PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 1, 1972

A CONSTRUCTIVE STUDY OF MEASURE THEORY

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In this paper we study measures on locally compact metric spaces. The constructive theory of a nonnegative measure has been treated in Bishop's book "Foundations of Constructive Analysis". Unfortunately, there is no constructive method to decompose a general signed measure into a difference of two nonnegative ones. In analogy to the classical development, we shall consider two ways to look at a signed measure, namely, as a function function (an integral) and as a set function (a set measure). From an integral on a locally compact metric space X we obtain compact subsets of X to which measures can be assigned. The set measure thus arrived at is shown to be in a weak sense additive, continuous, and of bounded variation. Next we study a set measure having these three properties defined on a large class of compact subsets of X. From such a set measure we derive a linear function on the space of test functions of X. This linear function is then shown to be an integral. Finally it is demonstrated that the set measure arising from an integral gives rise in this manner to an integral which is equal to the original one. In particular, every integral is the integral arising from some measure (Riesz Representation Theorem).

We shall make use of concepts and results in Bishop's book (reformed to hereafter as C.A.), in which one can find a presentation of the constructive viewpoint and the constructive methods.

1. Compact subsets of a boundedly compact metric space. A metric space (X, d) is said to be totally bounded if, given any $\varepsilon > 0$, there is a finite, possibly empty, sequence of points in X which forms an ε-net for X. A metric space (X, d) is said to be compact if it is totally bounded and complete. A boundedly compact space[1] is one in which every bounded subset is contained in some compact subset. Hereafter let (X, d) denote such a space. For each compact subset A of X, and each $x \in X$, we let $d(x, A)$ stand for the number min (1, inf $\{d(x, y): y \in A\}$). Here the infimum is easily proved to exist (C.A.) if A is nonempty, and is taken to be ∞ if A is empty. Given any compact subsets A and B of X, write

$$d'(A, B) \equiv \max (0, \sup\{d(x, B): x \in A\}, \sup \{d(x, A): x, B\}) .$$

Here the supremum of an empty set is taken to be $-\infty$. d' can easily be shown to be a metric on the family of compact subsets of X.

[1] Called locally compact space in C. A.

The proof of the following theorem is almost a verbatim reproduction of one in C.A., and so will not be given here.

THEOREM 1.1. *Assume (X, d) is compact. Let f be a continuous function on (X, d). Then we can find a countable subset $A = A_f$ of the real line, with the following properties.*

(1) *The set $(a \leqq f \leqq b) \equiv \{x \in X: a \leqq f(x) \leqq b\}$ is compact whenever the numbers a and b are in $-A$ with $a < b$.*

(2) *Suppose a and b are in $-A$ with $a < b$. Then for numbers a' and b' in $-A$ which are close enough to a and b respectively, the distance $d'((a \leqq f \leqq b), (a' \leqq f \leqq b'))$ is arbitrarily small.*

DEFINITION 1.2. Let A and B be subsets of X. We call the closure of $A \cap B$ (resp. $A \cup B$) in X the closed intersection (resp. union) of A and B, and denote it by $A \wedge B$ (resp. $A \vee B$).

Let K and L be compact subsets of X. We say K and L are compatible if $K \wedge L$ is compact, and if for each $\varepsilon > 0$, there exists $\varepsilon > 0$ such that

$$d(x, K \wedge L) < \varepsilon \text{ for every } x \in X \text{ with } d(x, K) < \delta \text{ and } d(x, L) < \delta .$$

We say K well contains L, and write $K \supset L$, if there is a positive constant a such that $d(x, L) > a$ whenever $d(x, K) > 0$.

PROPOSITION 1.3. *Assume that K, L, and H are compact subsets and that the sets generated from them by the operations $\wedge$ and $\vee$ are compact and pairwise compatible. Then the following equalities hold.*

(i) $K \vee L = L \vee K, K \wedge L = L \wedge K$.

(ii) $(K \vee L) \vee H = K \vee (L \vee H), (K \wedge L) \wedge H = K \wedge (L \wedge H)$.

(iii) $(K \vee L) \wedge H = (K \wedge H) \vee (L \wedge H)$, $(K \wedge L) \vee H = (K \vee H) \wedge (L \vee H)$.

The proof is straightforward and is omitted.

DEFINITION 1.4. For any pair of continuous functions f and g on X we shall let $f \wedge g$ (resp. $f \vee g$) denote the function $\min(f, g)$ (resp. $\max(f, g)$).

Suppose $f_1, \cdots, f_n$ are continuous functions on X. We shall let $\Psi = \Psi(f_1, \cdots, f_n)$ denote the (finite) family of functions generated from $f_1, \cdots, f_n$ by the operations $\wedge$ and $\vee$. We shall say a (Lebesgue) null subset A of R is exceptional for the functions $f_1, \cdots, f_n$ if the following three conditions are satisfied.

(i) For each $a \in -A$, the sets

$$(f \geqq a) \equiv \{x \in X: f(x) \geqq a\} \qquad (f \in \Psi)$$

are compact.

(ii) For each $a \in -A$ and $f \in \Psi$, by choosing the number $b \in -A$ near enough to a, we can make $d'((f \geqq a), (f \geqq b))$ arbitrarily small.

(iii) For each $a \in -A$, the sets $(f \geqq a)$ $(f \in \Psi)$ are pairwise compatible.

The following proposition and its corollary show the abundance of compatible compact sets.

PROPOSITION 1.5. *Assume (X, d) is compact. Given continuous functions $f_1, \cdots, f_n$, there exists an exceptional set A of real numbers for $f_1, \cdots, f_n$.*

Proof. Since $\Psi = \Psi(f_1, \cdots, f_n)$ is a finite family, using 1.1 we can find a countable subset A of R such that conditions 1.4 (i)—(ii) hold. Now let ω be a common modulus of continuity for the functions in Ψ. Consider any $a \in -A$, and functions f and g in Ψ. Let $\varepsilon > 0$ be arbitrary. By 1.4 (ii) we can choose $b \in -A$ with $b < a$ such that

$$d'((f \wedge g \geqq a), (f \wedge g \geqq b)) < \varepsilon .$$

Let $\delta = \min(1, \omega(a - b))$. Suppose $x \in X$ is such that $d(x, (f \geqq a)) < \delta$ and $d(x, (g \geqq a)) < \delta$. Then there is some y in $(f \geqq a)$ with $d(x, y) < \delta$, and so $f(x) \geqq b$. Similarly $g(x) \geqq b$. Consequently $x \in (f \wedge g \geqq b)$ and so $d(x, (f \wedge g \geqq a)) < \varepsilon$. But $(f \wedge g \geqq a) = (f \geqq a) \wedge (g \geqq a)$. It follows that

$$d(x, (f \geqq a) \wedge (g \geqq a)) < \varepsilon .$$

Therefore $(f \geqq a)$ and $(g \geqq a)$ are compatible. We have thus verified 1.4 (iii).

DEFINITION 1.6. A continuous function h on (X, d) is said to be proper if for all real numbers a and b with $a < b$, the set $(a \leqq h \leqq b)$ is contained in some compact subset of X.

COROLLARY 1.7. *Let $f_1, \cdots f_n$ be proper functions on X which are bounded above. Then there is a null subset A of R which is exceptional for $f_1, \cdots, f_n$.*

Proof. For each natural number k, choose a real number a_k such that $a_k < -k$, and such that the set $L_k \equiv (f_1 \vee \cdots \vee f_n \geqq a_k)$ is compact. Choose a compact set K_k which well contains L_k. Then by 1.5 there is a null set A_k of real numbers which is exceptional for the functions $f_1 | K_k, \cdots, f_n | K_k$. Since compact sets in L_k which are compatible as compact subsets of K_k are also compatible in X, the set of real numbers $\bigcup_{k=1}^{\infty} A_k$ can easily be verified to be exceptional for the functions $f_1, \cdots f_n$.

PROPOSITION 1.8. *Suppose the null set A is exceptional for the functions $f_1, \cdots, f_n$. Then for all f and g in $\Psi(f_1, \cdots, f_n)$ and $a \in -A$, we have $(f \wedge g \geqq a) = (f \geqq a) \wedge (g \geqq a)$ and $(f \vee g \geqq a) = (f \geqq a) \vee (g \geqq a)$.*

Proof. The first equality is easily verified. Take any x in $(f \vee g \geqq a)$. For every natural number k choose b_k in $-A$ with $b_k < a$ and $d'((f \geqq b_k), (f \geqq a)) < k^{-1}$. Then either $x \in (f < a)$ or $x \in (f \geqq b_k)$. In either case $d(x, (f \geqq a) \vee (g \geqq a))$ is less than k^{-1}. Thus $x \in (f \geqq a) \vee (g \geqq a)$. So $(f \vee g \geqq a)$ is contained in $(f \geqq a) \vee (g \geqq a)$. Containment in the other direction is obvious.

2. Measure induced by an integral. Hereafter $C(X)$ will denote the space of test functions on (X, d), namely, continuous functions on X with compact supports.

DEFINITION 2.1. A (signed) integral μ on X is a linear function on $C(X)$ whose value at an element f of $C(X)$ is written $\int f du$, such that for each $f \in C(X)$ there is $M_f \geqq 0$ with $\left|\int h d\mu\right| \leqq M_f$ whenever $h \in C(X)$ and $|h| \leqq |f|$.

DEFINITION 2.2. A sequence $\{f_n\}$ of test functions is said to belong to a compact subset K of X if for all n,

(i) $0 \leqq f_n \leqq 1$,

(ii) $f_n(x) = 0$ if $d(x, K) \geqq n^{-1}$,

and (iii) $f_n(x) = 1$ if $x \in K$.

In the following, let μ be a given integral on X.

DEFINITION 2.3. A compact subset K of X is said to be strongly measurable with respect to μ if there is a sequence $\{f_n\}$ of test functions belonging to K such that $\int f_n g d\mu$ converges for all test functions g.

The following lemma is proved in C.A. (P. 173).

LEMMA 2.4. *Let K be a compact subset of X. Suppose $\{f_n\}$ belongs to K and suppose $\int f_n d\mu$ converges. Then the limit is independent of the sequence $\{f_n\}$. Thus if $\{f_n\}$ is in addition such that $\int f_n g d\mu$ converges for all $g \in C(X)$, then, for given g, the limit $\lim_{n\to\infty}\int f_n g d\mu$ is independent of $\{f_n\}$.*

DEFINITION 2.5. Let K be strongly measurable and let $g \in C(X)$. Define $\mu(K) \equiv \lim_{n\to\infty}\int f_n d\mu$ and $\int_K g d\mu \equiv \lim_{n\to\infty}\int f_n g d\mu$, where $\{f_n\}$ is some sequence of test functions belonging to K such that these limits exist. (By 2.4 these are well defined). $\mu(K)$ is called the strong measure of K.

PROPOSITION 2.6. *Let K be strongly measurable and g a test function. From every sequence $\{f'_n\}$ of test functions belonging to K, we can select a subsequence $\{f''_n\}$ such that*

$$\lim_{n\to\infty}\int f''_n g d\mu = \int_K g d\mu .$$

Proof. Let $\{f_n\}$ be a sequence of test functions belonging to K such that $\int f_n h d\mu$ converges for all $h \in C(X)$. Since $|f_n - f'_n|$ vanishes on K and on $\{x \in X: d(x, K) \geqq n^{-1}\}$, we can select a sequence of integers $\{n_i\}$ such that $n_i < n_{i+1}$ and

$$|f_{n_1} - f'_{n_1}| + \cdots + |f_{n_i} - f'_{n_i}| \leqq 2 \quad \text{for all } i .$$

By abuse of notation we again write f_i for f_{n_i}, and f'_i for f'_{n_i}. Let $h_i = f_i - f'_i$ $(i = 1, 2, \cdots)$. Then clearly for every choice of integers j and $i_1 < i_2 < \cdots < i_j$ we have

$$|h_{i_1} + \cdots + h_{i_j}| \leqq 2 \qquad \text{on } X .$$

Therefore

$$|h_{i_1} g + \cdots + h_{i_j} g| \leqq 2|g| \qquad \text{on } X ,$$

and so, by the definition of an integral, there is $M \geqq 0$ such that

$$\left|\int h_{i_1} g d\mu + \cdots + \int h_{i_j} g d\mu\right| \leqq M$$

for all integers j and $i_1 < i_2 < \cdots < i_j$. Thus, by passing to a subsequence, we may assume

$$\int h_i g d\mu \to 0 \qquad \text{as } i \to \infty .$$

If follows that $\int f'_i g d\mu \to \lim_{n\to\infty} \int f_n g d\mu$.

Thus $\int f'_n g d\mu$ converges in a weak sense to $\int_K g d\mu$ (Given any subsequence of $\int f'_n g d\mu$ we can select a subsequence which converges to $\int_K g d\mu$). This is the strongest result we can expect to get, as is shown by the following example.

Let $\{a_n\}$ be a given sequence of 0 or 1's, containing at most one 1, but it is not known whether there is a 1. For each n write r_n for n^{-1}. Define integrals μ_n on the real line by $\mu_n = \varepsilon_{2n} - \varepsilon_{2n+1}$ where ε_k

denotes the integral defined by $\int f d\varepsilon_k = f(r_k)$ for all $f \in C(R)$. Let μ be the integral on R defined by

$$\int f d\mu = \sum_{n=1}^{\infty} a_n \int f d\mu_n \qquad \text{for all } f \in C(R) \text{ .}$$

The last series converges by the continuity of f. For each n, let f_n be a continuous function on R, having values in $[0, 1]$, supported by $[-r_{2n-1}, r_{2n-1}]$, and equal to 1 on $[-r_{2n}, r_{2n}]$. Let f'_n be a continuous function on R, having values in $[0, 1]$, supported by $[-r_{2n}, r_{2n}]$, and equal to 1 on $[-r_{2n+1}, r_{2n+1}]$. Clearly f_n and f'_n belong to the compact set $K = \{0\}$. For each n and m it is also clear that $\int f_n d\mu_m = 0$. Thus $\int f_n d\mu = 0$ for all n. Given any test function g, since g behaves like a constant near 0, we can prove $\int f_n g d\mu \to 0$. Thus K is strongly measurable with respect to μ. On the other hand for any n and m we have $\int f'_n d\mu_m = -\delta_{mn}$ (δ_{mn} the Kronecker delta). Hence $\int f'_n d\mu = -a_n$. We cannot, however, tell whether $-a_n$ converges in the ordinary sense.

LEMMA 2.7. *If K is strongly measurable, and g and g' are test functions which coincide on K, then $\int_K g d\mu = \int_K g' d\mu$.*

Proof. Let $\{f_n\}$ be a sequence of test functions belonging to K such that $\int f_n h d\mu$ converges for all $h \in C(X)$.

There exists a compact set L outside which g and g' vanish. Let f be a test function which equals 1 on L. Then by the definition of an integral, there is $M_f \geqq 0$ such that

$$\left| \int h d\mu \right| \leqq M_f \| h \|$$

wherever $h \in C(X)$ is supported by L. Now $g - g'$ vanishes on K and so $\| (g - g') f_n \|$ is arbitrarily small if n is large enough. But

$$\left| \int (g - g') f_n d\mu \right| \leqq M_f \| (g - g') f_n \| \text{ .}$$

The desired result follows.

DEFINITION 2.8. Let K be strongly measurable. For each continous function $\bar{g}$ on K, write

$$\mu_K(\bar{g}) \equiv \int \bar{g} d\mu_K \equiv \int_K g d\mu$$

where g is some test function on X which extends $\bar{g}$. μ_K is a function by 2.7. Clearly μ_K is an integral. We call μ_K the restriction of μ to K.

PROPOSITION 2.9. *Suppose K is a compact subset of X which is strongly measurable with respect to μ. Suppose L is a compact subset of X which is well containd in K and strongly measurable with respect to μ_K. Then L is strongly measurable with respect to μ, and $\mu(L) = \mu_K(L)$.*

Proof. Let $\{f_n\}$ be a sequence of test functions on X belonging to K, such that $\int f_n g d\mu$ converges for all $g \in C(X)$. Let $\{f'_n\}$ be a sequence of test functions on K bearing a similar relationship to L and μ_K. Since $K \supset\supset L$, for each n we may assume that f'_n is the restriction to K of some test function f''_n on X which is supported by K; indeed we may assume that the sequence $\{f''_n\}$ belongs to L. Now, for any $g \in C(X)$, let g_K be the restriction of g to K. Then by assumption the sequence $\int f'_n g_K d\mu_K$ converges. But, for each n,

$$\int f'_n g_K d\mu_K = \lim_{m\to\infty} \int f_m f''_n g d\mu = \int f''_n g d\mu .$$

The first equality holds by the definition of μ_K; the second because $f''_n g$ is supported by K, and $f_m = 1$ on K. Thus $\int f''_n g d\mu$ converges. Moreover

$$\mu(L) = \lim_{n\to\infty} \int f''_n d\mu = \lim_{n\to\infty} \int f'_n d\mu_K = \mu_K(L) .$$

The following theorem, which is a generalization of one in C.A., shows that strongly measurable sets are abundant.

THEOREM 2.10. *Let h be a proper function on X. Then there exists a* (Lebesgue) *null set B of real numbers such that the set $(u \leqq h \leqq v)$ is strongly measurable for all u and v in $-B$ with $u < v$.*

Proof. In view of 2.9, we may assume that X is compact. It suffices to show that for $[0, 1]$ (and similarly for any interval) there is a null set B such that the set $(u \leqq h \leqq v)$ is strongly measurable for all u and v in $[0, 1] - B$ with $u < v$. Without loss of generality, assume that $\left|\int f d\mu\right| \leqq \|f\|$ for every $f \in C(X)$.

For each natural number n write

$$G_n \equiv \{g \in C(X): \|g\| \leqq 1 \quad \text{and} \quad |g(x) - g(y)| \leqq nd(x, y)\} ,$$

and write for each k $(0 \leqq k \leqq n^4 - 1)$,

$$S_{nk} \equiv \{f \in C(R): \|f\| \leqq 1; \quad f \text{ is supported by } [kn^{-4}, (k+1)n^{-4}]; \quad \text{and} \quad |f(s) - f(t)| \leqq 2(n+1)^6 |s-t|\} .$$

Then, by Ascoli's Theorem (C.A.), the sets G_n and S_{nk} are compact with respect to the supremum norms. Therefore, for given natural numbers m, n, and $k(0 \leqq k \leqq n^4 - 1)$, the number

$$a_{mnk} \equiv \sup\left\{\int (f \circ h) g d\mu: f \in S_{nk}, g \in G_m\right\}$$

is well defined. For given m and n, let U_{mn} and V_{mn} be a partition of $\{0, 1, \cdots, n^4 - 1\}$ such that

$$a_{mnk} > n^{-2} \quad \text{if} \quad k \in U_{mn}; \quad a_{mnk} < 2n^{-2} \quad \text{if} \quad k \in V_{mn} .$$

For each $k \in U_{mn}$, choose $f_k \in S_{nk}$ and $g_k \in G_m$ with

$$\int (f_k \circ h) g_k d\mu > n^{-2} .$$

Let $f = \sum_{k \in U_{mn}} (f_k \circ h) g_k$ Then $\|f\| \leqq 1$ by the definition of the f_k's. Thus

$$1 \geqq \int f d\mu = \sum_{k \in U_{mn}} \int (f_k \circ h) g_k d\mu > n^{-2} \quad \text{card } (U_{mn}) .$$

Or card $(U_{mn}) \leqq n^2$. (card (U_{mn}) is the number of elements in U_{mn}).

Now construct a countable subset A_h of R which is related to h as A_f is to f in 1.1. For all natural numbers m and n, let B_{mn} be the union of the set

$$\bigcup_{k \in U_{mn}} [kn^{-4}, (k+1)n^{-4}]$$

and the set

$$\bigcup_{0 \leqq k \leqq n^4} ([kn^{-4} - n^{-6}, kn^{-4} + n^{-6}] \cap [0, 1]) .$$

Then the Lebesgue measure of B_{mn} is at most

$$n^{-4} \quad \text{card} \quad (U_{mn}) + 2n^4 \cdot n^{-6} \leqq 2n^{-2} .$$

Thus $B \equiv A_h \cup (\bigcup_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} B_{mn})$ has Lebesgue measure zero. Now suppose $u, v \in [0, 1] - B$ with $u < v$. We shall show that the set $(u \leqq h \leqq v)$ is strongly measurable.

For each n, construct $f_n \in C(R)$ with support $[u - n^{-6}, v + n^{-6}]$ such that $f_n = 1$ on $[u, v]$ and f_n is linear on each of the intervals $[u - n^{-6}, u]$ and $[v, v + n^{-6}]$. Let m be an arbitrary natural number, and $g \in G_m$. There is a natural number j such that $u, v \in -B_{mn}$ for all $n \geqq j$. Fix any n with $n \geqq j$. It follows from the definition of B_{mn} that

$$u \in [k'n^{-4} + n^{-6}, (k' + 1)n^{-4} - n^{-6}] \quad \text{for some } k' \in V_{mn} ;$$
$$v \in [k''n^{-4} + n^{-6}, (k'' + 1)n^{-4} - n^{-6}] \quad \text{for some } k'' \in V_{mn} (k' \leqq k'') .$$

Since $f_n - f_{n+1}$ is supported by $[u - n^{-6}, u] \cup [v, v + n^{-6}]$ and since $u < v$, we can write $f_n - f_{n+1} = f' + f''$ where f' equals $f_n - f_{n+1}$ on $[u - n^{-6}, u]$ and zero elsewhere, and where f'' equals $f_n - f_{n+1}$ on $[v, v + n^{-6}]$ and zero elsewhere. In particular, f' is supported by $[k'n^{-4}, (k' + 1)n^{-4}]$. For all real numbers s and t,

$$\begin{aligned} |f'(s) - f'(t)| &\leqq |f_n(s) - f_n(t)| + |f_{n+1}(s) - f_{n+1}(t)| \\ &\leqq 2(n + 1)^6 |s - t| . \end{aligned}$$

Therefore $f' \in S_{nk'}$. But $k' \in V_{mn}$. Thus $a_{mnk'} < 2^{n-2}$. Consequently, $\left|\int (f' \circ h) g d\mu\right| \leqq 2n^{-2}$. Similarly we show $\left|\int (f'' \circ h) g d\mu\right| \leqq 2n^{-2}$. Combining, we see that

$$\begin{aligned} &\left|\int (f_n \circ h) g d\mu - \int (f_{n+1} \circ h) g d\mu\right| \\ &\quad \leqq \left|\int (f' \circ h) g d\mu\right| + \left|\int (f'' \circ h) g d\mu\right| \leqq 4n^{-2} \quad (n \geqq j) . \end{aligned}$$

Therefore $\int (f_n \circ h) g d\mu$ converges for all $g \in G_m$. Since by the Stone-Weierstrass Theorem (C.A.), the linear span of $\bigcup_{m=1}^{\infty} G_m$ is dense in $C(X)$ under the supremum norm, we see that $\int (f_n \circ h) g d\mu$ convergers for all $g \in C(X)$. It remains to choose from the sequence $\{f_n \circ h\}$ a subsequence which belongs to the set $(u \leqq h \leqq v)$. For every natural number i, by the definition of B and A_h, we can find u' and v' in $[0, 1] - B$ such that $u' < u < v < v'$ and

$$d'((u \leqq h \leqq v), (u' \leqq h \leqq v')) \leqq i^{-1} .$$

Let n_i be a natural number greater than $(u - u')^{-1}$ and $(v' - v)^{-1}$. For every x such that $d(x, (u \leqq h \leqq v)) > i^{-1}$, we have $h(x) \leqq u'$ or $h(x) \geqq v'$. It follows that $f_{n_i} \circ h$ vanishes at this x. Clearly $f_{n_i} \circ h = 1$ on the set $(u \leqq h \leqq v)$. Combining, we see that $\{f_{n_i} \circ h\}$ belongs to $(u \leqq h \leqq v)$.

PROPOSITION 2.11. *If $\{K_n\}$ is a sequence of strongly measurable subsets of X, if K is a strongly measurable subset contained in each K_n, and if $d'(K, K_n) \leqq n^{-1}$ for each n, then $\mu(K'_n)$ converges to $\mu(K)$ for some subsequence $\{K'_n\}$ of $\{K_n\}$.*

Proof. Choose a sequence $\{g_n\}$ of test functions such that $0 \leqq g_n \leqq 1$, $g_n = 1$ on K_n, $g_n(x) = 0$ if $d(x, K_n) \geqq n^{-1}$, and $\left|\int g_n d\mu - \mu(K_n)\right| \leqq n^{-1}$. Since $d'(K, K_n) \leqq n^{-1}$, the sequence $\{g_{2n}\}$ belongs to K. Hence by 2.7 we can choose a subsequence $\{g'_n\}$ of $\{g_{2n}\}$ such that $\int g'_n d\mu \to \mu(K)$ as $n \to \infty$. Take the corresponding subsequence $\{K'_n\}$ of $\{K_{2n}\}$. Obviously $\mu(K'_n) \to \mu(K)$.

LEMMA 2.12. *Suppose K is strongly measurable. Then there exists a constant $A_k \geqq 0$ such that*

$$\sum_{i=1}^{m} |\mu(L_i) - \mu(K_i)| \leqq A_K$$

for every sequence $L_1, K_1, L_2, K_2, \cdots, L_m, K_m$ of strongly measurable sets with $K \supset\supset L_1 \supset\supset K_1 \supset\supset \cdots \supset\supset L_m \supset\supset K_m$.

Proof. By the definition of an integral, there exists a constant $A_K \geqq 0$ such that $\left|\int f d\mu\right| \leqq A_K \|f\|$ for each $f \in C(X)$ supported by K. Now let

$$(*) \qquad K \supset\supset L_1 \supset\supset K_1 \supset\supset \cdots \supset\supset L_m \supset\supset K_m$$

be a sequence of strongly measurable sets. Choose sequences of test functions $\{f_n^i\}$ and $\{g_n^i\}$ belonging to K_i respectively L_i such that

$$\lim_{n\to\infty} \int f_n^i d\mu = \mu(K_i); \quad \lim_{n\to\infty} \int g_n^i d\mu = \mu(L_i) \quad (1 \leqq i \leqq m) .$$

Let $\delta > 0$ be arbitrary. Partition $1, \cdots, m$ into subsets P and Q such that

$$u(L_i) - \mu(K_i) < \delta \quad \text{if} \quad i \in P; \quad \mu(L_i) - \mu(K_i) > -\delta \quad \text{if} \quad i \in Q .$$

Then

$$\begin{aligned}
\sum_{i=1}^{m} |(L_i) - \mu(K_i)| &= \sum_{i \in P} |\mu(L_i) - \mu(K_i)| + \sum_{i \in Q} |\mu(L_i) - \mu(K_i)| \\
&\leqq \sum_{i \in P} \{-(\mu(L_i) - \mu(K_i)) + 2\delta\} + \sum_{i \in Q} \{\mu(L_i) - \mu(K_i) + 2\delta\} \\
&= 2m\delta + \lim_{n\to\infty} \left\{ \sum_{i \in P} \int (f_n^i - g_n^i) d\mu + \sum_{i \in Q} \int (g_n^i - f_n^i) d\mu \right\} .
\end{aligned}$$

But, from the well containment $(*)$, it follows that if n is large enough, the function $\sum_{i \in P} (f_n^i - g_n^i) + \sum_{i \in Q} (g_n^i - f_n^i)$ is supported by K and has values in $[-1, 1]$. Therefore by the definition of A_K, the integral of this function

$$\sum_{i \in P} \int (f_n^i - g_n^i) d\mu + \sum_{i \in Q} \int (g_n^i - f_n^i) d\mu$$

is at most A_K if n is large enough.
Consequently

$$\sum_{i=1}^{m} | \mu(L_i) - \mu(K_i) | \leqq 2m\delta + A_K .$$

But δ was arbitrary. The desired inequality follows.

PROPOSITION 2.13. *Suppose K is strongly measurable. Then there exists a constant $B_K > 0$ such that*

$$\sum_{i=1}^{m-1} | \mu(K_i) - \mu(K_{i+1}) | \leqq B_K$$

for every sequence of strongly measurable sets $K \supset K_1 \supset \cdots \supset K_m$.

Proof. Choose a positive real number a_0 such that the set

$$K' \equiv \{x \in X: \quad d(x, K) \leqq a_0\}$$

is strongly measurable. Such a choice is possible by 2.10. Let $A_{K'} \geqq 0$ be a constant associated to K' as in the above lemma. Let $B_K = 2A_{K'} + 1$. Now consider any sequence of strongly measurable sets $K \supset K_1 \supset K_2 \supset \cdots \supset K_m$. Inductively (on $i = 1, \cdots, m$), let $a_i > 0$ be so small that $a_i < a_{i-1}$ and such that

$$K_i' \equiv \{x \in X: \quad d(x, K_i) \leqq a_i\}$$

is strongly measurable with $| \mu(K_i) - \mu(K_i') | < (2m)^{-1}$. Then it can easily be shown that

$$K' \supset\supset K_1' \supset\supset K_2' \supset\supset \cdots \supset\supset K_m' .$$

Therefore, by the definition of $A_{K'}$,

$$\sum_{i=1}^{m-1} | \mu(K_i') - \mu(K_{i+1}') | \leqq 2A_{K'} .$$

But then

$$\sum_{i=1}^{m-1} | \mu(K_i) - \mu(K_{i+1}) | \leqq 2A_{K'} + 1 = B_K .$$

PROPOSITION 2.14. *If two strongly measurable sets K_1 and K_2 are compatible, and $K_1 \wedge K_2$ as well as $K_1 \vee K_2$ are strongly measurable, then*

$$\mu(K_1) + \mu(K_2) = \mu(K_1 \wedge K_2) + \mu(K_1 \vee K_2) .$$

Proof. Let $\{f_n\}$, $\{g_n\}$ be sequences of test functions belonging to K_1 and K_2 respectively. Clearly the sequence $\{f_n \vee g_n\}$ belongs to $K_1 \vee K_2$. Since K_1 and K_2 are compatible we can select a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers such that $d(x, K_1 \wedge K_2) > k^{-1}$ implies $d(x, K_1) > n_k^{-1}$ or $d(x, K_2) > n_k^{-1}$. Then the sequence $\{f_{n_k} \wedge g_{n_k}\}$ belongs to $K_1 \wedge K_2$. Therefore, by 2.6 and by passing to a subsequence, we have

$$\mu(K_1 \wedge K_2) + \mu(K_1 \vee K_2) = \lim_{k\to\infty} \left(\int f_{n_k} \wedge g_{n_k} d\mu + \int f_{n_k} \vee g_{n_k} d\mu\right)$$
$$= \lim_{k\to\infty} \int (f_{n_k} + g_{n_k}) d\mu = \mu(K_1) + \mu(K_2) .$$

PROPOSITION 2.15. *The empty set $\varnothing$ is strongly measurable, and $\mu(\varnothing) = 0$.*

Proof. Just consider the sequence of test functions $\{f_n\}$ where $f_n \equiv 0$ for each n.

3. Measure spaces and integration.

DEFINITION 3.1. Let (X, d) be a boundedly compact metric space. Let F be a family of compact subsets of X such that, for every proper function f on X, there is a (Lebesgue) null subset A_f of R such that the set $(a \leqq f \leqq b)$ is in F for all a and b in $-A_f$ with $a < b$. Suppose further ν is a real valued function on F such that

(i) ν is additive in the sense that

$$\nu(K_1) + \nu(K_2) = \nu(K_1 \wedge K_2) + \nu(K_1 \vee K_2)$$

whenever $K_1, K_2 \in F$ are compatible and $K_1 \wedge K_2$, $K_1 \vee K_2$ are in F,

(ii) ν is continuous in the sense that, given compact sets $K, K_n \in F (n \in N)$ with $K \subset K_n$ and $d'(K, K_n) \to 0$, we can choose a subsequence $\{K_n'\}$ of $\{K_n\}$ such that $\nu(K_n') \to \nu(K)$,

(iii) ν is of bounded variation in the sense that, for every K in F there exists $B_K \geqq 0$ such that for any sequence $K \supset K_1 \supset K_2 \supset \cdots \supset K_m$ of elements of F we have

$$\sum_{i=1}^{m-1} |\nu(K_i) - \nu(K_{i+1})| \leqq B_K ,$$

(iv) $\nu(\varnothing) = 0$, where $\varnothing$ is the empty set.

Then we say that ν is a signed measure on (X, d) and (X, d, F, ν) is a signed measure space. When no confusion is likely, we call a signed measure simply a measure, and a signed measure space simply a measure space. Members of F are said to be measurable.

In what follows, (X, d, F, ν) will denote a given measure space, unless otherwise is explicitly stated.

LEMMA 3.2. *Let K and L in F be such that $\nu(K) \neq \nu(L)$. Then* $d'(K, L) > 0$.

Proof. Construct a sequence $\{a_n\}$ of 0 or 1's such that

$$d'(K, L) < n^{-1} \quad \text{if} \quad a_n = 0; \quad d'(K, L) > 0 \quad \text{if} \quad a_n = 1 .$$

For each natural number n such that $a_n = 0$, choose a positive real number b_n with $b_n < n^{-1}$ such that the set

$$K_n \equiv L_n \equiv \{x \in X: \quad d(x, K) \leqq b_n\}$$

is compact and belongs to F. By the definition of F this choice is possible. For each n such that $a_n = 1$, let $K_n = K$ and $L_n = L$. Obviously $d'(K, K_n) \leqq n^{-1}$ and $d'(L, L_n) \leqq 2n^{-1}$ for all n. Therefore by 3.1 (ii) we can find an n such that

$$|\nu(K_n) - \nu(K)| < 2^{-1}|\nu(L) - \nu(K)|$$

and

$$|\nu(L_n) - \nu(L)| < 2^{-1}|\nu(L) - \nu(K)| .$$

For this n, we cannot have $a_n = 0$, because this would imply $K_n = L_n$, contradicting the above inequalities. But then $a_n = 1$ and so $d'(K, L) > 0$.

In the following let g be a test function and $\alpha > 0$ be some real number such that $(|g| \geqq \alpha)$ is compact. Suppose $x_1, \cdots, x_n$ form a β-net for $(|g| \geqq \alpha)$ where β is some positive real number such shat $\beta < \alpha$ and $K_i = \{x: d(x, x_1) \wedge \cdots \wedge d(x, x_i) \leqq \beta\}$ belong to F for every $i = 1, \cdots, n$. If $-\beta$ belongs to the complement of some exceptional set for $-d(\cdot, x_1)$ $(i = 1, \cdots, n)$, write

$$S = \sum_{i=1}^{n} g(x_i)(\nu(K_i) - \nu(K_{i-1})) ,$$

where for convenience we write K_0 for the empty set. We will show that as α approaches 0, the sum S converges.

PROPOSITION 3.3. *With notation as above, the limit* $\lim_{\alpha\to 0} S$ *exists.*

Proof. Consider two sums $S = \sum_{i=1}^{n} g(x_i)(\nu(K_i) - \nu(K_{i-1}))$ where $x_1, \cdots, x_n$ form a β-net for $(|g| \geqq \alpha)$, and $S' = \sum_{j=1}^{m} (g'_j)(\nu(K'_j) - \nu(K'_{j-1}))$ where $x'_1, \cdots, x'_m$ form a β'-net *for* $(|g| \geqq \alpha')$. In proving that $|S - S'|$ is arbitrarily small if α and α' are small enough, we may assume that

K_i and K'_j $(i = 1, \cdots, n;\ j = 1, \cdots, m)$ are such that the compact sets obtained from them via the operations $\wedge$ and $\vee$ belong to F and are pairwise compatible. (If necessary we can replace β and β' by $\beta + \theta$ and $\beta' + \theta$ respectively, where θ is chosen according to the following restrictions. Firstly $-\theta$ belongs to the complement of some exceptional set for the functions generated via $\wedge$ and $\vee$ from the functions $\beta - d(\cdot, x_1) \wedge \cdots \wedge d(\cdot, x_i)$ and $\beta' - d(\cdot, x'_1) \wedge \cdots \wedge d(\cdot, x'_j)$, $(i = 1, \cdots, n;$ $j = 1, \cdots, m)$. Secondly, for every function f generated in this way, the compact set $(f \geqq -\theta)$ belongs to F. Furthur, we know by assumption that $-\beta$ belongs to the complement of some exceptional set A for the functions $d(\cdot, x_1), \cdots, d(\cdot, x_n)$, and we choose θ so that $-\beta - \theta$ belongs to $-A$ also; a similar relation is to hold for $-\beta'$ and $-\beta' - \theta$. Then the compact sets $\bar{K}_i \equiv \{x: d(x, x_1) \wedge \cdots \wedge d(x, x_i) \leqq \beta + \theta\} = \{x: \beta - d(x, x_1) \wedge \cdots \wedge d(x, x_i) \geqq -\theta\}$ and the compact sets $\bar{K}'_j \equiv \{x: d(x, x'_1) \wedge \cdots \wedge d(x, x'_j) \leqq \beta' + \theta\} = \{x: \beta' - d(x, x'_1) \wedge \cdots \wedge d(x, x'_j) \geqq -\theta\}$ will have the desired properties; namely, compact sets generated from them via $\wedge$ and $\vee$ belong to F and are pairwise compatible. From 3.1(ii) we see that the numbers θ, $|S - \bar{S}|$, and $|S' - \bar{S}'|$ can be made arbitrarily small.)

Let L be a compact set outside which g vanishes. Choose a number $\gamma > 1$ such that $K \equiv \{x: d(x, L) \leqq \gamma)$ belongs to F. Since we will be concerned only with small values of α and α', we may assume $K_n \subset K$ and $K'_m \subset K$. Now suppose $\varepsilon > 0$ is given. Let $\delta \in (0, \varepsilon)$ be so small that $d(x, x') < \delta$ implies $|g(x) - g(x')| < \varepsilon$. To estimate $|S - S'|$, write

$$S = \sum_{i=1}^{n} g(x_i)[\nu(K_i) - \nu(K_{i-1}) - \nu(K_i \wedge K'_m) + \nu(K_{i-1} \wedge K'_m)]$$
$$+ \sum_{i=1}^{n} g(x_i) \sum_{j=1}^{m} [\nu(K_i \wedge K'_j) - \nu(K_i \wedge K'_{j-1}) - \nu(K_{i-1} \wedge K'_j) + \nu(K_{j-1} \wedge K'_{j-1})]\ ,$$

with a similar expression for S'. Then, for α, $\alpha' < \delta/2$, we have

$$\begin{aligned}|S - S'| \leqq & \sum_{i=1}^{n} |g(x_i)[\nu(K_i) - \nu(K_{i-1}) - \nu(K_i \wedge K'_m) + \nu(K_{i-1} \wedge K'_m)]| \\
& + \sum_{i=1}^{n} \sum_{j=1}^{m} |(g(x_i) - g(x'_j))[\nu(K_i \wedge K'_j) - \nu(K_i \wedge K'_{j-1}) \\
& - \nu(K_{i-1} \wedge K'_j) + \nu(K_{i-1} \wedge K'_{j-1})]| \\
& + \sum_{j=1}^{m} |g(x'_j)[\nu(K'_j) - \nu(K'_{j-1}) - \nu(K'_j \wedge K_n) + \nu(K'_{j-1} \wedge K_n)]| \\
= & \sum_{i=1}^{n} |g(x_i)[\nu(K_i \vee K'_m) - \nu(K_{i-1} \vee K'_m)]| \\
& + \sum_{i=1}^{n} \sum_{j=1}^{m} |(g(x_i) - g(x'_j))[\nu((K_i \wedge K'_j) \vee (K_{i-1} \wedge K'_m)) \\
& - \nu((K_i \wedge K'_{j-1}) \vee (K_{i-1} \wedge K'_m))]|\end{aligned}$$

$$+ \sum_{j=1}^{m} |g(x'_j)[\nu(K'_j \vee K_n) - \nu(K'_{j-1} \vee K_n)]| \,.$$

The equality follows easily from 3.1(i). Suppose the ith term in the first sum is positive. Then, in view of 3.2, we can find x in $K_i \vee K'_m$ with $d(x, K_{i-1} \vee K'_m) > 0$. Obviously $d(x, x_j) \leqq \beta$ and $d(x, K'_m) > 0$. Since $K'_m \supset (|g| \geqq \alpha')$, we must have $|g(x)| \leqq \alpha' < \delta$ and so $|g(x_i)| \leqq \delta + \varepsilon < 2\varepsilon$. Hence the first sum in the bound for $|S - S'|$ is no greater than

$$2\varepsilon \sum_{i=1}^{n} |\nu(K_i \vee K'_m) - \nu(K_{i-1} \vee K'_m)| \leqq 2\varepsilon B_K \,,$$

where B_K is the constant associated to K as in 3.1(iii). Similarly, the third sum is bounded by $2\varepsilon B_K$. As for the second sum, if the summand indexed by (i, j) is positive, then 3.2 again implies the existence of some y in $(K_i \wedge K'_j) \vee (K_{i-1} \wedge K_m)$ with $d(y, (K_i \wedge K'_{j-1}) \vee (K_{i-1} \wedge K'_m)) > 0$. It follows from this that $d(y, x_i) \leqq \beta$ and $d(y, x'_j) \leqq \beta'$, and so $d(x_i, x'_j) \leqq \beta + \beta' < 2\alpha < \delta$. Therefore $|g(x_i) - g(x'_j)| \leqq \varepsilon$. Hence the second sum is bounded by

$$\varepsilon \sum_{i=1}^{n} \sum_{j=1}^{m} |\nu((K_i \wedge K'_j) \vee (K_{i-1} \wedge K'_m)) - \nu((K_i \wedge K'_{j-1}) \vee (K_{i-1} \wedge K'_m))| \leqq \varepsilon B_K \,.$$

(Here we used the fact that for every i,

$$K_i \wedge K'_m \supset \cdots \supset (K_i \wedge K'_j) \vee (K_{i-1} \wedge K'_m) \supset (K_i \wedge K'_{j-1}) \vee (K_{i-1} \wedge K'_m) \supset \cdots \supset (K_{i-1} \wedge K'_m) \,.)$$

Summing up, we have $|S - S'| \leqq 5\varepsilon B_K$. Since $\varepsilon > 0$ was arbitrary, the desired convergence follows.

PROPOSITION 3.4. *If we define a function* $\bar{\nu}$ *on* $C(X)$ *by* $\bar{\nu}(g) \equiv \int g d\bar{\nu} \equiv \lim_{\alpha \to 0} S$, *then* $\bar{\nu}$ *is an integral.*

Proof. The linearity of $\bar{\nu}$ is obvious from the definition of S. Suppose now $f \in C(X)$ is supported by the compact set L. Let K be associated to L as in the proof of 3.3. Let g be any test function such that $|g| \leqq |f|$. Then g is also supported by L, and by the definition of S we have $|S| \leqq \|g\| B_K$ as soon as α is so small that $K_n \subset K$. Thus $\left|\int g d\bar{\nu}\right| \leqq \|g\| B_K \leqq \|f\| B_K$. Therefore 2.1 is satisfied.

We now prove the following Riesz Representation Theorem for a signed integral on (X, d). This theorem also shows that the family of measure spaces as defined in 3.1 is not vacuous.

PROPOSITION 3.5. *Let* μ *be an integral on* (X, d). *Let* F *be the* *family of compact subsets of X which are strongly measurable with respect to μ, and let $\nu(K)$ be the strong measure of K for each K in F. Then (X, d, F, ν) is a measure space, and for each g in $C(X)$ we have $\int g d\mu = \int g d\bar{\nu}$, where $\bar{\nu}$ is the integral defined in* 3.4.

Proof. That (X, d, F, ν) is a measure space follows from 2.10, 2.11, 2.13, 2.14, and 2.15. To show that μ and $\bar{\nu}$ coincide on $C(X)$ consider any continuous function g which vanishes outside some compact set L. Consider a sum $S = \sum_{i=1}^{n} g(x_i)(\nu(K_i) - \nu(K_{i-1}))$ where $x_1, \cdots, x_n$ form a β-net for $(|g| \geqq \alpha)$ with $\beta < \alpha$. For each i let g_m^i be a sequence of test functions belonging to K_i such that $\int g_m^i d\mu$ converges to $\nu(K_i)$ as m approaches infinity. We may assume that $g_m^{i-1} \leqq g_m^i \leqq f$ where f is a nonnegative test function with $f(x) = 1$ whenever $d(x, L) \leqq 1$. Suppose $\varepsilon > 0$ is given. Let $\alpha < \varepsilon$ be so small, and m so large, that $d(x, x') \leqq m^{-1} + \alpha$ implies $|g(x) - g(x')| < \varepsilon$. Consider a point x in K_n. We have $\sum_{i=1}^{n} (g_m^i(x) - g_m^{i-1}(x)) = g_m^n(x) = 1$. (Here $g_m^0 \equiv 0$). Therefore

$$g(x) - \sum_{i=1}^{n} g(x_i)(g_m^i(x) - g_m^{i-1}(x)) = \sum_{i=1}^{n} (g(x) - g(x_i))(g_m^i(x) - g_m^{i-1}(x)) \, .$$

If the ith term is unequal to 0, then we must have $d(x, x_i) \leqq m^{-1} + \beta$ and so $|g(x) - g(x_i)| \leqq \varepsilon$. Therefore the above sum is bounded in absolute value by $\varepsilon \sum_{i=1}^{n} (g_m^i(x) - g_m^{i-1}(x)) = \varepsilon$. Next consider a point x with $d(x, K_n) > 0$. Then $|g(x)| \leqq \alpha < \varepsilon$. If the ith term in $\sum_{i=1}^{n} g(x_i)(g_m^i(x) - g_m^{i-1}(x))$ is unequal to 0, then we must have $d(x, x_i) < m^{-1} + \beta$ and so $|g(x_i)| \leqq |g(x)| + \varepsilon < 2\varepsilon$. Hence $|g(x) - \sum_{i=1}^{n} g(x_i)(g_m^i(x) - g_m^{i-1}(x))| < 3\varepsilon$.

In view of the continuity of g and the functions g_m^i, we conclude that the function $g - \sum_{i=1}^{n} g(x_i)(g_m^i - g_m^{i-1})$ is always bounded in absolute value by 3ε. This function is therefore bounded in absolute value by the function $3\varepsilon f$. Its integral $\int g d\mu - \int \sum_{i=1}^{n} g(x_i)(g_m^i - g_m^{i-1}) d\mu$ must then be bounded in absolute value by $3\varepsilon M_f$. (M_f is the constant associated to f in 2.1). Letting $m \to \infty$, we have $\left|\int g d\mu - S\right| \leqq 3\varepsilon M_f$. Then, letting $\alpha \to 0$, we have $\left|\int g d\mu - \int g d\bar{\nu}\right| \leqq 3\varepsilon M_f$. But ε was arbitrary. The integrals μ and $\bar{\nu}$ are equal.

The author wishes to thank Professor E. A. Bishop for his guidance, advice, and encouragement during the time when this research was being done.

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Received June 29, 1970. Part of a dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics at the University of California, San Diego. This work was partially supported by N.S.F. Grant G.P. 5094.

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