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RATIONAL WHITEHEAD PRODUCTS AND A SPECTRAL SEQUENCE OF QUILLEN

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A third order rational Whitehead product is defined in terms of the appropriate differential graded Lie algebra. The product is used to calculate the second differential in Quillen's rational reverse Adams spectral sequence. Some facts about a fourth order product are stated, and conjectures are made concerning higher order products. The products of this paper are compared to those defined by Zeeman, Hardie, and Porter.

In the paper "Rational homotopy theory" [7], Quillen introduces a very interesting rational unstable reverse Adams spectral sequence. He also introduces a functor, which assigns to each one-connected topological space a differential graded Lie algebra, whose homology is the rational homotopy Lie algebra of the space. In this paper we show, first, how higher order rational Whitehead products may be defined using the DG Lie algebra, secondly, how these Whitehead products may be used to calculate the differentials in Quillen's spectral sequence, and, thirdly, how these Whitehead products compare to those defined by Zeeman, Hardie, Porter, et al.

In the first section we review some of the notation and material of [7]. In the second section we define the third order rational Whitehead product, and show how it and the ordinary (second order) rational Whitehead product fit into Quillen's spectral sequence. Further, we indicate how the higher order products may be defined, but an explicit definition is included only for the fourth order product. In the final section a theorem is established, which provides a mechanism for comparing our rational products with those defined classically, and this comparison is carried out for the third order product.

1. Review of material and notation from [7]. We are concerned with the following three categories: $\mathscr{T}_2$, the category of 1-connected pointed topological spaces, and basepoint preserving continuous maps, whose weak equivalences are those maps which induce isomorphisms on rational homotopy.

$(DGL)_1$, the category of reduced differential graded Lie algebras over Q and DG Lie algebra morphisms, whose weak equivalences are those morphisms which induce isomorphisms on homology.

$(DGC)_2$, the category of 2-reduced differential graded coalgebras over

Q and DG coalgebra morphisms, whose weak equivalences are, again, those morphisms which induce isomorphisms on homology.

Quillen defines, implicitly, a functor

$$\mu\colon \mathscr{T}_2 \longrightarrow (DGL)_1 .$$

For a space X in $\mathscr{T}_2$, the homology of $\mu(X)$ is the rational homotopy Lie algebra of X, $L_*(X)$. The latter may be obtained by setting $L_n(X) = \pi_{n+1}(X) \otimes Q$, and defining the Lie bracket of $\alpha \in L_n(X)$ and $\beta \in L_m(X)$ by the formula

$$[\alpha, \beta] = (-1)^{n+1}[a, b] \otimes rs \in \pi_{n+m+1}(X) \otimes Q = L_{n+m}(X) ,$$

where r and s are rational numbers such that $\alpha = a \otimes r$ and $\beta = b \otimes s$, and where the bracket on the right is the conventional Whitehead product in $\pi_*(X)$. With this definition $L_*(X)$ is isomorphic to $\pi_*(\Omega X) \otimes Q$, endowed with the Samelson product.

Quillen also defines a functor

$$\mathscr{C}\colon (DGL)_1 \longrightarrow (DGC)_2 ,$$

which has the property that, for a space X in $\mathscr{T}_2$, the homology of $\mathscr{C}\mu(X)$ is precisely the rational singular homology coalgebra of X.

For an object of $\mathscr{T}_2$, X, say, by considering the primitive filtration on $\mathscr{C}\mu(X)$, Quillen obtains a coalgebra spectral sequence,

$$E^1 = S(\Sigma L_*(X)) \Rightarrow H_*(X; Q) ,$$

where Σ is the functor which raises the grading degree by one (e.g. $(\Sigma L_*(X))_n = L_{n-1}(X)$), and S is the symmetric algebra functor. The construction is functorial, and so the edge map

$$\pi_n(X) \otimes Q \cong E^1_{1,n-1} \xrightarrow{\text{onto}} E^\infty_{1,n-1} \xrightarrow{\subseteq} H_n(X; Q)$$

is the rational Hurewicz homomorphism.

In order to establish the notation needed for the calculations of §2, below, it is necessary to review in detail Quillen's construction of the functor $\mathscr{C}$. To this end let L be a reduced DG Lie algebra, and define a DG Lie algebra $\Sigma L \sharp L$. As a graded vector space $\Sigma L \sharp L$ is the direct sum of ΣL and L. The inclusion maps $L \to \Sigma L \to \Sigma L \sharp L$ and $L \to \Sigma L \sharp L$ are denoted by Σ and θ respectively. (In addition, if x is an element of degree n in L, we shall denote by Σx the corresponding element of degree $n+1$ in ΣL.) The Lie bracket on $\Sigma L \sharp L$ is defined by requiring that, for all x and y in L,

(i) $[\Sigma x, \Sigma y] = 0$

(ii) $[\Sigma x, \theta y] = \Sigma[x, y]$

and

(iii) $[\theta x, \theta y] = \theta[x, y]$.

The differential on $\Sigma L \sharp L$ is defined by requiring that, for all x in L,

(i) $d\Sigma x = \theta x - \Sigma d'x$,

and

(ii) $d\theta x = \theta d'x$,

where d' is the differential on L. Letting U denote the universal enveloping algebra functor, θ induces a right $U(L)$-module structure on $U(\Sigma L \sharp L)$. $\mathscr{C}(L)$ is then the coalgebra $U(\Sigma L \sharp L) \otimes_{U(L)} Q$, and the differential on $\mathscr{C}$ is obtained by requiring that the natural surjection, π: $U(\Sigma L \sharp L) \to \mathscr{C}(L)$, be a DG coalgebra morphism. If i: $S(\Sigma L) \to U(\Sigma L \sharp L)$ is the Hopf algebra map induced by the inclusion $\Sigma L \to \Sigma L \sharp L$, then πi: $S(\Sigma L) \to \mathscr{C}(L)$ is a coalgebra isomorphism.

2. Higher order Whitehead products and the spectral sequence.

First we show how the ordinary (second order) Whitehead product determines the differential, d^1, of the spectral sequence.

THEOREM 2.1. *Let X be a one-connected pointed topological space, and let a_i be an element of $L_{n_i}(x)$, for $i = 1, 2$. Then in Quillen's spectral sequence for X,*

$$d^1(\Sigma a_1 \Sigma a_2) = (-1)^{n_1}\Sigma[a_1, a_2] .$$

Proof. Let $\alpha_i \in \mu_{n_i}(x)$ be a cycle which represents a_i $(i = 1, 2)$. Then in $U(\Sigma\mu(x) \sharp \mu(x))$ we have

$$d(\Sigma\alpha_1\Sigma\alpha_2) = \theta\alpha_1\Sigma\alpha_2 + (-1)^{n_1+1}\Sigma\alpha_1\theta\alpha_2 .$$

Now $[\theta\alpha_1, \Sigma\alpha_2] = \theta\alpha_1\Sigma\alpha_2 - (-1)^{n_1(n_2+1)}\Sigma\alpha_2\theta\alpha_1$. Hence $\pi d(\Sigma\alpha_1\Sigma\alpha_2) = \pi[\theta\alpha_1, \Sigma\alpha_2] = (-1)^{n_1}\pi\Sigma[\alpha_1, \alpha_2]$. I.e., where d' denotes the differential on $\mathscr{C}\mu(X)$,

$$d'\pi i(\Sigma\alpha_1\Sigma\alpha_2) = (-1)^{n_1}\pi i\Sigma[\alpha_1, \alpha_2] .$$

The result now follows, because of the identifications made in constructing the spectral sequence.

Now suppose that $a_i \in L_{n_i}(X)$, for $i = 1, 2, 3$, and that $[a_2, a_3]$, $[a_3, a_1]$, and $[a_1, a_2]$ are all zero. Let α_i be a cycle in $\mu_{n_i}(X)$, for $i = 1, 2, 3$, such that α_i represents a_i. For $1 \leqq i, j \leqq 3$, and $i \neq j$, let ξ_{ij} be an element of $\mu_{n_i+n_j+1}(X)$, such that in $\mu(X)$, $d\xi_{ij} = [\alpha_i, \alpha_j]$. Then by the Jacobi identity,

$$(-1)^{n_1n_3}[\xi_{12}, \alpha_3] + (-1)^{n_2n_1}[\xi_{23}, \alpha_1] + (-1)^{n_3n_2}[\xi_{31}, \alpha_2]$$

is a cycle in $\mu(X)$. Call this cycle $\langle\alpha_1, \alpha_2, \alpha_3; \xi_{ij}\rangle$.

DEFINITION. With the notation and conditions of the above paragraph, we define the third order rational Whitehead product of a_1, a_2, a_3, denoted by $[a_1, a_2, a_3]$, to be the set of all elements of $L_{n_1+n_2+n_3+1}(X)$ represented by cycles of the form $\langle \alpha_1, \alpha_2, \alpha_3; \xi_{ij} \rangle$.

Let $n_1 + n_2 + n_3 = n$, and let $J_{n+1}(a_1, a_2, a_3)$ be the vector subspace of $L_{n+1}(X)$ spanned by all elements of the form $[a_1, x]$, $[a_2, y]$, and $[a_3, z]$, for all $x \in L_{n+1-n_1}(X)$, $y \in L_{n+1-n_2}(X)$, and $z \in L_{n+1-n_3}(X)$. The following proposition shows that $J_{n+1}(a_1, a_2, a_3)$ is the indeterminancy of $[a_1, a_2, a_3]$.

PROPOSITION 2.2. *With the above notation,* $[a_1, a_2, a_3]$ *is a coset of* $J_{n+1}(a_1, a_2, a_3)$ *in* $L_{n+1}(X)$.

Proof. Let a be an element of $[a_1, a_2, a_3] \subseteqq L_{n+1}(X)$, and let $x \in L_{n+1-n_1}(X)$. Suppose a is represented by $\langle \alpha_1, \alpha_2, \alpha_3; \xi_{ij} \rangle$, and let $\xi \in \mu_{n_2+n_3+1}(X)$ be a cycle representing x. Then $[\xi, \alpha_1]$ represents $[x, a_1]$, and $\langle \alpha_1, \alpha_2, \alpha_3; \xi_{ij} \rangle + [\xi, \alpha_1]$ represents $a + [x, a_1]$. But $\langle \alpha_1, \alpha_2, \alpha_3; \xi_{ij} \rangle + [\xi, \alpha_1] = \langle \alpha_1, \alpha_2, \alpha_3; \xi'_{ij} \rangle$, where $\xi'_{12} = \xi_{12}$, $\xi'_{23} = \xi_{23} + (-1)^{n_2 n_1}\xi$, and $\xi'_{31} = \xi_{31}$.
$\therefore \quad a + [x, a_1] \in [a_1, a_2, a_3]$.
$\therefore \quad a + J_{n+1}(a_1, a_2, a_3) \subseteqq [a_1, a_2, a_3]$.

Now let $b \in [a_1, a_2, a_3]$ be represented by a cycle $\langle \alpha'_1, \alpha'_2, \alpha'_3; \xi'_{ij} \rangle$. Since α'_i represents a_i, there must exist $\beta_i \in \mu_{n_i+1}(X)$, such that $\alpha'_i = \alpha_i + d\beta_i$. Suppose that β_2 and β_3 are zero. Then

$$\begin{aligned} d\xi'_{12} &= [\alpha'_1, \alpha'_2] = [\alpha_1, \alpha_2] + [d\beta_1, \alpha_2] \\ d\xi'_{31} &= [\alpha'_3, \alpha'_1] = [\alpha_3, \alpha_1] + [\alpha_3, d\beta_1] \\ d\xi'_{23} &= [\alpha'_2, \alpha'_3] = [\alpha_2, \alpha_3] \end{aligned}$$

$$\begin{aligned} \therefore \quad & \xi_{12} - \xi'_{12} + [\beta_1, \alpha_2] \ , \\ & \xi_{31} - \xi'_{31} + (-1)^{n_3}[\alpha_3, \beta_1] \ , \end{aligned}$$

and $\xi_{23} - \xi'_{23}$ are cycles. Call these cycles ζ_{12}, ζ_{31}, and ζ_{23}, respectively. So

$$\begin{aligned} \langle \alpha'_1, \alpha'_2, \alpha'_3; \xi'_{ij} \rangle &= \langle \alpha_1, \alpha_2, \alpha_3; \xi_{ij} \rangle + (-1)^{n_1 n_3}[[\beta_1, \alpha_2], \alpha_3] \\ &\quad - (-1)^{n_1 n_3}[\zeta_{12}, \alpha_3] + (-1)^{n_2 n_1}[\xi_{23}, d\beta_1] - (-1)^{n_2 n_1}[\zeta_{23}, \alpha'_1] \\ &\quad + (-1)^{n_3 n_2 + n_3}[[\alpha_3, \beta_1], \alpha_2] - (-1)^{n_3 n_2}[\zeta_{31}, \alpha_2] \\ &= \langle \alpha_1, \alpha_2, \alpha_3; \xi_{ij} \rangle - (-1)^{n_1 n_3}[\zeta_{12}, \alpha_3] \\ &\quad - (-1)^{n_3 n_2}[\zeta_{31}, \alpha_2] - (-1)^{n_1 n_2}[\zeta_{23}, \alpha'_1] \\ &\quad + (-1)^{n_2 + n_3 + n_1 n_2 + 1} d[\xi_{23}, \beta_1] \ . \end{aligned}$$

Let the homology classes of ζ_{23}, ζ_{31}, and ζ_{12} be x, y, and z, respectively. Then

$$b = a - (-1)^{n_1 n_3}[z, a_3] - (-1)^{n_3 n_2}[y, a_2] - (-1)^{n_1 n_2}[x, a_1] \ .$$

Hence, for all $a, b \in [a_1, a_2, a_3]$, $a - b \in J_{n+1}(a_1, a_2, a_3)$.

In the following, if $\{E^r_{ij}\}$ is the Quillen spectral sequence of some space, and if x is an element of E^r_{ij} such that $d^r(x) = 0$, then we shall let $[x]^{(r+1)}$ denote the class in E^{r+1}_{ij} which is represented by x.

COROLLARY 2.3. *Let X be a space of $\mathscr{T}_2$, let $a_i \in L_{n_i}(X)$, $i = 1, 2, 3$, be such that $[a_2, a_3]$, $[a_3, a_1]$, and $[a_1, a_2]$ are all zero. Let $a, b \in [a_1, a_2, a_3] \subseteq L_{n+1}(X)$, where $n = n_1 + n_2 + n_3$. Then in the Quillen spectral sequence of X, $[\Sigma a]^{(2)} = [\Sigma b]^{(2)}$ in $E^2_{1,n+1}$.*

THEOREM 2.4. *With X, a_1, a_2, a_3 and a as in the above corollary, then in the Quillen spectral sequence of X, $d^1(\Sigma a_1 \Sigma a_2 \Sigma a_3) = 0$, and*

$$d^2[\Sigma a_1 \Sigma a_2 \Sigma a_3]^{(2)} = (-1)^{n_1 n_3 + n_2 + 1}[\Sigma a]^{(2)} .$$

Proof. We calculate as in the proof of Theorem 2.1. Let $\alpha_i \in \mu_{n_i}(X)$ be a cycle which represents a_i $(i = 1, 2, 3)$. Then in $U(\Sigma\mu(X) \# \mu(X))$ we have

$$\begin{aligned} d(\Sigma\alpha_1\Sigma\alpha_1\Sigma\alpha_3) &= d(\Sigma\alpha_1\Sigma\alpha_2)\Sigma\alpha_3 + (-1)^{n_1+n_2}\Sigma\alpha_1\Sigma\alpha_2 d(\Sigma\alpha_3) \\ &= \theta\alpha_1\Sigma\alpha_2\Sigma\alpha_3 + (-1)^{n_1+1}\Sigma\alpha_1\theta\alpha_2\Sigma\alpha_3 + (-1)^{n_1+n_2}\Sigma\alpha_1\Sigma\alpha_2\Sigma\alpha_3 \end{aligned}$$

$$\begin{aligned} \therefore \quad \pi d(\Sigma\alpha_1\Sigma\alpha_2\Sigma\alpha_3) &= \pi([\theta\alpha_1, \Sigma\alpha_2]\Sigma\alpha_3 + (-1)^{n_1(n_2+1)}\Sigma\alpha_2\Sigma\alpha_1\Sigma\alpha_3 \\ &\quad + (-1)^{n_1+1}\Sigma\alpha_1[\theta\alpha_2, \Sigma\alpha_3]) \\ &= \pi((-1)^{n_1}\Sigma[\alpha_1, \alpha_2]\Sigma\alpha_3 + (-1)^{n_1 n_2}\Sigma\alpha_2\Sigma[\alpha_1, \alpha_3] \\ &\quad + (-1)^{n_1+n_2+1}\Sigma\alpha_1\Sigma[\alpha_2, \alpha_3]) . \end{aligned}$$

Now

$$\begin{aligned} \Sigma[\alpha_1, \alpha_j]\Sigma\alpha_k &= \Sigma d\xi_{ij}\Sigma\alpha_k \\ &= \theta\xi_{ij}\Sigma\alpha_k - d(\Sigma\xi_{ij})\Sigma\alpha_k \\ &= \theta\xi_{ij}\Sigma\alpha_k - d(\Sigma\xi_{ij}\Sigma\alpha_k) + (-1)^{n_i+n_j+1}\Sigma\xi_{ij}d\Sigma\alpha_k \\ &= \theta\xi_{ij}\Sigma\alpha_k - d(\Sigma\xi_{ij}\Sigma\alpha_k) + (-1)^{n_i+n_j+1}\Sigma\xi_{ij}\theta\alpha_k , \end{aligned}$$

and

$$\begin{aligned} \Sigma\alpha_i\Sigma[\alpha_j, \alpha_k] &= \Sigma\alpha_i\theta\xi_{jk} - \Sigma\alpha_i d(\Sigma\xi_{jk}) \\ &= \Sigma\alpha_i\theta_{jk} + (-1)^{n_i}d(\Sigma\alpha_i\Sigma\xi_{jk}) + (-1)^{n_i+1}\theta\alpha_i\Sigma\xi_{jk} , \end{aligned}$$

where ξ_{ij} is such that $d\xi_{ij} = [\alpha_i, \alpha_j]$. Again let d' be the differential in $\mathscr{C}\mu(X)$. Then

$$\begin{aligned} d'\pi i(\Sigma\alpha_1\Sigma\alpha_2\Sigma\alpha_3) &= \pi i((-1)^{n_2+1}\Sigma[\xi_{12}, \alpha_3] \\ &\quad + (-1)^{n_1 n_2+1}\Sigma[\alpha_2, \xi_{13}] + (-1)^{n_1 n_2}\Sigma[\alpha_1, \xi_{23}]) \end{aligned}$$

$$
\begin{aligned}
&+ d'\pi i((-1)^{n_1+1}\Sigma\xi_{12}\Sigma\alpha_3 + (-1)^{n_1n_2+n_2}\Sigma\alpha_2\Sigma\xi_{13} \\
&+ (-1)^{n_2+1}\Sigma\alpha_1\Sigma\xi_{23}) \\
= {}& \pi i((-1)^{n_2+1}\Sigma[\xi_{12}, \alpha_3] + (-1)^{n_1n_3+n_2n_3+n_2+1}\Sigma[\xi_{31}, \alpha_2] \\
&+ (-1)^{n_1n_2+n_1n_3+n_2+1}\Sigma[\xi_{23}, \alpha_1]) + d'\pi i((-1)^{n_1+1}\Sigma\xi_{12}\Sigma\alpha_3 \\
&+ (-1)^{n_1n_3+n_2n_3+n_1+n_2+n_3+1}\Sigma\xi_{31}\Sigma\alpha_2 \\
&+ (-1)^{n_1n_2+n_1n_3+n_3+1}\Sigma\xi_{23}\Sigma\alpha_1) \\
= {}& (-1)^{n_1n_3+n_2+1}\pi i\Sigma\langle\alpha_1, \alpha_2, \alpha_3; \xi_{ij}\rangle + d'\pi i\zeta ,
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta = {}& (-1)^{n_1+1}\Sigma\xi_{12}\Sigma\alpha_3 + (-1)^{n_1n_3+n_2n_3+n_1+n_2+n_3+1}\Sigma\xi_{31}\Sigma\alpha_2 \\
&+ (-1)^{n_1n_2+n_1n_3+n_3+1}\Sigma\xi_{23}\Sigma\alpha_1 .
\end{aligned}
$$

Again the result follows because of the identifications made in constructing the spectral sequence. In particular, the term $d'\pi i\zeta$ is explained away by formula (3.8) of MacLane [3], Chapter XI, §3, applied to $E^2_{3,n}$.

REMARK 1. It is clear from the definition that the third order product has the usual functoriality, and the usual trilinearity but different change of sign under permutation.

Specifically, if X and Y are spaces of $\mathscr{T}_2$, if $f: X \to Y$ is a base-point preserving map, and if $[a_1, a_2, a_3]$ is a well defined third order product in $L_*(X)$, then $[f_*(a_1), f_*(a_2), f_*(a_3)]$ is well defined in $L_*(Y)$, and

$$
f_*([a_1a_2a_3]) \subseteqq [f_*(a_1), f_*(a_2), f_*(a_3)] .
$$

Also if $[a_1, a_2, a_3]$ and $[a_1, a_2, a_3']$ are both well defined, then so is $[a_1, a_2, a_3 + a_3']$, and

$$
[a_1, a_2, a_3 + a_3'] \subseteqq [a_1, a_2, a_3] + [a_1, a_2, a_3'] .
$$

Similarly for the other variables.

However, $[a_1, a_2, a_3] = (-1)^{n_1n_2+n_1n_3+n_2n_3+1}[a_2, a_1, a_3] = [a_2, a_3, a_1]$, and so on. (Cf. Hardie [2].)

REMARK 2. The general definition for kth order products seems hard to calculate. For example, let X be a space in $\mathscr{T}_2$, let $a_i \in L_{n_i}(X)$, $i = 1, 2, 3, 4$, be elements such that

(i) $i \neq j \Rightarrow [a_i, a_j] = 0$,

and

(ii) i, j, k all distinct $\Rightarrow 0 \in [a_i, a_j, a_k]$.

Let ξ_{ij}, $i \neq j$, and ξ_{ijk}, i, j, k all distinct, be such that

(i) $d\xi_{ij} = [\alpha_i, \alpha_j]$

and

(ii) $d\xi_{ijk} = \langle \alpha_i, \alpha_j, \alpha_k; \xi_{ij} \rangle$, where α_i is a cycle in $\mu_{n_i}(X)$, representing a_i, $i = 1, 2, 3, 4$. Then a calculation similar to the above yields the following result:

$$d^3[\Sigma a_1 \Sigma a_2 \Sigma a_3 \Sigma a_4]^{(3)} = [\Sigma a]^{(3)} ,$$

where $a \in L_{n_1+n_2+n_3+n_4+2}(X)$ is represented by the cycle

$$\begin{aligned} &(-1)^{n_1n_3+n_1+n_3+1}[\xi_{123}, \alpha_4] + (-1)^{n_1n_4+n_3n_4+n_1+n_3}[\xi_{124}, \alpha_3] \\ &+ (-1)^{n_1n_4+n_2n_3+n_2n_4+n_1+n_3+1}[\xi_{134}, \alpha_2] + (-1)^{n_1n_2+n_1n_3+n_1n_4+n_2n_4+n_1+n_3}[\xi_{234}, \alpha_1] \\ &+ (-1)^{n_2+n_3+1}[\xi_{12}, \xi_{34}] + (-1)^{n_2n_3}[\xi_{13}, \xi_{24}] + (-1)^{n_2n_4+n_3n_4+n_3+n_4+1}[\xi_{14}, \xi_{23}] . \end{aligned}$$

Before stating a conjecture for the general form of the kth order product, we recall from May [4], for example, that if $p + q = k$, then (μ, ν) is a (p, q)-shuffle if $\mu(i) = \pi(i)$ for $1 \leqq i \leqq p$ and $\nu(j) = \pi(p + j)$ for $1 \leqq j \leqq q$, where π is a permutation of $\{1, \cdots, k\}$ such that $\pi(i) < \pi(j)$ for $1 \leqq i < j \leqq p$ or for $p + 1 \leqq i < j \leqq k$. (μ, ν) is a (p, q)-shuffle of type II relative to 1, if $\mu(1) = 1$. Now suppose that $x_i \in L_{n_i}(X)$, $i = 1, \cdots, k$, are elements such that $1 \leqq i_1 < \cdots < i_r \leqq k$ and $2 \leqq r \leqq k - 1$ imply that $[x_{i_1}, \cdots, x_{i_r}]$ is defined and includes zero. Let ξ_i be a cycle in $\mu_{n_i}(X)$ which represents x_i, $1 \leqq i \leqq k$, and for $1 \leqq i_1 < \cdots < i_r \leqq k$ and $2 \leqq r \leqq k - 1$, let $\xi_{i_1 \cdots i_r}$ be an element of $\mu(X)$, such that $d\xi_{i_1 \cdots i_r}$ represents the element of $[x_{i_1}, \cdots, x_{i_r}]$, represented by the cycle $\langle \xi_{i_1}, \xi_{i_1 i_2}, \cdots, \xi_{i_1 \cdots i_{r-1}} \rangle$ constructed recursively as follows.

For $k = 2$, $\langle \xi_{i_1} \rangle = (-1)^{n_1}[\xi_1, \xi_2]$.

For $k = 3$, $\langle \xi_{i_1}, \xi_{i_1 i_2} \rangle = (-1)^{n_1n_3+n_2+1} \langle \xi_1, \xi_2, \xi_3; \xi_{i_1 i_2} \rangle$, as above.

$$\langle \xi_{i_1}, \xi_{i_1 i_2}, \cdots, \xi_{i_1 \cdots i_{k-1}} \rangle = \sum_{p=1}^{k-1} \sum \pm [\xi_\mu, \xi_\nu] ,$$

where the second sum is taken over all (p, q)-shuffles, (μ, ν), of type II relative to 1, with $p + q = k$, and where $\xi_\mu = \xi_{\mu(1) \cdots \mu(p)}$ and $\xi_\nu = \xi_{\nu(1) \cdots \nu(q)}$. The conjecture is that the signs of the terms, $[\xi_\mu, \xi_\nu]$, can be chosen to make $\langle \xi_{i_1}, \xi_{i_1 i_2}, \cdots, \xi_{i_1 \cdots i_{k-1}} \rangle$ into a cycle, representing an element x, say, of $L_{n+k-2}(X)$, where $n = \sum_{i=1}^k n_i$; and that such elements define the kth order product $[x_1, \cdots, x_k]$. Furthermore, we make the conjecture that if x is such an element of $[x_1, \cdots, x_k]$, then in the Quillen spectral sequence for X, $[\Sigma x_1 \cdots \Sigma x_k]^{(k-1)}$ exists, and

$$d^{k-1}[\Sigma x_1 \cdots \Sigma x_k]^{(k-1)} = [\Sigma x]^{(k-1)} .$$

3. Comparison with the classical definition. We recall from Hardie [2], Porter [6], or more specifically, Arkowitz [1], the following definition of a kth order Whitehead product in ordinary homotopy. Let X be a space in $\mathscr{T}_2$, and let $a_i \in \pi_{n_i+1}$, $i = 1, \cdots, k$, be homotopy classes. Let P be the product, $S^{n_1+1} \times \cdots \times S^{n_k+1}$, let S be the (lean) wedge, $S^{n_1+1} \vee \cdots \vee S^{n_k+1}$, and let T be the fat wedge, $T(S^{n_1+1}, \cdots, S^{n_k+1})$, consisting of k-tuples from P, which have at least one coordinate at the base point. Let $g: S \to X$ be the map induced by the a_i. That is, we choose a particular representative $f_i: S^{n_i+1} \to X$, $i = 1, \cdots, k$, for each a_i; and g is the unique map such that $g \circ in_i = f_i$, $i = 1, \cdots, k$, where $in_i: S^{n_i+1} \to S$ is the natural inclusion. Let $n = \sum_{i=1}^{k} n_i$, and choose a generator $z \in H_{n+k}(P, T)$. Suppose there exists an extension of g, $\hat{g}: T \to X$. Then we have a diagram

$$H_{n+k}(P, T) \xleftarrow{h} \pi_{n+k}(P, T) \xrightarrow{\partial} \pi_{n+k-1}(T) \xrightarrow{\hat{g}_\#} \pi_{n+k-1}(X) ,$$

where the Hurewicz map h is an isomorphism. The kth order Whitehead product, $[a_1, \cdots, a_k]$, is defined to be $\{\hat{g}_\# \partial h^{-1}(z): \hat{g}$ is an extension of $g\} \subseteqq \pi_{n+k-1}(X)$. If no extensions exist, then $[a_1, \cdots, a_k]$ is empty.

In order to obtain kth order Whitehead products in rational homotopy, we repeat the above construction after localizing the spaces concerned at the empty set of primes. (Here we follow a suggestion of the referee.) This localization will be denoted, as usual, by the subscript zero. Thus for $a_i \in L_{n_i}(X) \cong \pi_{n_i+1}(X_0)$, $i = 1, \cdots, k$, we obtain a map $g_0: S_0 \to X_0$, and define the kth order rational Whitehead product, $[a_1, \cdots, a_k]'$, to be $\{\hat{g}_{0\#} \partial h^{-1}(z_0): \hat{g}_0$ is an extension of g_0 to $T_0\}$. The element z_0 is just the image of z, above, under the localization map, and ∂h^{-1} comes from the diagram

$$H_{n+k}(P_0, T_0) \xleftarrow{h} \pi_{n+k}(P_0, T_0) \xrightarrow{\partial} \pi_{n+k-1}(T_0) .$$

We have, then, $[a_1, \cdots, a_k]' \subseteqq \pi_{n+k-1}(X_0) \cong L_{n+k-2}(X)$.

Throughout the following, we shall assume that $\min\{n_i: 1 \leqq i \leqq k\} \geqq 1$, so that S, T, and P are all one-connected. We shall assume also that $k \geqq 2$.

The following lemma shows that the classical rational Whitehead products are not trivial.

LEMMA 3.1. *With the above notation,*

$$\partial h^{-1}(z_0) \neq 0 \in \pi_{n+k-1}(T_0) .$$

Proof. The inclusion, $S \to P$, induces an epimorphism, $\pi_n(S) \to \pi_n(P)$, for all n. Hence the inclusion, $T \to P$, induces an epimorphism, $\pi_n(T) \to \pi_n(P)$, for all n. The map, $\partial: \pi_{n+k}(P, T) \to \pi_{n+k-1}(T)$, is, therefore a monomorphism. $\pi_{n+k}(P, T) \cong H_{n+k}(P, T) \cong Z$, and the result follows.

The next proposition will be useful for analyzing the Quillen spectral sequence of P, and hence of T, too.

PROPOSITION 3.2. *Let $X_1, \cdots, X_n$ be spaces of $\mathscr{T}_2$. Then the Quillen spectral sequence of $X_1 \times \cdots \times X_n$, $E^r(X_1 \times \cdots \times X_n)$, say, is the tensor product of the spectral sequences of $X_1, \cdots, X_n$, $E^r(X_1), \cdots, E^r(X_n)$, say. That is, for all $r \geqq 1$, there is a DG-coalgebra isomorphism*

$$E^r(X_1 \times \cdots \times X_n) \cong E^r(X_1) \otimes \cdots \otimes E^r(X_n) .$$

Proof. Using induction it is clearly enough to show the result for the product of two spaces, X and Y, say.

For DG Lie algebras, L and L', let $L \times L'$ denote their product. We recall that as a vector space $L \times L'$ is just $L \oplus L'$, and the differential and Lie bracket are given by the obvious formulae:

$$d(x, x') = (dx, d'x') ,$$
$$[(x, x'), (y, y')] = ([x, y], [x', y']) .$$

This is clearly a product in the category $(DGL)_1$.

First we show that there exists a weak equivalence

$$\theta\colon \mu(X \times Y) \longrightarrow \mu(X) \times \mu(Y) .$$

Let p_1, p_2 be the projections of $X \times Y$ onto X, Y respectively. Then we have a diagram in $(DGL)_1$:

$$\begin{tikzcd} \mu(X) & \mu(X \times Y) \arrow[l, "\mu(p_1)"'] \arrow[r, "\mu(p_2)"] & \mu(Y) \\ & \mu(X) \times \mu(Y) \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

where π_1 and π_2 are the projections. Then there exists a unique

$$\theta\colon \mu(X \times Y) \longrightarrow \mu(X) \times \mu(Y)$$

making the diagram commute. To see that θ is a weak equivalence, take homology. We then have the commutative diagram:

$$\begin{tikzcd} L_*(X) & L_*(X \times Y) \arrow[l, "L_*(p_1)"'] \arrow[r, "L_*(p_2)"] \arrow[d, "H(\theta)"] & L_*(Y) \\ & L_*(X) \times L_*(Y) . \arrow[ul, "H(\pi_1)"] \arrow[ur, "H(\pi_2)"'] & \end{tikzcd}$$

$H(\theta)$ must be an isomorphism, since it is unique in making the diagram commute, and it is known from elementary homotopy theory that there is an isomorphism, which makes this diagram commute.

For DG coalgebras A and B, denote their product by $A \prod B$. The functor $\mathscr{C}: (DGL)_1 \to (DGC)_2$ is a right adjoint, and so preserves products. (Pareigis [5].) Also $\mathscr{C}$ carries weak equivalences into weak equivalences. Hence there is a weak equivalence in $(DGC)_2$,

$$\mathscr{C}(\theta): \mathscr{C}\mu(X \times Y) \longrightarrow \mathscr{C}\mu(X) \prod \mathscr{C}\mu(Y) .$$

As a DG coalgebra, $\mathscr{C}\mu(X) \prod \mathscr{C}\mu(Y) \cong \mathscr{C}\mu(X) \otimes \mathscr{C}\mu(Y)$; and, where F_m, $m \geqq 0$, denotes the primitive filtration, it is clear that

$$F_m(\mathscr{C}\mu(X) \otimes \mathscr{C}\mu(Y)) = \sum_{p+q=m} F_p(\mathscr{C}\mu(X)) \otimes F_q(\mathscr{C}\mu(Y)) .$$

Then $\mathscr{C}(\theta)$ induces a weak equivalence

$$\mathscr{C}(\theta)^\circ: E^\circ(X \times Y) \longrightarrow E^\circ(X) \otimes E^\circ(Y) ,$$

and, hence, $\mathscr{C}(\theta)$ induces isomorphisms

$$\mathscr{C}(\theta)^r: E^r(X \times Y) \longrightarrow E^r(X) \otimes E^r(Y)$$

for $r \geqq 1$.

COROLLARY 3.3. *In the Quillen spectral sequence of P, the differential d^r is trivial for $r \geqq 2$.*

Proof. The Quillen spectral sequence is trivial for odd dimensional spheres, and has d^1 as the only nontrivial differential for even dimensional spheres.

In the following theorem we shall continue to use the above notation. Let $a_i \in L_{n_i}(T)$ be the homotopy classes induced by the inclusions, $S^{n_i+1} \to T$, $i = 1, \cdots, k$, and let

$$a = \partial h^{-1}(z_0) \in \pi_{n+k-1}(T_0) \cong L_{n+k-2}(T) .$$

PROPOSITION 3.4. *In the Quillen spectral sequence for T, $[\Sigma a_1 \cdots \Sigma a_k]^{(k-1)}$ exists, $[\Sigma a]^{(k-1)}$ is nonzero, and there is a nonzero rational number α, such that*

$$d^{k-1}[\Sigma a_1 \cdots \Sigma a_k]^{(k-1)} = \alpha[\Sigma a]^{(k-1)} .$$

Proof. We compare the spectral sequence of T with that of P via the inclusion map $i: T \to P$. For $r \geqq 1$, i induces an isomorphism on all terms generated by $\sum_{m=1}^{n+k-3} E^1_{1,m}(T)$. It is clear then, that

(i) $d^r[\Sigma a_1 \cdots \Sigma a_k]^{(r)} = 0$ for $1 \leqq r < k - 1$,

and

(ii) $[\Sigma a]^{(k-1)} \neq 0$.

Furthermore, if $[\Sigma a]^{(k-1)}$ and $[\Sigma a_1 \cdots \Sigma a_k]^{(k-1)}$ survive to $E^k(T)$, they must survive to $E^\infty(T)$, since there are no terms to kill them. But killed they must be, since $\dim T \leqq n + k - 2$. The result follows.

Now let X be a space in $\mathscr{T}_2$, and let $x_i \in \pi_{n_i+1}(X)$, $i = 1, 2, 3$, be homotopy classes such that $[x_2, x_3]$, $[x_3, x_1]$ and $[x_1, x_2]$ are all of finite order. Let $a_i = x_i \otimes 1 \in L_{n_i}(X)$, $i = 1, 2, 3$, be the corresponding rational classes, and so $[a_1, a_2, a_3]'$ is the classical third order rational product. The indeterminacy of $[a_1, a_2, a_3]'$ is $J_{n+1}(a_1, a_2, a_3)$, the same as for $[a_1, a_2, a_3]$. (Hardie [2].) We have the following corollary of Proposition 3.4.

COROLLARY 3.5. *For $a \in [a_1, a_2, a_3]$ and $b \in [a_1, a_2, a_3]'$, there is a nonzero rational number α, such that $a - \alpha b$ is a sum of rational (second order) Whitehead products in $L_{n+1}(X)$.*

Proof. Let $\hat{g}_0$ be an extension of $g_0\colon S_0 \to X_0$ to T_0, where $g\colon S \to X$ is determined by x_i, $i = 1, 2, 3$. Then, if $\partial h^{-1}(z_0) = c$, $\hat{g}_{0\#}(c)$ is a representative of $[a_1, a_2, a_3]'$. Thus $b - \hat{g}_{0\#}(c)$ is a sum of rational Whitehead products.

Now, from Proposition 3.4,

$$d^2[\Sigma a_1 \Sigma a_2 \Sigma a_3]^{(2)} = \beta[\Sigma \hat{g}_{0\#}(c)]^{(2)} ,$$

where β is a nonzero rational number.

But, from Theorem 2.4,

$$d^2[\Sigma a_1 \Sigma a_2 \Sigma a_3]^{(2)} = (-1)^{n_1 n_3 + n_2 + 1}[\Sigma a]^{(2)} .$$

Thus $(-1)^{n_1 n_3 + n_2 + 1} a - \beta \hat{g}_{0\#}(c)$ is a sum of rational Whitehead products by Theorem 2.1, and the result follows.

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