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A CHARACTERIZATION OF A CLASS OF UNIFORM SPACES THAT ADMIT AN INVARIANT INTEGRAL

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In this paper we consider a class of uniform spaces that we temporarily call equihomogeneous spaces. These spaces were first considered by Y. Mibu in a paper On Measures Invariant Under Given Homeomorphism Groups of A Uniform Space. The reason for considering equihomogeneous spaces is that one can easily show the existence of a Haar type integral on them just by using an obvious modification of a standard existence type proof for the Haar integral on locally compact topological groups. We show that these spaces coincide in the locally compact case with the class of uniformly locally compact spaces considered by I. E. Segal in his paper Invariant Measures on Locally Compact Spaces that appeared in 1949. Our main theorem is that a locally compact equihomogeneous space is a locally compact topological homogeneous space and hence is a quotient of locally compact topological groups. We are therefore able to use the theory of A. Weil to deduce existence and uniqueness of an invariant integral for these spaces. These results seem to explain why no examples of spaces, satisfying Segal's or Mibu's conditions, aside from topological groups and their quotients have been found to date.

1. Introduction and definitions. We first became aware of the existence of an invariant measure on certain locally compact uniform spaces when we were looking for conditions for existence of a continuous measure on such spaces. It was natural to attempt to imitate the existence proof for the Haar integral in locally compact groups in the more general case of uniform spaces. We found that an obvious modification of a proof found in Hewitt and Ross [1] worked in the case of homogeneous uniform spaces provided two conditions were satisfied by the group of homeomorphisms acting on the space.

Afterwards we became aware of two papers, one by I. E. Segal [5], which seemed to be more general in nature then our observation, and the other by Y. Mibu [3], which considered exactly the conditions we considered. Both papers used a set theoretic-measure theoretic approach adopted first by Haar, while we used a functional analytic approach first considered by A. Weil. The problem with these results, is that no examples of uniform spaces have been found satisfying these conditions except, for the obvious ones. We will show in this paper that the obvious spaces are the only ones.

Let $(X, \mathscr{U})$ be a uniform space.

DEFINITION 1.1. A function $f: X \to X$ is nonexpansive with respect to a base $\mathscr{B}$ for $\mathscr{U}$ if for each $U \in \mathscr{B}$ and $(x, y) \in U$, the relation $(f(x), f(y)) \in U$, also holds.

DEFINITION 1.2. By a $\mathscr{B}$-nonexpansive homeomorphism f of a uniform space $(X, \mathscr{U})$ onto itself, we mean a homeomorphism f of X onto itself such that f is nonexpansive with respect to a base $\mathscr{B}$ for the uniformity $\mathscr{U}$.

DEFINITION 1.3. A uniform space $(X, \mathscr{U})$ will be called an equihomogeneous space if there is a group G of homeomorphisms acting on X such that (i) G is transitive (i.e., given $p, q \in X$, there is $g \in G$ such that $gp = q$), and (ii) there is a base $\mathscr{B}$ for $\mathscr{U}$ such that G is a group of $\mathscr{B}$-nonexpansive homeomorphisms of the uniform space.

COROLLARY 1.4. (i) *Each* $g \in G$ *satisfies* $gU[x] = U[gx]$. ($U[x] = \{y: (x, y) \in U\}$). (ii) *Each* $g \in G$ *is uniformly continuous hence a unimorphism.*

Convention. From now on we will assume that $\mathscr{B}$ is a symmetric base for $\mathscr{U}$; that is, if $U \in \mathscr{B}$ then $U^{-1} = U$(i.e., if $(x, y) \in U$ than $(y, x) \in U$). Our notation will be the standard notation of Kelley [2].

THEOREM 1.5. *Let* G *be a group of homeomorphisms acting on the uniform space* $(X, \mathscr{U})$, *then the following are equivalent*:

(i) *there is a base* $\mathscr{B}$ *for* $\mathscr{U}$ *such that* G *is a group of* $\mathscr{B}$-*nonexpansive homeomorphisms of* $(X, \mathscr{U})$.

(ii) G *is an equicontinuous group of unimorphisms on the uniform space* $(X, \mathscr{U})$.

Proof. (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (i) Let $g^\#$ be the map induced on the structure space by the element $g \in G$ via the relation

$$g^\#(x, y) = (gx, gy) .$$

Let $U \in \mathscr{U}$. Since G is equicontinuous, there exists $V \in \mathscr{U}$ such that $g^\# V \subset U$, for all $g \in G$. Let

$$U^\# = \bigcup_{g \in G} g^\# V \subset U .$$

For each $U \in \mathscr{U}$ select one $U^\#$ as above. Then $\mathscr{B} = \{U^\#: U \in \mathscr{U}\}$ is a base for $\mathscr{U}$ and G is nonexpansive relative to $\mathscr{B}$. To see this, observe that if $g_0 \in G$, then

$$g_0^\# U^\# = g_0^\# \bigcup_{g \in G} g^\# V = \bigcup_{g \in G} g_0^\# g^\# V = \bigcup_{g \in G} g^\# V = U^\# .$$

NOTATION. If $(X, \mathscr{U})$ is an equi-homogeneous space then we shall use the notation $(X, G, \mathscr{U}, \mathscr{B})$ to indicate that G is a group of $\mathscr{B}$-nonexpansive homeomorphisms acting on $(X, \mathscr{U})$.

THEOREM 1.6. *If $(X, G, \mathscr{U}, \mathscr{B})$ is locally compact, then $(X, \mathscr{U})$ is uniformly locally compact.*

Proof. Let $p \in X$. Then there is $U \in \mathscr{B}$ such that $\bar{U}[p]$ is compact. But then $g\bar{U}[p] = \bar{U}[gp]$ is compact for each $g \in G$. Since G is transitive, $\bar{U}[x]$ is compact for each $x \in X$, so $(X, \mathscr{U})$ is uniformly locally compact.

REMARK. This last theorem shows that the conditions in Segal's paper may be weakened. That is, local compactness may be substituted for uniform local compactness.

2. $(X, G, \mathscr{U}, \mathscr{B})$ is a **topological homogeneous space.** Let $(X, G, \mathscr{U}, \mathscr{B})$ be an equi-homogeneous locally compact space. Let $e \in G$ denote the identity of G (and of X). We introduce a topology τ on G as follows: The basic open sets around $e \in G$ are of the form

$$\mathscr{A}(U, F) = \{g \in G: \textit{for} \text{ each } p \in F,\ gp \in U[p]\}$$

where $F \subset X$ is compact and $U \in \mathscr{B}$.

REMARK. This topology was introduced by Y. Mibu in [3], where he showed that these sets were a base for a topology on G. We remark that he did not show that these sets are actually open in the topology. We will show that each of the sets $\mathscr{A}(U, F)$ are open, and give a different proof that they form a base for a topology on G.

LEMMA 2.1. *Let $g \in \mathscr{A}(U, F)$. Then there is $V \in \mathscr{B}$ such that*

$$V[gp] \subset U[p], \textit{ for all } p \in F .$$

Proof. For $x \in X$, define $\psi(x, x) = (x, g(x))$. Then ψ is a continuous map of X into $X \times X$; and $\psi(F \times F) \subset U$. Since ψ is continuous, there is $V \in \mathscr{B}$ such that $\psi(F \times V[F]) \subset U$. Thus

$$\{(x, gp): x \in F,\ p \in V[F]\} \subset U .$$

In particular, we have $(x, gp) \in U$ for all $p \in V[x]$, $x \in F$. Thus $gp \in U[x]$

for all $p \in V[x]$, and hence

$$V[gx] = gV[x] \subset U[x],\ x \in F\ .$$

THEOREM 2.2. *(G, τ) is a Hausdorff topological group.*

Proof. To show that (G, τ) is topological group we need to verify (i)—(v) of Theorem 4.5 in Hewitt and Ross [1]. The obvious identity

$$\{e\} = \cap \{\mathscr{A}(U, F) \colon U \in \mathscr{B},\ F \text{ compact}\}$$

implies that (G, τ) is Hausdorff.

(i) Given $\mathscr{A}(U, F)$ there is $\mathscr{A}(W, F')$ such that $[\mathscr{A}(W, F')]^2 \subset \mathscr{A}(U, F)$.

Without loss of generality, we may assume that $\bar{U}[p]$ is compact for all $p \in X$. Since F is compact, it follows that there is a finite collection $p_1, \cdots, p_n \in F$ such that

$$F' = \bigcup_{i=1}^{n} U[p_i] \supset F\ .$$

Since F' is a neighborhood of F, there is $V \in \mathscr{B}$, $V \subset U$, such that $V[F] \subset F'$ (Kelley [2], Theorem 33, p. 199). Let $W \in \mathscr{B}$ be such that $W \subset V$, and $W \circ W \subset V$.

Let $g, g' \in \mathscr{A}(W, F')$. Then for each $p \in F$ we have

$$gp \in W[p] \subseteqq W[F] \subseteqq F'\ .$$

Thus it follows from the definition of $\mathscr{A}(W, F')$ that

$$g'g[p] \in W[gp] \subset W \circ W[p] \subset V[p] \subset U[p]\ .$$

But then $g'g \in \mathscr{A}(U, F)$.

(ii) Given $\mathscr{A}(U, F)$, there is $\mathscr{A}(V, F')$ such that $\mathscr{A}(V, F')^{-1} \subset \mathscr{A}(U, F)$.

This is clear because $\mathscr{A}(U, F) = \mathscr{A}(U, F)^{-1}$. This last identity follows from the observation that if $gp \in U[p]$ then

$$p = g^{-1}gp \in U[g^{-1}p]\ .$$

Since U is symmetric $g^{-1}p \in U[p]$.

(iii) For each $\mathscr{A}(U, F)$ and $g \in \mathscr{A}(U, F)$, there is $\mathscr{A}(V, F')$ such that

$$g\mathscr{A}(V, F') \subset \mathscr{A}(U, F)\ .$$

Let $g \in \mathscr{A}(U, F)$. Then by Lemma 2.1, there is V such that $V[gp] \subset U[p]$, for all $p \in F$. Clearly if $g_0 \in \mathscr{A}(V, F)$, then $g_0p \in V[p]$, if $p \in F$ so that

$$gg_0p \in gV[p] = V[gp] \subset U[p] \, .$$

Thus $g\mathscr{A}(V, F) \subset \mathscr{A}(U, F)$.

(iv) For each $\mathscr{A}(U, F)$ and each $g \in G$, there is a $\mathscr{A}(V, F')$ such that $g^{-1}\mathscr{A}(V, F')g \subset \mathscr{A}(U, F)$.

If we take $F' = gF$, and let $U = V$, then

$$\begin{aligned} g^{-1}\mathscr{A}(U, gF)g &= \{g^{-1}g_0g \colon g_0[p] \in U[p], \text{ for all } p \in gF\} \\ &= \{g^{-1}h_0 \colon h_0[p] \in U[gp] \text{ for all } p \in F\} \\ &= \{k_0 \colon k_0[p] \in U[p] \text{ for all } p \in F\} \\ &= \mathscr{A}(U, F) \, , \end{aligned}$$

so (iv) is satisfied.

(v) If $\mathscr{A}_1 = \mathscr{A}(U_1, F_1)$, $\mathscr{A}_2 = \mathscr{A}(U_2, F_2)$ there is $\mathscr{A}(U_3, F_3) \subset \mathscr{A}_1 \cap \mathscr{A}_2$.

Just choose $U_3 \subset U_1 \cap U_2$, $F_3 = F_1 \cup F_2$.

REMARK. Condition (iii) implies that each $\mathscr{A}(U, F)$ is open, since there is a neighborhood of each of its points contained properly in it.

LEMMA 2.3. *The map $G \times X \rightarrow X$ defined by $(g, x) \mapsto gx$ is continuous.*

Proof. Let $U \in \mathscr{B}$ and let $x \in X$ be arbitrary. Then let $V \in \mathscr{B}$ be such that $V \circ V \subset U$. Let $g \in \mathscr{A}(V, \{x\})$. Then

$$gV[x] = V[gx] \subset V \circ V[x] \subset U[x] \, .$$

Thus $\mathscr{A}(V, \{x\})V[x] \subset U[x]$.

More generally, consider $U[gx]$. Let $V \in \mathscr{B}$ be such that $V \circ V \subset U$. Then

$$[g\mathscr{A}(V, \{x\})]V[x] \subset U[gx] \, ,$$

since if $g_0 \in \mathscr{A}(V, \{x\})$, then

$$gg_0V[x] = gV[g_0x] \subset gV \circ V[x] \subset gU[x] = U[gx] \, .$$

LEMMA 2.4. *If $W \subset G$ is open, and $C \subset X$, then $gWg^{-1}C$ is homeomorphic with WC $(g \in G)$.*

Proof. Let $\psi_g(g_0) = gg_0g^{-1}$, $g_0 \in G$. Then ψ_g is an inner automorphism and homeomorphism of G onto itself. Let $f \colon WC \rightarrow gWg^{-1}C$ *be* defined by

$$f(g_0x) = [\psi_g(g_0)](x) \, .$$

Then f is $1-1$ and onto, since if $gg_0g^{-1}x = gg_0g^{-1}y$ then $x = y$ since $gg_0g^{-1} \in G$, and hence $g_0x = g_0y$.

We show now that f is continuous. Let $U \in \mathscr{B}$ and consider

$U[fg_0x]$. By continuity of the map $G \times X \to X$, there is a neighborhood $W_0 \subset G$ of fg_0 and a neighborhood $V \in \mathscr{B}$ such that

$$W_0 V[x] \subset U[fg_0x] .$$

Since ψ_g is continuous, there is $V_0 \subset G$ containing g_0 such that $\psi_g(V_0) \subset W_0$. But then

$$f(V[g_0x]) = f(g_0 V[x]) = \psi_g(g_0) V[x] \subset [\psi_g(V_0)] V[x] \subset U[fg_0x] ,$$

so that f is continuous.

We observe that f^{-1} is defined since f is $1-1$ and onto. Also f^{-1} is defined by $f^{-1}g_0x = \psi_g^{-1}(g_0x)$. Here $\psi_g^{-1} = \psi_{g^{-1}}$ since

$$g_0 = \psi_g^{-1}\psi_g(g_0) = \psi_g^{-1}[gg_0g^{-1}] = g^{-1}gg_0g^{-1}g = \psi_{g^{-1}} \circ \psi_g(g_0) .$$

Thus a similar proof shows that f^{-1} is continuous so that f is a homeomorphism.

Y. Mibu in [3], proved the following powerfull theorem.

THEOREM 2.5. *Let $W \subset G$ be open. Then W is totally bounded iff for each compact $F \subseteqq X$ the set $W(F)$ is totally bounded.*

In the conditions for this theorem Mibu states that X is σ-compact (σ-bounded), however he does not use this fact in the proof. It is a corollary of this theorem, and of Lemma 2.4 that if X is locally compact, then G is locally bounded. Here we say that G is locally bounded if each $g \in G$ has a totally bounded neighborhood. Our proof is a modification of a proof in Mibu's paper which shows that if X is locally compact and connected then G is locally bounded.

NOTATION. Let $\mathscr{A}(U, x) \equiv \mathscr{A}(U, \{x\})$, $x \in X$.

THEOREM 2.6. *If X* is *locally compact, then G is locally bounded.*

Proof. Let $W_0 = \mathscr{A}(U, p_0)$ where $p_0 \in X$ is fixed, and $\bar{U}[p]$ is compact for all $p \in X$. We observe that $U^n[p_0]$ is totally bounded for each $n \geqq 1$. Let C be compact, $C \subset \bigcup_{n=1}^{\infty} U^n[p_0]$. Since C is compact, there is n_0 such that

$$C \subset \bigcup_{k=1}^{n_0} U^k[p_0]. \quad (U^k = U \circ U^{k-1}, k = 2, 3, \cdots, n_0) .$$

Now each $g \in W_0$ satisfies $gC \subset \bigcup_{k=1}^{n_0+1} U^k[p_0]$. Therefore,

$$W_0[C] \subset \bigcup_{k=1}^{n_0+1} U^k[p_0] ,$$

and so $W_0[C]$ is totally bounded. Observe that $\Omega = \bigcup_{n=1}^{\infty} U^n[p_0]$ is open and closed in X.

Let $C \subset X$ be compact. Then there are a finite number of elements of G, say $g_1, \cdots, g_k$ such that $C \subset \bigcup_{n=1}^{k} g_n \Omega$. Observe that each $C_n = C \cap g_n\Omega$ is compact, since each $g_n\Omega$ is open and closed.

By Lemma 2.4, W_0C_n is homeomorphic to $g_n W_0 g_n^{-1} C_n$. Now $g_n^{-1}C_n = C'_n \subset \Omega$ and is compact, so the above argument shows that $W_0 g_n^{-1} C_n$ is totally bounded. But then so is $g_n W_0 g_n^{-1} C_n$ and also W_0C_n. It is now clear that

$$W_0C = \bigcup_{n=1}^{k} (W_0C_n)$$

is totally bounded. Since C was arbitrary in X, it follows from Theorem 2.5 that W_0 is totally bounded.

THEOREM 2.7. *Let F be compact and $U \in \mathscr{B}$. Then $\mathscr{A}(U, F)x$ is open for each $x \in X$.*

Proof. We observe first that if $U \in \mathscr{B}$, $x \in X$, then

$$\mathscr{A}(U, x)x = U[x] .$$

This is because if $y \in U[x]$ then there exists $g \in \mathscr{A}(U, \{x\})$ such that $gx = y$. Also if $g \in \mathscr{A}(U, x)$ then $gx \in U[x]$.

Now if $y \in X$ is arbitrary and $g \in G$ is such that $gx = y$ then

$$\begin{aligned} g\mathscr{A}(U, x)g^{-1} &= \{gg_0g^{-1}: g_0x \in U[x]\} \\ &= \{gh_0: h_0gx \in U[x]\} \\ &= \{k_0: k_0gx \in U[gx]\} = \mathscr{A}(U, gx) . \end{aligned}$$

Therefore $g\mathscr{A}(U, x)g^{-1}y = \mathscr{A}(U, y)y = U[y]$, which is open. But then $\mathscr{A}(U, \{x\})y$ is homeomorphic to $U[y]$, by Lemma 2.4, so it is open.

Let $\mathscr{A}(U, F)$ be arbitrary, where F is compact, $U \in \mathscr{B}$. Choose V such that $V \circ V \circ V \subset U$. Then the collection $\{V[x]: x \in F\}$ is an open cover of F and so is reducible to a finite subcover $V[x_1], \cdots, V[x_k]$ of F. Consider now $\bigcap_{i=1}^{k} \mathscr{A}(V, \{x_i\})$. Let $g \in \cap \mathscr{A}(V, \{x_i\})$. Then if $x \in F$ we have $x \in V[x_i]$, for some i. But then

$$gx \in V[gx_i] \subset V \circ V[x_i] \subset V \circ V \circ V[x] \subset U[x] ,$$

so that $g \in \mathscr{A}(U, F)$. Thus

$$\bigcap_{i=1}^{k} \mathscr{A}(V, \{x_i\}) \subset \mathscr{A}(U, F) .$$

Now $\bigcap_{i=1}^{k} \mathscr{A}(V, \{x_i\})\, x$ is open, so that $\mathscr{A}(U, F)x$ is a neighborhood of x. Let $y \in \mathscr{A}(U, F)x$, then $y = gx$ for some $g \in \mathscr{A}(U, F)$.

By part (iii) of Theorem 2.2 there is a neighborhood $\mathscr{A}(V, F')$ such that

$$g\mathscr{A}(V, F') \subset \mathscr{A}(U, F) .$$

Clearly $gx \in g\mathscr{A}(V, F')x \subset \mathscr{A}(U, F)x$, and $g\mathscr{A}(V, F')x$ is a neighborhood of gx since $\mathscr{A}(V, F')x$ is a neighborhood of x. This shows that y is in the interior of $\mathscr{A}(U, F)x$, so that $\mathscr{A}(U, F)x$ is open.

COROLLARY 2.8. *The sets* $\mathscr{A}(U, x)$, $U \in \mathscr{B}$, $x \in X$, *are a subbasis for the topology of* G.

COROLLARY 2.9. (X, G, τ) *is a topological homogeneous space.*

Proof. This follows from the definition of a topological homogeneous space (see Nachbin [4] page 128).

COROLLARY 2.10. *Let* $p \in X$. *Let* $G_p \subset G$ *be the group of homeomorphisms in* G *leaving* p *fixed. Then* X *is topologically isomorphic* (homeomorphic as a homogeneous space) *with* G/G_p.

Proof. This is Proposition 1 on page 133 of Nachbin [4].

3. The connection with the Weil theory of homogeneous spaces. We may observe that in the proofs of the above theorems we only assumed two facts, namely:

(a) G was transitive

(b) If $g \in G$ then $gU[x] = U[gx]$ for each $x \in X$.

The transitivity of G was used in 2.6 and 2.7. It is clear therefore if we extend G to a larger group G' satisfying (b) (and automatically (a)) that the theory goes through as in §2, so that again X is topologically isomorphic with G'/G'_p.

LEMMA 3.1. *If* $\{g_\alpha\}$ *is a Cauchy net in* G *then* $\{g_\alpha x\}$ *converges for each* $x \in X$.

Proof. Since $\{g_\alpha\}$ is Cauchy, it follows that there exists $\alpha_0 = \alpha_0(U, F)$ such that if $\alpha, \beta \geqq \alpha_0$ then $g_\beta^{-1} g_\alpha \in \mathscr{A}(U, F)$. Thus $g_\alpha \in g_{\alpha_0}\mathscr{A}(U, F)$ if $\alpha \geqq \alpha_0$. If $x \in X$, then $y_\alpha = g_\alpha x \in g_\alpha \mathscr{A}(U, F)x$, a totally bounded set. This means that there is a subnet $y_{\varphi(\gamma)}$ converging to some point $y \in g_{\alpha_0}\overline{\mathscr{A}(U, F)x}$. We claim that $y_\alpha \to y$. To see this, consider $V[y]$. Let W be such that $W \circ W \subset V$. Now $\mathscr{A}(W, y)W[y] \subset V[y]$. Thus there is α_1 such that if $\alpha, \beta \geqq \alpha_1$, then $g_\alpha g_\beta^{-1} \in \mathscr{A}(W, y)$, since $\{g_\alpha\}$ is Cauchy. Also there is γ_1 such that if $\gamma \geqq \gamma_1$ then $g_{\varphi(\gamma)}x = y_{\varphi(\gamma)} \in W[y]$. Choose $\alpha > \alpha_1$ and γ_2 such that $\gamma_2 \geqq \gamma_1$, $\varphi(\gamma_2) \geqq \alpha_1$. Then if $\gamma \geqq \gamma_2$ and $\alpha > \alpha_1$ we have

$$g_\alpha x = g_\alpha g_{\varphi(\gamma)}^{-1} g_{\varphi(\gamma)} x \in \mathscr{A}(W, y) W[y] \subset V[y] .$$

Since $V \in \mathscr{B}$ was arbitrary, $g_\alpha x \to y$. The above shows that the limit of each Cauchy sequence of functions is a well defined function of X into X.

NOTATION. Let $\bar{G} = \{g\colon g_\alpha \to g,\, g_\alpha \in G,\, \{g_\alpha\}$ a Cauchy sequence in $G\}$. Then $\bar{G}$ is the Weil completion of G.

THEOREM 3.2. (Weil) *$\bar{G}$ is locally compact if G is locally bounded.*

THEOREM 3.3. *Let $\bar{G}$ be the Weil completion of G, then each $g \in \bar{G}$ is a homeomorphism of X satisfying $gU[x] = U[gx]$.*

Proof. (a) g is $1-1$.

If $x \neq y$, there is $V \in \mathscr{B}$ such that $x \notin V[y]$. Let $W \subset V$ be such that $W \circ W \subset V$. Then $W[x] \cap W[y] = \emptyset$, and so $W[g_\alpha x] \cap W[g_\alpha y] = \emptyset$ for all α. If $gx = gy$, then $g_\alpha x \in W[gx]$ and $g_\alpha y \in W[gx]$ for $\alpha \geqq \alpha_0$ (for some α_0). But then $gx \in W[g_\alpha y] \cap W[g_\alpha x]$, a contradiction.

(b) g is onto because $g^{-1}x = \lim_\alpha g_\alpha^{-1} x$ is defined for all $x \in X$. See Kelley [2], page 212, the note after Exercise $Q(d)$.

(c) g is continuous and open and $gU[x] = U[gx]$.

We shall show that $gU[x] \subset \bar{U}[gx]$, for each $U \in \mathscr{B}$, and $x \in X$. Thus if $V \subset U$ is such that $\bar{V}[gx] \subset U[gx]$, then $gV[x] \subset U[gx]$, so that g is continuous. Similarly g^{-1} is continuous, so that g is open. But then $gU[x] \subset U[gx]$, and $g^{-1}U[x] \subset U[g^{-1}x]$. Thus

$$U[x] \subset g^{-1}U[gx] \subset U[x] ,$$

and therefore $gU[x] = U[gx]$.

To see that $gU[x] \subset \overline{U[gx]}$, for each $U \in \mathscr{B}$, and $x \in X$, let $y \in U[x]$, for some $x \in X$, $U \in \mathscr{B}$. Then choose $V \subset U$ such that $\bar{V}[y] \subset U[x]$. Since $g_\alpha y \to gy$, there is α_0, such that $\alpha > \alpha_0$ implies that $g_\alpha y \in V[gy]$. But then $gy \in V[g_\alpha y] \subset U[g_\alpha x]$. Therefore $g_\alpha x \in U[gy]$ for $\alpha > \alpha_0$. This shows that $gx \in \bar{U}[gy]$ so $gy \in \bar{U}[gx]$.

COROLLARY 3.4. *The space $(X, \bar{G})$ is a locally compact topological homogeneous space, where $\bar{G}$ is the Weil completion of G. Thus X is homeomorphic as a homogeneous space to $\bar{G}/\bar{G}_p$, where $\bar{G}_p \subset \bar{G}$ is the stability group of $p \in X$* (i.e., $\bar{G}_p = \{g \in \bar{G}\colon gp = p\}$.).

Proof. $\bar{G}$ satisfies conditions (a) and (b) of the beginning of this section, so this Corollary follows from 2.9 and 2.10.

DEFINITION 3.5. Let G be a locally compact topological group with

Haar measure λ. The modular function $\Delta_r^G(s)$ is defined by

$$\Delta_r^G(s) = \frac{\int_G f(xs^{-1})d\lambda(x)}{\int_G f(x)d\lambda(x)}, \quad f \in C_{00}^+(G) .$$

DEFINITION 3.6. A positive integral $\mu \neq 0$ on a locally compact group G is relatively left invariant if for every $s \in G$, there is a real number $\Delta(s) > 0$ such that $\mu(f \circ s) = \Delta(s)\mu(f)$ for every $f \in C_{00}^+(G) = \{f: f \geqq 0$, and f has compact support$\}$. μ is invariant if $\Delta \equiv 1$.

The following theorems are well-known and may be found in Nachbin [4]. In these theorems the groups G, H are locally compact.

THEOREM 3.7. *Δ_r^G is a continuous homomorphism of G into the multiplicative group $\boldsymbol{R}_+^*$ of the strictly positive real numbers.*

THEOREM 3.8. (Weil) *The following are equivalent for a topological homogeneous space (X, G):*

(i) *There exists a positive integral $\mu \neq 0$ on X, which is relatively invariant under G and having modulus Δ, where Δ: $G \rightarrow \boldsymbol{R}_+^*$ is a continuous homomorphism.*

(ii) *$\Delta_r^H(t) = \Delta(t)\Delta_r^G(t)$, for all $t \in H$. Here $H = G_p$ for some fixed $p \in X$.*

If either (i) *or* (ii) *occurs then μ is unique up to a multiplicative constant.*

THEOREM 3.9. (Weil) *In order that there exist at least one invariant positive integral $\mu \neq 0$ on X, it is necessary and sufficient that $\Delta_r^H(t) = \Delta_r^G(t)$, for $t \in H$.* (Here $H = G_p$, for some fixed $p \in X$).

Note. Theorem 3.8 can be found in [4], page 138, and Theorem 3.9 in [4], page 140.

NOTATION. Let $W(G) = \bar{G} =$ Weil completion of G, and let $H = \bar{G}_p = [W(G)]_p$ for some fixed $p \in X$.

LEMMA 3.10. *$\Delta_r^{W(G)}$ is constant on H. In fact $\Delta_r^{W(G)}(H) = \{1\}$.*

Proof. We observe that $e \in H \subset \mathscr{A}(U, p)$, for each $U \in \mathscr{B}$. Thus $H \subset \cap \{\mathscr{A}(U, p): U \in \mathscr{B}\}$. (Note that we are considering $\mathscr{A}(U, p) \subset W(G)$.).

Let now $U \in \mathscr{B}$ be fixed and let $\bar{U}[x]$ be compact for all $x \in X$. Let $V \subset U$ be such that $V \circ V \subset U$. Then an elementary computation shows that $\mathscr{A}(V, p)\mathscr{A}(V, p) \subset \mathscr{A}(\bar{U}, p)$. By 3.7, $\Delta_r^{W(G)}$ is a continuous homomorphism. Since $\mathscr{A}(\bar{U}, p)$ is compact, it follows that

$$D_u = \Delta_r^{W(G)} \mathscr{A}(\bar{U}, p)$$

is compact in $\boldsymbol{R}_+^*$. Now

$$\cap \{D_u: U \in \mathscr{B}\} = \{1\}$$

since $a^{1/2^n} \to 1$ for all $a \in \boldsymbol{R}_+^*$. Therefore $\Delta_r^{W(G)}(H) = \{1\}$.

THEOREM 3.11. *There exists an invariant integral on X, unique up to a multiplicative constant.*

Proof. Since $H \subset \mathscr{A}(U, p)$, where $\bar{U}[x]$ is compact for all $x \in X$, it follows from the proof of Theorem 2.6 that H is compact. Therefore H is unimodular so that $\Delta_r^H \equiv 1$. Also from Lemma 3.10 it follows that

$$1 = \Delta_r^H(t) = \Delta_r^{W(G)}(t),\ t \in H\,.$$

Thus 3.9 implies that there exists an invariant positive integral $\mu \neq 0$ on X. But then 3.8 implies that μ is unique up to a multiplicative constant.

From this point on, we will assume that G is weakly transitive on X, and that G is $\mathscr{B}$-nonexpansive. By weakly transitive we mean that Gx is a dense subset of X for some $x \in X$ (hence for each $x \in X$). This definition agrees with the one appearing in Segal [5].

LEMMA 3.12. $(U, \{x\})x \subset U[x]$ *as a dense subset.*

Proof. Obvious from definition of weak transitivity.

LEMMA 3.13. *Let $y \in U[x]$, then there is a net $x_\alpha = g_\alpha x \to y$. The net $\{g_\alpha\}$ is a Cauchy net in G.*

Proof. Consider the product net $\{g_\beta^{-1}g_\alpha\}$ where $g_\beta \in \{g_\alpha\}$. We will prove that $\lim_{\alpha,\beta} g_\beta^{-1}g_\alpha x = x$. To see this, consider $W[x]$. Let $V \circ V \subset W$. Then there exists α_0 such that $\alpha > \alpha_0$ implies that $g_\alpha x \in V[y]$. But then $y \in V[g_\alpha x]$ so that $g_\alpha^{-1}y \in V[x]$. Now let $\alpha, \beta > \alpha_0$. Then

$$g_\beta^{-1}g_\alpha x \in g_\beta^{-1}V[y] \subset V[g_\beta^{-1}y] \subset V \circ V[x] \subset W[x]\,.$$

Since $W \in \mathscr{B}$ is arbitrary, it follows that $\lim_{\alpha\beta} g_\beta^{-1}g_\alpha x = x$.

Now let $y = gx$ for some $g \in G$. Then $\{gg_\alpha g^{-1}y = gg_\alpha x\}$ is a Cauchy net in X. Thus Lemma 2.4 shows that $\{g_\alpha y\}$ is convergent. Hence as above $\lim_{\alpha,\beta} g_\beta^{-1}g_\alpha y = y$. This shows that $\lim_{\alpha,\beta} g_\beta^{-1}g_\alpha y = y$ on a dense set $D \subset X$. Thus $g_\beta^{-1}g_\alpha$ is eventually in $\mathscr{A}(U, y)$ for each $U \in \mathscr{B}$, $y \in D$.

Consider now $\mathscr{A}(U, F)$. Let $V \in \mathscr{B}$ be such that $V \circ V \circ V \subset U$. For each $x \in F$, let $y_x \in D \cap V[x]$. Then $x \in V[y_x]$. Since F is compact, there is $y_1, \cdots, y_n \in D$ such that $F \subset \bigcup_{i=1}^n V[y_i]$. Consider now $g \in \bigcap_{i=1}^n \mathscr{A}(V, y_i)$. If $x \in F$, then $x \in V[y_i]$ for some i. But then

$$gx \in gV[y_i] = V[gy_i] \subset V \circ V[y_i] \subset V \circ V \circ V[x] \subset U[x] .$$

Thus

$$\bigcap_{i=1}^n \mathscr{A}(V, y_i) \subset \mathscr{A}(U, F) .$$

Since $g_\beta^{-1} g_\alpha$ is eventually in every $\mathscr{A}(V, x)$, $x \in D$, $V \in \mathscr{B}$, it follows that $g_\beta^{-1} g_\alpha$ is eventually in every $\mathscr{A}(U, F)$ and so $\{g_\alpha\}$ is Cauchy in G.

COROLLARY 3.14. *The following are equivalent for a net $g_\alpha \in G$:*
(i) *$\{g_\alpha\}$ is Cauchy in G.*
(ii) *There exists $x \in X$ such that $\{g_\alpha x\}$ is Cauchy in X.*
(iii) *The net $\{g_\alpha\}$ is pointwise convergent on X.*
(iv) *The net $\{g_\alpha\}$ converges uniformly on compacta in X.*

Proof. (iii) $\Leftrightarrow$ (iv) is Theorem 15 on page 232 of Kelley [2]. (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) follows from 3.1 and 3.13.

COROLLARY 3.15. *G has the topology of uniform convergence on compacta. $W(G)$ is the closure, in the collection of all continuous functions of X into X, with respect to the topology of uniform convergence on compacta.*

COROLLARY 3.16. *If there exists a weakly transitive uniformly equicontinuous group of homeomorphisms on a locally compact uniform space $(X, \mathscr{U})$, then X is a quotient of locally compact topological groups.*

COROLLARY 3.17. *Let $(X, \mathscr{U})$ be a locally compact uniform space. If $(X, \mathscr{U})$ admits a weakly transitive group of uniformly equicontinuous homeomorphisms, then the group of all uniformly equicontinuous homeomorphisms is locally compact in the topology of uniform convergence on compacta.*

3.18. Second proof of Theorem 2.6.

According to the Ascoli Theorem (Kelley, Theorem 17, p. 233 [2]), $W_0 \subset G$ is totally bounded iff $W_0[x]$ is totally bounded for each $x \in X$.

Letting $W_0 = \mathscr{A}(U, p_0)$, where $\bar{U}[x]$ is compact for all $x \in X$, we see that $W_0 p_0 = U[p_0]$ and $\bar{U}[p_0]$ is compact. Let $p \in X$, then by 2.4, $W_0 p$ is homeomorphic with $gW_0 g^{-1} p$. Clearly if $g \in G$ is such that

$g^{-1}p = p_0$ we have that $W_0 g^{-1}p = W_0 p_0$ has compact closure, hence so does $W_0 p$. Thus W_0 is totally bounded, i.e., the closure of W_0 in $W(G)$ is compact.

COROLLARY 3.19. *Let X be a locally compact metric space. If there exists a weakly transitive group of isometries on X, then X is a quotient of locally compact topological groups.*

COROLLARY 3.20. (i) *$W(G)$ the Weil completion of G is transitive on X.*

(ii) *There exists a G-invariant measure μ on X.*

(iii) *μ is unique up to a multiplicative constant.*

Proof. (i) $W(G)$ consists of all limits of Cauchy sequences in G.

(ii) $(X, W(G))$ is a topological homogeneous space so existence follows from Theorem 3.11.

(iii) Let $\{g_\alpha\}$ be a Cauchy sequence in G. Then g_α converges uniformly on compacta to $g \in W(G)$. Let $f \in C_{00}^{+}(X)$, so that support $f = F$ is compact. We will show that

$$\int f \circ g(x) d\mu(x) = \lim_{\alpha} \int f \circ g_\alpha(x) d\mu(x)$$

where μ is a G-invariant regular Borel measure on X. Thus μ extends to a $W(G)$-invariant regular Borel measure on X and so is unique. Since $g_\alpha \to g$ in the topology of $W(G)$, there exists α_0 such that $\alpha \geqq \alpha_0$ implies that

$$g g_\alpha^{-1} \in \mathscr{A}(U, F) .$$

Therefore if $\alpha \geqq \alpha_0$, we have $g g_\alpha^{-1} F \subset U[F]$ so that $g_\alpha^{-1} F \subset U[g^{-1}F]$. Clearly if $x \in \complement U[g^{-1}F]$ then $x \in \complement g_\alpha^{-1} F$ for each $\alpha \geqq \alpha_0$. Thus $g_\alpha x \in \complement F$ for each $\alpha \geqq \alpha_0$, so that

$$f \circ g_\alpha(x) = 0, \quad \alpha \geqq \alpha_0 .$$

This shows that $U[g^{-1}F]$ contains the support of $f \circ g_\alpha$ for $\alpha \geqq \alpha_0$. If we take $U \in \mathscr{B}$ such that $\bar{U}[x]$ is compact for all $x \in X$, it follows that $\bar{U}[g^{-1}F]$ is compact. Also since f is uniformly continuous, the family $\{h_\alpha : h_\alpha = f \circ g_\alpha\}$ is equicontinuous and pointwise convergent to $h = f \circ g$. Therefore $h_\alpha \to h$ uniformly on compacta. Thus if $\alpha \geqq \alpha_0$ we have

$$\int_G f \circ g_\alpha(x) d\mu(x) = \int_{\bar{U}[g^{-1}F]} f \circ g_\alpha(x) d\mu(x) \to \int_{\bar{U}[g^{-1}F]} f \circ g(x) d\mu(x)$$
$$= \int_G f \circ g(x) d\mu(x) .$$

This proves that

$$\int_G f(x)d\mu(x) = \int_G f\circ g_\alpha(x)d\mu(x) \to \int_G f\circ g(x)d\mu(x)$$

so that μ is $W(G)$-invariant.

COROLLARY 3.21. (Segal [5]) *μ is unique iff G is weakly transitive.*

Proof. See Theorem 7, page 126, paragraph 1, for the case in which G is not weakly transitive. We have shown the converse.

COROLLARY 3.22. *Let X be a locally compact metric space. If there is a weakly transitive group G of isometries of X, then there is a unique G-invariant regular Borel measure on X.*

4. Existence proof of a G-invariant measure according to the method of A. Weil. For completeness, we give here a direct proof for existence of a G-invariant measure on $(X, G, \mathscr{U}, \mathscr{B})$. The proof is almost identical with one of the standard proofs of existence of the Haar measure on a locally compact topological group.

In what follows $(X, G, \mathscr{U}, \mathscr{B})$ will be an equihomogeneous locally compact space, that is, G is a transitive group of $\mathscr{B}$-nonexpansive homeomorphisms acting on the uniform space $(X, \mathscr{U})$.

LEMMA 4.1. *Let f, $\phi \in C_{00}^+(X)$. Then there exist elements $s_1, s_2, \cdots, s_n \in G$ and $c_1, \cdots, c_n > 0$* (positive real numbers) *such that*

$$f \leqq \sum_{r=1}^{n} c_r \phi\circ s_r .$$

Proof. Let $F \subset X$ be a compact set such that $f = 0$ on $\complement F$. Let $a \in X$ be such that $\phi(a) > 0$. Since ϕ is continuous, there exists a $\mu > 0$, and a neighborhood $U \in \mathscr{B}$ such that $\phi(x) \geqq \mu$ for all $x \in U[a]$. By compactness, we can cover F by a finite number of neighborhoods $U[a_i] (F \subset \bigcup_{i=1}^n U[a_i])$. For each i, let $g_i \in G$ be such that $g_i(a_i) = a$. Let $c_r = ||f||_\infty/\mu$. Then

$$f(x) \leqq \sum_{r=1}^{n} \frac{||f||_\infty}{\mu} \phi\circ g_i(x) ,$$

for all x, proving the Lemma.

DEFINITION. Let f, $\phi \in C_{00}^+(X)$. Let $(f: \phi) = \inf \{\sum_{r=1}^n c_r$: there exists a sequence $g_1, \cdots, g_n \in G$ such that

$$f \leqq \sum_{r=1}^{n} c_r \phi \circ g_r \} \ .$$

LEMMA 4.2. (a) $\|f\|_\infty / \|\phi\|_\infty \leqq (f\colon \phi)$
(b) $(f \circ a\colon \phi) = (f\colon \phi)$
(c) $(\alpha f\colon \phi) = \alpha(f\colon \phi),\ \alpha \geqq 0$
(d) $(f_1 + f_2\colon \phi) \leqq (f_1\colon \phi) + (f_2\colon \phi)$
(e) $(f\colon \psi) \leqq (f\colon \phi)(\phi\colon \psi)$

Proof. (a) We note that

$$f(x) \leqq \sum c_r \phi \circ g_r(x) \leqq \sum c_r \|\phi\|_\infty$$

for each x, so that

$$\frac{\|f\|_\infty}{\|\phi\|_\infty} \leqq \sum c_r$$

for each covering sum.

(b) If $f \circ a \leqq \sum_{r=1}^{n} c_r \phi \circ g_r$, then

$$f = f \circ a \circ a^{-1} \leqq \sum_{r=1}^{n} c_r \phi \circ g_r \circ a^{-1} = \sum_{r=1}^{n} c_r \phi \circ g'_r$$

so that

$$(f\colon \phi) \leqq \sum c_r$$

and hence $(f\colon \phi) \leqq (f \circ a\colon \phi)$ by taking infimums of covering sums. Similarly

$$(f \circ a\colon \phi) \leqq (f\colon \phi) \ .$$

(c) obvious
(d) If $f_1 \leqq \sum_{r=1}^{n} c_r \phi \circ g_r$ and $f_2 \leqq \sum_{j=1}^{m} d_j \phi \circ h_j$, then

$$f_1 + f_2 \leqq \sum_{r=1}^{n} c_r \phi \circ g_r + \sum_{j=1}^{m} d_j \phi \circ h_j$$

so that

$$(f_1 + f_2\colon \phi) \leqq \sum_{r=1}^{n} c_r + \sum_{j=1}^{m} d_j \ .$$

Taking infimum we get

$$(f_1 + f_2\colon \phi) \leqq (f_1\colon \phi) + (f_2\colon \phi) \ .$$

(e) If we apply Lemma 1 to f, ϕ, ψ we get

$$f \leqq \sum_{r=1}^{n} c_r \phi \circ g_r ,\ \phi \leqq \sum_{j=1}^{m} d_j \psi \circ h_j \ .$$

Therefore for any x, we have

$$\begin{aligned} f(x) \leqq \sum_{r=1}^{n} c_r \phi \circ g_r(x) &\leqq \sum_{r=1}^{n} c_r [\sum_{j=1}^{m} d_j \psi \circ h_j \circ g_r(x)] \\ &= \sum_{r=1}^{n} \sum_{j=1}^{m} c_r d_j \psi \circ h_j \circ g_r(x) \ . \end{aligned}$$

Therefore $(f\colon \psi) \leqq (\sum_{r=1}^{n} c_r)(\sum_{j=1}^{m} d_j)$. Again taking infimum over both $\sum c_r$, $\sum d_j$ we get

$$(f\colon \psi) \leqq (f\colon \phi)(\phi\colon \psi) \ .$$

From now on f_0 will be a fixed nonzero element of $C_{00}^{+}(X)$.

DEFINITION. Let $\phi \in C_{00}^{+}(X)$ for $f \in C_{00}^{+}(X)$ define

$$I_\phi(f) = \frac{(f\colon \phi)}{(f_0\colon \phi)} \ .$$

LEMMA 4.3. (a) $I_\phi(0) = 0$.
(b) $I_\phi(f \circ a) = I(f)$ *for all* $a \in G$.
(c) $I_\phi(\alpha f) = \alpha I_\phi(f) \quad (\alpha > 0)$.
(d) $I_\phi(f_1 + f_2) \leqq I_\phi(f_1) + I_\phi(f_2)$.
(e) $I_\phi(f_1) \leqq I_\phi(f_2)$ *if* $f_1 \leqq f_2$.
(f) $1/(f_0\colon f) \leqq I_\phi(f) \leqq (f\colon f_0)$.

Proof of (f). Since $(f\colon \phi) \leqq (f\colon f_0)(f_0\colon \phi)$ it follows that $I_\phi(f) \leqq (f\colon f_0)$.

Also

$$(f_0\colon \phi) \leqq (f_0\colon f)(f\colon \phi)$$

so

$$\frac{1}{(f_0\colon f)} \leqq \frac{(f\colon \phi)}{(f_0\colon \phi)} = I_\phi(f) \ .$$

LEMMA 4.4. *Let* $p \in X$ *be fixed. Let* $f_1, f_2 \in C_{00}^{+}(X)$ *and let* $\varepsilon > 0$. *Then there exists* $U \in \mathscr{B}$ *such that each* $\phi \in C_{00}^{+}(X)$ *having support in* $U[p]$ *satisfies*

$$I_\phi(f_1) + I_\phi(f_2) \leqq I_\phi(f_1 + f_2) + \varepsilon \ .$$

Proof. Let F be a compact set on which $f_1(x) + f_2(x) = 0$ for all

$x \notin F$. Choose $\psi \in C_{00}^{+}(X)$ so that $\psi(f) = 1$ $\psi(X) \subset [0, 1]$. Let $\varepsilon > 0$.

In the following we will be using numbers δ, η depending on f_1, f_2, f_0, ε, ψ in a manner to be determined later on in the proof.

Let $f = f_1 + f_2 + \delta\psi$. Define

$$h_1(x) = \begin{cases} \dfrac{f_1(x)}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

$$h_2(x) = \begin{cases} \dfrac{f_2(x)}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \,. \end{cases}$$

Thus $h_1, h_2 \in C_{00}^{+}(X)$.

Let $U \in \mathscr{B}$ be such that $|h_1(x) - h_1(y)| < \eta$ and $|h_2(x) - h_2(y)| < \eta$ whenever $(x, y) \in U$. (By Kelley [2], Theorem 31, h_1, h_2 are uniformly continuous). Let $\phi \in C_{00}^{+}(X)$ be such that $\phi(x) = 0$ for $x \notin U[p]$. Then as was shown in Lemma 1, there exists a sequence $s_1, \cdots, s_n \in G$ and positive real numbers $c_1, \cdots, c_n$ such that $f \leqq \sum_{r=1}^{n} c_r \phi \circ s_r$.

If $\phi \circ s_r(x) \neq 0$, then $s_r(x) \in U[p]$ so that $x \in U[s_r^{-1}p]$ and hence (.ie., $(x, s_r^{-1}p) \in U$)

$$|h_1(x) - h_1(s_r^{-1}p)| < \eta \,.$$

Thus we can write

$$h_1(x) = h_1(x) - h_1(s_r^{-1}p) + h_1(s_r^{-1}p) \leqq \eta + h_1(s_r^{-1}p) \,,$$

and hence

$$(*) \qquad \phi(s_r x)h_1(x) \leqq \phi(s_r x)[\eta + h_1[s_r^{-1}p)] \text{ when } \phi(s_r x) \neq 0 \,.$$

However (*) holds even if $\phi \circ s_r(x) = 0$ and so (*) holds for all x.

We observe now that (*) implies that

$$\begin{aligned} f_1(x) = h_1(x)f(x) &\leqq h_1(x) \sum_{r=1}^{n} c_r \phi \circ s_r(x) \\ &= \sum_{r=1}^{n} c_r \phi \circ s_r(x) h_1(x) \leqq \sum_{r=1}^{n} c_r \phi \circ s_r(x)[h_1(s_r^{-1}p) + \eta] \,. \end{aligned}$$

Thus

$$(f_1 \colon \phi) \leqq \sum_{r=1}^{n} c_r [h_1 \circ s_r^{-1}(p) + \eta] \,,$$

and similarly

$$(f_2 \colon \phi) \leqq \sum_{r=1}^{n} c_r [h_2 \circ s_r^{-1}(p) + \eta] \,.$$

Since

$$h_1 + h_2 = \frac{f_1 + f_2}{f} \leqq \frac{f_1 + f_2 + \delta\psi}{f} = 1 ,$$

it follows that

$$(f_1 : \phi) + (f_2 : \phi) \leqq \sum_{r=1}^{n} c_r[h_1 \circ s_r^{-1}(p) + h_2 \circ s_r^{-1}(p) + 2\eta] \leqq \sum_{r=1}^{n} c_r[1 + 2\eta] .$$

Taking infimum of Σc_r we have

$$(f_1 : \phi) + (f_2 : \phi) \leqq (f : \phi)[1 + 2\eta] ,$$

and dividing by $(f_0 : \phi)$ we get

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) &\leqq I_\phi(f)[1 + 2\eta] \\ &\leqq [I_\phi(f_1 + f_2) + I_\phi(\delta\psi)][1 + 2\eta] \\ &= [I_\phi(f_1 + f_2) + \delta I_\phi(\psi)][1 + 2\eta] \\ &= I_\phi(f_1 + f_2) + 2\eta I_\phi(f_1 + f_2) + \delta I_\phi(\psi)[1 + 2\eta] \\ &\leqq I_\phi(f_1 + f_2) + 2\eta(f_1 + f_2 : f_0) + \delta(\psi : f_0)[1 + 2\eta] . \end{aligned}$$

Now choose δ, η so that $2\eta(f_1 + f_2 : f_0) + \delta(\psi : f_0)[1 + 2\eta] < \varepsilon$, and the Lemma is proved.

THEOREM 4.5. *If $(X, G, \mathscr{U}, \mathscr{B})$ is an equi-homogeneous locally compact space then there exists a G invariant integral* (measure) *I on $C_{00}^+(X)$. The measure defined by I is a regular G-invariant Borel measure.*

Proof. Let $f \in C_{00}^+(X)$. Define $T_f = [1/(f_0 : f), (f : f_0)]$ (T_f is a compact interval in $\boldsymbol{R}^+$). Let $T = \boldsymbol{P}_f T_f$. Then T is compact. We observe now that each I_ϕ can be regarded as an element of T, by associating I_ϕ with $\{t_f\} \in T$ where $t_f = I_\phi(f)$.

Let $\mathscr{B}$ be the base for $\mathscr{U}$ associated with the group of homeomorphisms G, and suppose $\mathscr{B}$ is directed by inclusion. Fix $p \in X$. For each $U \in \mathscr{B}$, choose exactly one nonzero $\phi_U \in C_{00}^+(X)$ such that $\phi_U(x) = 0$ if $x \in \complement U[p]$. Now $\{I_{\phi_u} : U \in \mathscr{B}\}$ is a net in the compact space T, and so there is a convergent subnet $\{I_\beta\}$ converging to an element I of T. It is clear that I has the following properties

(i) $I \neq 0$.
(ii) $I(f) > 0$ whenever $f \in C_{00}^+(X)$, $f \neq 0$.
(iii) $I(f_1 + f_2) = I(f_1) + I(f_2)$.
(iv) $I(f \circ a) = I(f)$ for all $f \in C_{00}^+(X)$, $a \in G$.
(v) $I(\alpha f) = \alpha I(f)$ for $\alpha \geqq 0$.

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