to show that G_ρ is also totally disconnected, thus reducing the lemma to the preceding one. Now for every neighborhood V of e in G_ρ there is a compact normal subgroup N in V such that G_ρ/N is a Lie group ([14, p. 175]). We have $G_\tau/N \sim G_\rho/N$ (Lemma 3.2), hence $G_\tau/N \simeq G_\rho/N$ (Proposition 54). But G_τ/N is totally disconnected (cf. 2.1), hence discrete, i.e., N is open in G_ρ. As $G_{\rho,0}$ is the intersection of all open subgroups of G_ρ, $G_{\rho,0} = \{e\}$.

PROPOSITION 6.3. *Let $G_\rho \sim G_\tau$, with G_ρ totally disconnected and σ-compact. Then $G_\tau \simeq G_\rho$.*

Proof. Let U be an open compact subgroup of G_ρ. Then U is closed in G_τ, (Lemma 3.2), and a countable number of left translates of U covers G_τ (Lemma 2.2 (iii)). According to the Baire category theorem, one of them and therefore all of them are open in G_τ. Letting U run through a basis at e of G_ρ, we see that τ is finer than ρ. The preceding lemma yields the result.

7. Comparable topologies on G.

PROPOSITION. *Let $G_\tau \sim G_\rho$ and $\tau \supseteqq \rho$. Then $G_\tau \simeq G_\rho$.*

Proof. $G_{\tau,0}$ is contained in $G_{\rho,0}$ and hence (Lemma 2.2 (iv)) it is σ-compact with respect to both topologies. Thus $G_{\tau,0,\tau} \simeq G_{\tau,0,\rho}$, and in view of Lemma 2.6, we may assume G_τ to be totally disconnected. That is, the proposition is reduced to Proposition 6.2.

8. σ-compact topologies.

PROPOSITION. *Let G_ρ be σ-compact and $G_\rho \sim G_\tau$. Then $G_\rho \simeq G_\tau$.*

Proof. We have $G_\tau/G_{\rho,0} \simeq G_\rho/G_{\rho,0}$ according to 6.3. This shows that an open almost connected subgroup of G_ρ is also open in G_τ, and we may assume G_ρ to be almost connected (cf. 3.2). Let V be a neighborhood of e in G_ρ and N a compact normal subgroup such that G_ρ/N is a Lie group. Then $G_\tau/N \simeq G_\rho/N$ (cf. 5.5) and, in particular $G_\tau/NG_{\tau,0} \simeq G_\rho/NG_{\tau,0}$. Section 2.1 shows that $G_\tau/NG_{\tau,0}$ is discrete, hence $G_{\rho,0} \subset NG_{\tau,0}$. Since V can be chosen arbitrarily small, $G_{\rho,0} \subset G_{\tau,0}$. As in the proof of Proposition 7.1, we divide by $G_{\rho,0}$ and thus reduce the proof to Proposition 6.3.

9. Summary.

THEOREM. *Let G be a locally compact group with respect to two topologies. If the irreducible unitary representations continuous with respect to either topology coincide, then so do the topologies in the following cases:*

(i) *The topogies are comparable;*

(ii) *There exists a normal subgroup N of G, open and σ-compact in one of the topologies.*

Indeed, (i) is Proposition 7, while (ii) follows from Proposition 8 and Corollary 3.2, since N is, in fact, open in both topologies (Proposition 1).

The class of groups covered by (ii) is slightly larger than the class of "manageable" groups treated by Greenleaf [7], Leptin [11], [12], and others in their investigations of properties equivalent to amenability.

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